

Kick-off meeting of the LHCPheNet Initial Training Network

Multi-Summation for Particle Physics

Carsten Schneider
Research Institute for Symbolic Computation, Linz

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A warm up example:

$$\text{GIVEN } F(N) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{e^{-e\gamma}}{\Gamma(\varepsilon + 1)} \times$$

$$\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} \right.$$

$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right).$$

$$f(N, k, j)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

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$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+N)\Gamma(k+j+2+\frac{\epsilon}{2})} \right).$$

$$\underbrace{\hspace{15em}}_{f(N, k, j)}$$

FIND the first coefficients of the ϵ -expansion

$$F(N) = F_0(N) + \epsilon F_1(N) + \dots$$

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$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\epsilon}{2})\Gamma(1+\frac{\epsilon}{2})\Gamma(j+1+\epsilon)\Gamma(j+1-\frac{\epsilon}{2})\Gamma(k+j+1+\frac{\epsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\epsilon}{2}+N)\Gamma(k+j+2+\frac{\epsilon}{2})} \right).$$

$$f(N, k, j)$$

Step 1: Compute the first coefficients of the ϵ -expansion

$$f(N, k, j) = f_0(N, k, j) + \epsilon f_1(N, k, j) + \dots$$

Arose in the context of

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A warm up example:

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$$\times \left(\frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(\frac{\varepsilon}{2})\Gamma(1-\frac{\varepsilon}{2})\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+1+\frac{\varepsilon}{2})\Gamma(k+j+1+N)}{\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(j+2+N)\Gamma(k+j+2)} \right.$$

$$\left. + \frac{\Gamma(k+1)}{\Gamma(k+2+N)} \frac{\Gamma(-\frac{\varepsilon}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(j+1+\varepsilon)\Gamma(j+1-\frac{\varepsilon}{2})\Gamma(k+j+1+\frac{\varepsilon}{2}+N)}{\Gamma(j+1)\Gamma(j+2+\frac{\varepsilon}{2}+N)\Gamma(k+j+2+\frac{\varepsilon}{2})} \right).$$

$$f(N, k, j)$$

Step 2: **Simplify** the sums in

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(N, k, j) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_0(N, k, j) + \varepsilon \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_1(N, k, j) + \dots$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, *Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals*. 2006

Simplify the constant term

$$f(N, k, j)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

where

$$S_1(N) = \sum_{i=1}^N \frac{1}{i}$$

Simplify the constant term

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overbrace{\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right)}^{f(N, k, j)} \\
 & + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \\
 & \sum_{j=0}^a f(N, k, j) = \text{▶ Sigma}
 \end{aligned}$$

Simplify the constant term

$$f(N, k, j)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \text{Sigma}$$

FIND $g(j)$:

$$f(N, k, j) = g(j+1) - g(j)$$

Simplify the constant term

$$f(N, k, j)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \text{Sigma}$$

FIND $g(j)$:

$$f(N, k, j) = g(j+1) - g(j)$$

Sigma (based on a refined version of M. Karr's difference fields (1981)) computes

$$g(j) = \frac{(j+k+1)(j+N+1)j!k!(j+k+N)!(S_1(j) - S_1(j+k) - S_1(j+N) + S_1(j+k+N))}{kN(j+k+1)!(j+N+1)!(k+N+1)!}$$

Simplify the constant term

$$f(N, k, j)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \text{Sigma}$$

FIND $g(j)$:

$$f(N, k, j) = g(j+1) - g(j)$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^a f(N, k, j) = g(a+1) - g(0)$$

Simplify the constant term

$$f(N, k, j)$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right. \\ \left. + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!} \right)$$

$$\sum_{j=0}^a f(N, k, j) = \frac{(a+1)!(k-1)!(a+k+N+1)!(S_1(a) - S_1(a+k) - S_1(a+N) + S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!} \\ + \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} + \frac{(2a+k+N+2)ak!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}$$

$a \rightarrow \infty$

Simplify the constant term

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overbrace{\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right)}^{f(N, k, j)} \\
 & + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}
 \end{aligned}$$

$$\sum_{j=0}^{\infty} f(N, k, j) = \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!}$$

Simplify the constant term

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overbrace{\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right)}^{f(N, k, j)} \\
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 \end{aligned}$$

$$\begin{aligned}
 \sum_{k=1}^a \sum_{j=0}^{\infty} f(N, k, j) &= \sum_{k=1}^a \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} \\
 &= \text{Sigma}
 \end{aligned}$$

Simplify the constant term

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overbrace{\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right)}^{f(N, k, j)} \\
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 \end{aligned}$$

$$\begin{aligned}
 \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(N, k, j) &= \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} \\
 &= \frac{S_1(N)^2 + S_2(N)}{2N(N+1)!}
 \end{aligned}$$

where

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a **definite** sum

$$S(n) = \sum_{k=0}^n f(n, k);$$

$f(n, k)$: indefinite nested product-sum in k ;
 n : extra parameter

FIND a **recurrence** for $S(n)$

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2. Recurrence solving

GIVEN a recurrence

$$a_0(n), \dots, a_d(n), h(n):$$

indefinite nested product-sum expressions.

$$a_0(n)S(n) + \dots + a_d(n)S(n+d) = h(n);$$

FIND **all solutions** expressible by indefinite nested products and sums

(Nörlund 24, Abramov/Petkovšek 94, Hendriks/Singer 99/Sigma 01)

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3. Find a “closed form”

$S(n)$ =combined solutions.

