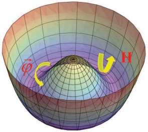

Effective Field Theory

Antonio Pich

IFIC, Univ. Valencia – CSIC

Outline

hep-ph/9806303
arXiv:1804.05664

1) General Aspects of EFT

- Dimensional Analysis
- Relevant, Irrelevant & Marginal
- Quantum Loops
- Decoupling. Matching
- Scaling

2) Nambu-Goldstone Bosons

- Sigma Model
- Goldstone Theorem
- Chiral Symmetry
- Effective Goldstone Theory
- Chiral Symmetry Breakings

3) QCD: χPT

- External Sources
- Lowest-Order χPT
- Weinberg's Power Counting
- Quantum Loops
- χPT at $\mathcal{O}(p^4)$ and Beyond

4) EW Effective Theory

- Higgs Mechanism
- Custodial Symmetry
- Equivalence Theorem
- EW Effective Theory
- Linear Realization

Outline (cont)

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5) Fingerprints of Heavy Scales

- CCWZ Formalism
- Heavy Fields
- Low-Energy Constants (LECs)
- QCD: Large-N_C Limit
- Asymptotic Behaviour
- EWET: Oblique Constraints
- Signals of Heavy Scales

Euler-Heisenberg Lagrangian

Light-by-light scattering in QED at very low energies $(\mathbf{E_\gamma \ll m_e})$

- Gauge, Lorentz, Charge Conjugation & Parity constraints
- Energy expansion (E_γ/m_e)

$$\mathcal{L}_{\rm eff} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{a}{m_e^4} \left(F^{\mu\nu} F_{\mu\nu}\right)^2 + \frac{b}{m_e^4} F^{\mu\nu} F_{\nu\sigma} F^{\sigma\rho} F_{\rho\mu} + \mathcal{O}(F^6/m_e^8)$$

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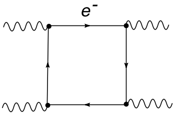

$$\longrightarrow \qquad a = -\frac{1}{36}\,\alpha^2 \quad , \quad b = \frac{7}{90}\,\alpha^2$$

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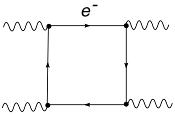

$$\Longrightarrow \qquad a = -\frac{1}{36}\,\alpha^2 \quad , \quad b = \frac{7}{90}\,\alpha^2$$

$$\sigma(\gamma\gamma \to \gamma\gamma) \;\propto\; \frac{\alpha^4 E^6}{m_e^8}$$

Why the sky looks blue?

Why the sky looks blue?

Rayleigh scattering

Low-energy scattering of photons with neutral atoms

$$E_\gamma \;\ll\; \Delta E \sim \alpha^2 m_e \;\ll\; a_0^{-1} \sim \alpha m_e \;\ll\; M_A$$

- Neutral atom + gauge invariance ➡ $F^{\mu\nu} = (\vec{E}, \vec{B})$
- Non-relativistic description: $\mathcal{L} = \psi^\dagger \left(i\,\partial_t + \frac{1}{2M}\,\vec{\nabla}^2 \right) \psi + \mathcal{L}_{\mathrm{int}}$

$$\mathcal{L}_{\mathrm{int}} \;=\; a_0^3\; \psi^\dagger\psi\, \left(c_1 \vec{E}^2 + c_2 \vec{B}^2 \right) + \ldots \qquad , \qquad c_i \sim \mathcal{O}(1)$$

$$\mathcal{M} \;\sim\; c_i\, a_0^3\, E_\gamma^2 \qquad \longrightarrow \qquad \boldsymbol{\sigma \;\propto\; a_0^6\, E_\gamma^4}$$

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Blue light is scattered more strongly than red one

Dimensions

$$S = \int d^4x \, \mathcal{L}(x) \quad \longrightarrow \quad [\,\mathcal{L}\,] = E^4$$

$$\mathcal{L}_{\rm KG} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \, \phi^\dagger \phi \quad \longrightarrow \quad [\,\phi\,] = [\,V^\mu\,] = [\,A^\mu\,] = E$$

$$\mathcal{L}_{\rm Dirac} = \bar{\psi} \, (i\gamma^\mu \partial_\mu - m) \, \psi \quad \longrightarrow \quad [\,\psi\,] = E^{3/2}$$

$$[\,\sigma\,] = E^{-2} \qquad , \qquad [\,\Gamma\,] = E$$

Scalar Field Theory

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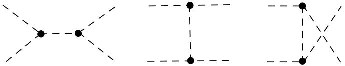

$$\sigma(1+2\to 3+4) \;\sim\; \frac{\lambda^4}{s^3}\,\left\{1+\mathcal{O}\left(\frac{\lambda^2}{s}\right)+\cdots\right\} \qquad (s \gg m^2)$$

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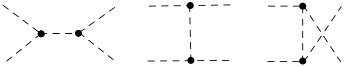

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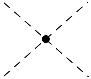

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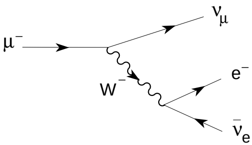
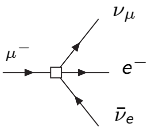
Fermi Theory

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8\, M_W^2}$$



$$\frac{-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2}}{q^2 - M_W^2} \quad \xrightarrow{q^2 \ll M_W^2} \quad \frac{g_{\mu\nu}}{M_W^2}$$



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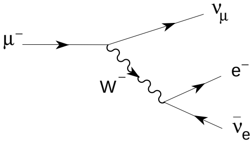

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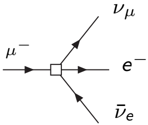

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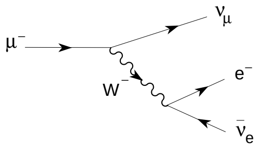

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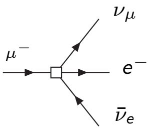

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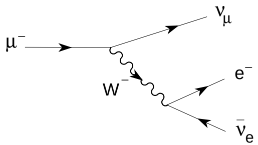

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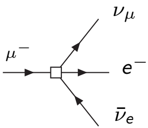

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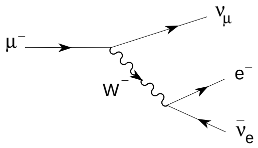

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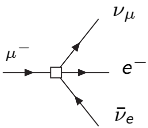

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Exp: $(17.82 \pm 0.04)\%$

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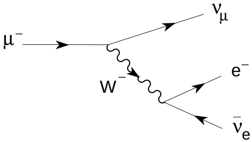

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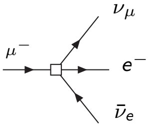

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- $\sigma(\nu_\mu e^- \to \mu^- \nu_e) \sim G_F^2\, s$

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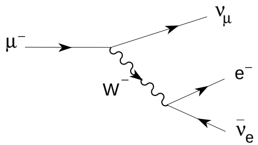

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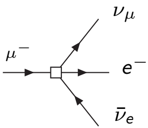

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Exp: $(17.82 \pm 0.04)\%$

- $\sigma(\nu_\mu e^- \to \mu^- \nu_e) \sim G_F^2\, s$ $\qquad$ **Violates unitarity at high energies**

Relevant, Irrelevant & Marginal

$$\mathcal{L} = \sum_i c_i \, O_i \qquad , \qquad [O_i] = d_i \quad \longrightarrow \quad c_i \sim \frac{1}{\Lambda^{d_i-4}}$$

Low-energy behaviour:

- **Relevant** $(\mathbf{d_i < 4})$**:** $\qquad I \, , \, \phi^2 \, , \, \phi^3 \, , \, \bar{\psi}\psi$

  Enhanced by $(\Lambda/E)^{4-d_i}$

- **Marginal** $(\mathbf{d_i = 4})$**:** $\qquad \phi^4 \, , \, \phi \, \bar{\psi}\psi \, , \, V_\mu \, \bar{\psi}\gamma^\mu\psi$

- **Irrelevant** $(\mathbf{d_i > 4})$**:** $\qquad \bar{\psi}\psi \, \bar{\psi}\psi \, , \, \partial_\mu\phi \, \bar{\psi}\gamma^\mu\psi \, , \, \phi^2 \, \bar{\psi}\psi \, , \, \cdots$

  Suppressed by $(E/\Lambda)^{d_i-4}$

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \beta_1 \frac{\alpha(Q_0^2)}{2\pi} \log\left(Q^2/Q_0^2\right)}$$

QED: $\beta_1 = \frac{2}{3} \sum_f Q_f^2 N_f > 0$ → $\lim_{Q^2\to 0} \alpha(Q^2) = 0$

Quantum corrections make QED irrelevant at low energies

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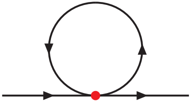
Quantum corrections make QED irrelevant at low energies

QCD: $\beta_1 = \frac{2\,N_F \,-\, 11\,N_C}{6} \;<\; 0$ ➡ $\lim_{Q^2\to 0}\alpha_s(Q^2) \;=\; \infty$

Quantum corrections make QCD relevant at low energies

Quantum Loops

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi - \frac{a}{\Lambda^2}\,(\bar{\psi}\psi)^2 - \frac{b}{\Lambda^4}\,(\bar{\psi}\,\Box\psi)(\bar{\psi}\psi) + \cdots$$



$$\delta m \;\sim\; 2i\,\frac{a}{\Lambda^2}\,m\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-m^2}$$

Quantum Loops

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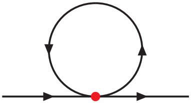

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- **Cut-off regularization:** $\delta m \sim m\,\frac{a}{\Lambda^2}\,\Lambda^2 \sim m\,a$ **Not suppressed!**

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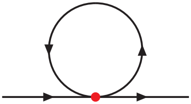

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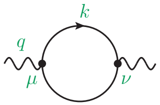
- **Cut-off regularization:** $\delta m \sim m\,\frac{a}{\Lambda^2}\,\Lambda^2 \sim m\,a$ **Not suppressed!**
- **Dimensional regularization:** **Mass independent**

$$\delta m \;\sim\; 2a\,m\,\frac{m^2}{16\pi^2\Lambda^2}\left\{\Delta_\infty(\mu) + \log\left(\frac{m^2}{\mu^2}\right) - 1 + \mathcal{O}(D-4)\right\}$$

Well-defined expansion $\Delta_\infty(\mu) = \frac{2\,\mu^{D-4}}{D-4} + \gamma_E - \log(4\pi)$

Vacuum Polarization $(\mathbf{m_f = 0})$



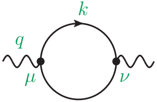
$$i\,\Pi^{\mu\nu}(q) \;=\; i\,\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\,\Pi(q^2)$$

$$\Pi(q^2) \;=\; -\frac{\alpha Q_f^2}{3\pi}\left\{\Delta_\infty(\mu) + \log\left(\frac{-q^2}{\mu^2}\right) - \frac{5}{3}\right\}$$

$$\equiv\; \Delta\Pi_\epsilon(\mu^2) \;+\; \Pi_R(q^2/\mu^2)$$

$$\Delta_\infty(\mu) = \frac{2\,\mu^{D-4}}{D-4} + \gamma_E - \log(4\pi)$$

Vacuum Polarization $(m_f = 0)$



$$i\,\Pi^{\mu\nu}(q) \;=\; i\,\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\,\Pi(q^2)$$

$$\Pi(q^2) \;=\; -\frac{\alpha Q_f^2}{3\pi}\left\{\Delta_\infty(\mu) + \log\left(\frac{-q^2}{\mu^2}\right) - \frac{5}{3}\right\}$$

$$\equiv\; \Delta\Pi_\epsilon(\mu^2) \;+\; \Pi_R(q^2/\mu^2)$$

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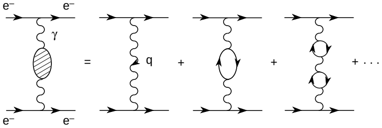

$$\alpha_0\,\left\{1 - \Delta\Pi_\epsilon(\mu^2) - \Pi_R(q^2/\mu^2)\right\}$$

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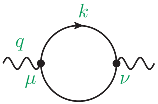

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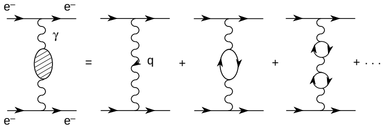

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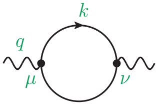
$$\frac{\mu}{\alpha}\,\frac{d\alpha}{d\mu} \equiv \beta(\alpha) = \beta_1\,\frac{\alpha}{\pi} + \cdots \quad\Longrightarrow\quad \alpha(Q^2) \;\approx\; \frac{\alpha(Q_0^2)}{1 - \beta_1\,\frac{\alpha(Q_0^2)}{2\pi}\;\log\left(Q^2/Q_0^2\right)}$$

Vacuum Polarization $(m_f \neq 0)$



$$i\,\Pi^{\mu\nu}(q) \;=\; i\,\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\,\Pi(q^2)$$

$$\Pi(q^2) \;=\; -\frac{\alpha Q_f^2}{3\pi}\left\{\Delta_\infty(\mu) + 6\int_0^1 dx\; x(1-x)\,\log\left(\frac{m_f^2 - q^2 x(1-x)}{\mu^2}\right)\right\}$$

$$\equiv\; \Delta\Pi_\epsilon(\mu^2) \;+\; \Pi_R(q^2/\mu^2)$$

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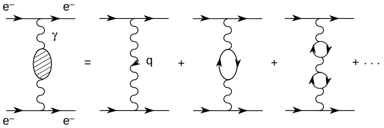

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$$\Pi_R(q^2/\mu^2) \; = \; -Q_f^2 \, \frac{\alpha}{3\pi} \, 6 \int_0^1 dx \, x(1-x) \, \log\left[\frac{m_f^2 - q^2 x(1-x)}{m_f^2 + \mu^2 x(1-x)}\right]$$

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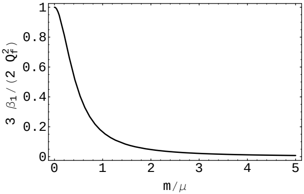

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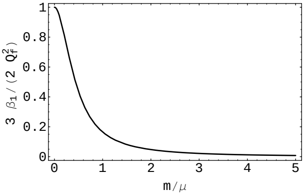

- $\mathbf{m_f^2} \ll \mu^2$, $\mathbf{q^2}$: $\quad \beta_1 = \frac{2}{3}\,Q_f^2 \quad , \quad \Pi_R(q^2/\mu^2) = -Q_f^2\,\frac{\alpha}{3\pi}\,\log\left(-q^2/\mu^2\right)$

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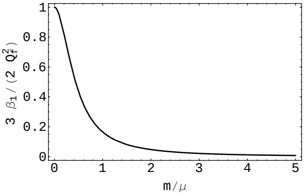

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- $\mathbf{m_f^2 \gg \mu^2, q^2}$: $\quad \beta_1 \sim \frac{2}{15} Q_f^2 \frac{\mu^2}{m_f^2} \quad , \quad \Pi_R(q^2/\mu^2) \sim Q_f^2 \frac{\alpha}{15\pi} \frac{q^2+\mu^2}{m_f^2}$

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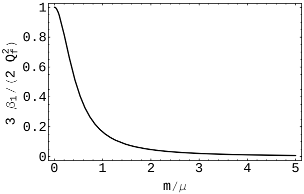

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→ **DECOUPLING** **(Appelquist-Carazzone Theorem)**

$\overline{\text{MS}}$ Scheme:

$$\Delta\Pi_\epsilon(\mu^2) \;\equiv\; -Q_f^2\,\frac{\alpha_0}{3\pi}\,\Delta_\infty(\mu)$$

$$\Pi_R(q^2/\mu^2) \;=\; -Q_f^2\,\frac{\alpha}{3\pi}\,6\int_0^1 dx\; x(1-x)\,\log\left[\frac{m_f^2 - q^2 x(1-x)}{\mu^2}\right]$$

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Heavy fermions do not decouple

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Perturbation theory breaks down

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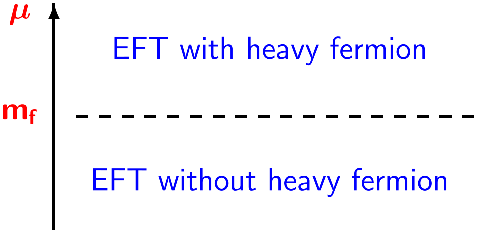
SOLUTION: Integrate Out Heavy Particles

Matching



$$\beta_1 = \frac{2}{3}\, Q_f^2 + \beta_1^{\rm light}$$

$$\beta_1 = \beta_1^{\rm light}$$

- **Two different EFTs (with and without the heavy fermion f)**
- **Same S-matrix elements for light-particle scattering at $\mu = m_f$**

Effective Field Theory

$$\mathcal{L}(\phi, \Phi) \; = \; \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\Phi^2 - \frac{\lambda}{2}\phi^2\Phi$$

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$$\sigma(\phi\phi \to \phi\phi) \; \sim \; \frac{1}{E^2} \; \times \; \begin{cases} (\lambda/E)^4 \; , & (m \ll M \ll E) \\ (\lambda/M)^4 \; , & (m, E \ll M) \end{cases}$$

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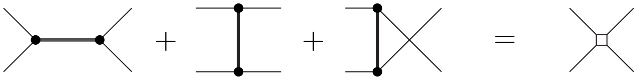

$\mathbf{E, m \ll M}$

$$\frac{\lambda^2}{s-M^2} = -\frac{\lambda^2}{M^2}\sum_{n=0}\frac{s}{M^2} \quad \Longrightarrow \quad \mathcal{L}_{\rm eff}(\phi) \;=\; \sum_i c_i\, O_i(\phi)$$

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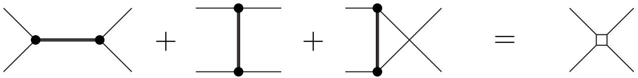

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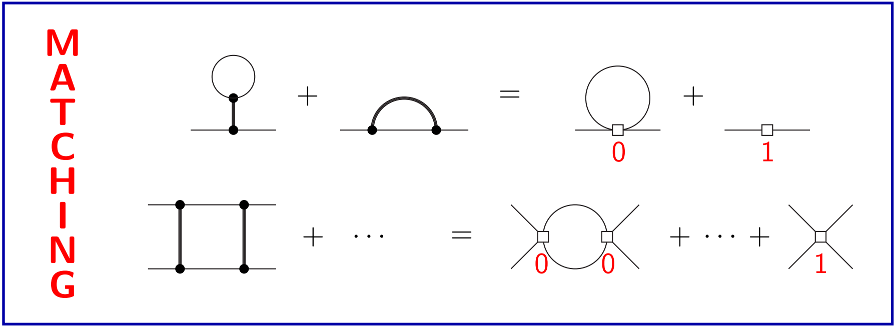
$$[O_i] = d_i \qquad ; \qquad c_i \sim \frac{\lambda^2}{M^2}\,\frac{1}{M^{d_i-4}}$$

One Loop:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\, a\, (\partial\phi)^2 - \frac{1}{2}\, b\, \phi^2 + c\, \frac{\lambda^2}{8M^2}\, \phi^4 + \cdots$$



$$a = 1 + a_1\, \frac{\lambda^2}{16\pi^2 M^2} + \cdots \qquad ; \qquad b = m^2 + b_1\, \frac{\lambda^2}{16\pi^2} + \cdots$$

$$c = 1 + c_1\, \frac{\lambda^2}{16\pi^2 M^2} + \cdots \qquad ; \qquad \cdots$$

Principles of Effective Field Theory

- **Low-energy dynamics** independent of details at high energies
- Appropriate physics description at the analyzed scale **(degrees of freedom)**
- **Energy gaps:** $\quad 0 \leftarrow m \ll E \ll M \rightarrow \infty$
- Non-local heavy-particle exchanges replaced by a **tower of local interactions** among the light particles
- **Accuracy:** $\quad (E/M)^{d_i-4} \gtrsim \epsilon \quad \Longleftrightarrow \quad d_i \lesssim 4 + \frac{\log(1/\epsilon)}{\log(M/E)}$
- **Same infrared** (but different ultraviolet) **behaviour** than the underlying fundamental theory
- The only remnants of the high-energy dynamics are in the **low-energy couplings** and in the **symmetries** of the EFT

Evolution from High to Low Scales

Large μ

$\mathcal{L}(\phi_i) + \mathcal{L}(\phi_i, \Phi)$ — ϕ_i, Φ

↓ **Renormalization Group**

$\mu = M$ - - - - - - **Matching**

$\mathcal{L}(\phi_i) + \delta\mathcal{L}(\phi_i)$ — ϕ_i

↓ **Renormalization Group**

Small μ

QCD Matching

$$(\mu > M) \qquad \mathcal{L}_{\rm QCD}^{(N_F)} \quad \Longleftrightarrow \quad \mathcal{L}_{\rm QCD}^{(N_F-1)} + \sum_{d_i>4} \frac{c_i}{M^{d_i-4}}\, O_i \qquad (\mu < M)$$

QCD Matching

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$$\alpha_s^{(N_F)}(\mu^2) \;=\; \alpha_s^{(N_F-1)}(\mu^2)\;\left\{ 1 + \sum_{k=1}^{\infty} C_k(L)\;\left[\frac{\alpha_s^{(N_F-1)}(\mu^2)}{\pi}\right]^k \right\}$$

$$L \equiv \ln\left(\mu^2/M^2\right)$$

$$m_q^{(N_F)}(\mu^2) \;=\; m_q^{(N_F-1)}(\mu^2)\;\left\{ 1 + \sum_{k=1}^{\infty} H_k(L)\;\left[\frac{\alpha_s^{(N_F-1)}(\mu^2)}{\pi}\right]^k \right\}$$

QCD Matching

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- **Matching conditions known to 4 loops:** $C_{1,2,3,4}$, $H_{1,2,3,4}$

  (Schroder-Steinhauser, Chetyrkin et al, Larin et al, Liu-Steinhauser)

- $\alpha_s(\mu^2)$ **is not continuous at threshold**

Wilson Coefficients: $\mathcal{L} = \sum_i \frac{c_i}{\Lambda^{d_i-4}} O_i$

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$$\langle O_i \rangle_B = Z_i(\epsilon, \mu) \, \langle O_i(\mu) \rangle_R \qquad ; \qquad \mu \frac{d}{d\mu} \langle O_i \rangle_B = 0$$

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$$\longrightarrow \qquad \begin{aligned} c_i(\mu) &= c_i(\mu_0)\,\exp\left\{\int_{\alpha_0}^{\alpha}\frac{d\alpha}{\alpha}\,\frac{\gamma_{O_i}(\alpha)}{\beta(\alpha)}\right\} \\ &= c_i(\mu_0)\left[\frac{\alpha(\mu^2)}{\alpha(\mu_0^2)}\right]^{\gamma_{O_i}^{(1)}/\beta_1}\left\{1+\cdots\right\}\end{aligned}$$

Operator Mixing: $\mathcal{L} = \sum_i \frac{c_i}{\Lambda^{d_i-4}} O_i$

$$\langle O_i \rangle_B = \sum_j \mathbf{Z}_{ij}(\epsilon,\mu)\, \langle O_j(\mu)\rangle_R \qquad ; \qquad \gamma_O \equiv \mathbf{Z}^{-1}\, \mu \frac{d}{d\mu}\, \mathbf{Z}$$

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$$\left(\mu \frac{d}{d\mu} + \gamma_O\right) \langle \vec{O} \rangle_R = 0 \qquad ; \qquad \left(\mu \frac{d}{d\mu} - \gamma_O^T\right) \langle \vec{c}\, \rangle_R = 0$$

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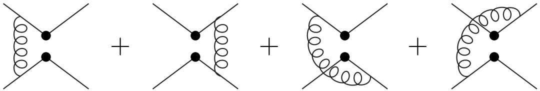
$$\rightarrow \qquad c_i(\mu) = \sum_{j,k} \mathbf{U}_{ij}\; \exp\left\{ \int_{\alpha_0}^{\alpha} \frac{d\alpha}{\alpha} \frac{\tilde{\gamma}_{O_j}(\alpha)}{\beta(\alpha)} \right\} \mathbf{U}^{-1}_{jk}\, c_k(\mu_0)$$

Wilson Coeff. in the Fermi EFT

$$\mathcal{L}_{\rm eff} = \frac{G_F}{\sqrt{2}} V_{12} V_{43}^* \, O_{\{1,2;3,4\}} \qquad ; \qquad O_{\{1,2;3,4\}} \equiv [\bar{q}_1 \gamma^\mu (1-\gamma_5) q_2] \, [\bar{q}_3 \gamma_\mu (1-\gamma_5) q_4]$$



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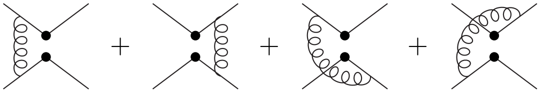

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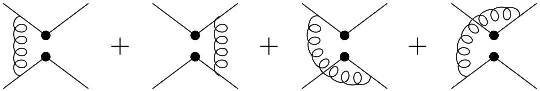

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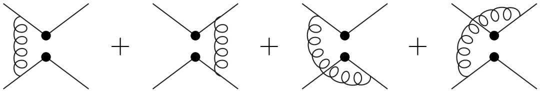

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$$\gamma_\pm = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \frac{\alpha_s}{\pi} \quad \Longrightarrow \quad \boxed{ c_\pm(\mu) \approx \left(\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)} \right)^{a_\pm} \quad , \quad a_\pm = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \frac{6}{33 - 2N_f} }$$

$K \to (2\pi)_I$

$$A[K^0 \to \pi^+\pi^-] \equiv A_0\, e^{i\chi_0} + \frac{1}{\sqrt{2}} A_2\, e^{i\chi_2}$$

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Bose: $I = 0, 2$ $\quad (\Delta I = \frac{1}{2}, \frac{3}{2})$

$$\omega \equiv \frac{\mathrm{Re}\,(A_2)}{\mathrm{Re}\,(A_0)} \approx \frac{1}{22}$$

$\Delta I = 1/2$ Rule

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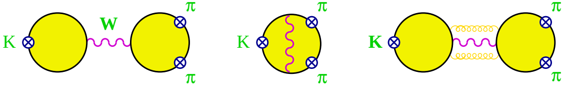
Weak Currents Factorize at Large N_C



$\mathcal{O}(N_C^2)$ $\qquad \mathcal{O}(N_C)$ $\qquad \mathcal{O}(1)$

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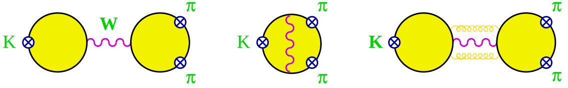

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Short-distance enhancement: $Q_\pm \sim [\bar{u}_L \gamma^\mu s_L]\, [\bar{d}_L \gamma_\mu u_L] \pm [\bar{u}_L \gamma^\mu u_L]\, [\bar{d}_L \gamma_\mu s_L]$

$$c_\pm(\mu) \approx \left(\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)}\right)^{a_\pm} \approx \begin{cases} 0.75 \\ 1.76 \end{cases} \qquad \text{at} \quad \mu = 1\ \mathrm{GeV}$$