Simplify the constant term

$$\begin{aligned}
 & \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overbrace{\left(\frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} \right)}^{f(N, k, j)} \\
 & + \frac{j!k!(j+k+N)!(-S_1(j) + S_1(j+k) + S_1(j+N) - S_1(j+k+N))}{(j+k+1)!(j+N+1)!(k+N+1)!}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} f(N, k, j) &= \sum_{k=1}^{\infty} \frac{S_1(k) + S_1(N) - S_1(k+N)}{kN(k+N+1)N!} \\
 &= \frac{S_1(N)^2 + S_2(N)}{2N(N+1)!}
 \end{aligned}$$

where

$$S_2(N) = \sum_{i=1}^N \frac{1}{i^2}$$

Automatic machinery

“Background”

- ▶ The following examples arise in the context of 2- and 3-loop massive single scale Feynman diagrams with operator insertion.
- ▶ These are related to the QCD anomalous dimensions and massive operator matrix elements.
- ▶ At 2-loop order all respective calculations are finished:

M. Buza, Y. Matiounine, J. Smith, R. Migneron, W.L. van Neerven, Nucl. Phys. **B472** (1996) 611;

I. Bierenbaum, J. Blümlein, S. Klein, Nucl. Phys. **B780** (2007) 40;

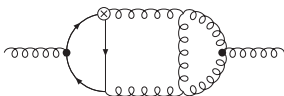
I. Bierenbaum, J. Blümlein, S. Klein, C. Schneider, Nucl.Phys. **B803** (2008)

and lead to representations in terms of harmonic sums.

Example 1: All N-Results for 3-Loop Ladder Graphs

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
A. Hasselhuhn (DESY), S. Klein (RWTH)

Consider, e.g., the diagram



(containing three massive fermion propagators)



Around 1000 sums have to be calculated

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

Simple sum

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \boxed{\sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}}$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \left[\sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!} \right]$$

||

$$\left(\binom{j+1}{r} \left(\frac{(-1)^r (-j+N-2)! r!}{(N+1)(-j+N+r-1)(-j+N+r)!} + \frac{(-1)^{N+r} (j+1)! (-j+N-2)! (-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)! (-j-1)_r (2-N)_j} \right) \right)$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+N-2)! r!}{(N+1)(-j+N+r-1)(-j+N+r)!} + \frac{(-1)^{N+r} (j+1)! (-j+N-2)! (-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)! (-j-1)_r (2-N)_j} \right) \right)$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\sum_{r=0}^{j+1} \binom{j+1}{r} \left(\frac{(-1)^r (-j+N-2)! r!}{(N+1)(-j+N+r-1)(-j+N+r)!} + \frac{(-1)^{N+r} (j+1)! (-j+N-2)! (-j+N-1)_r r!}{(N-1)N(N+1)(-j+N+r)! (-j-1)_r (2-N)_j} \right) \right)$$

||

$$\left(\frac{N^2 - N + 1}{(N-1)^2 N^2 (N+1)(2-N)_j} + \frac{\sum_{i=1}^j \frac{(2-N)_i}{(-i+N-1)^2 (i+1)!}}{(N+1)(2-N)_j} + \frac{(-1)^{j+N} (-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2 N!} \right) (j+1)! - \frac{1}{(N+1)^2 (-j+N-1)}$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\left(\frac{N^2 - N + 1}{(N-1)^2 N^2 (N+1)(2-N)_j} + \frac{\sum_{i=1}^j \frac{(2-N)_i}{(-i+N-1)^2 (i+1)!}}{(N+1)(2-N)_j} + \frac{(-1)^{j+N} (-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2 N!} \right) (j+1)! - \frac{1}{(N+1)^2 (-j+N-1)} \right)$$

$$\sum_{j=0}^{N-2} \sum_{r=0}^{j+1} \sum_{s=0}^{N-j+s-2} \frac{(-1)^{r+s} \binom{j+1}{r} \binom{-j+N+r-2}{s} (-j+N-2)! r!}{(N-s)(s+1)(-j+N+r)!}$$

||

$$\sum_{j=0}^{N-2} \left(\left(\frac{N^2 - N + 1}{(N-1)^2 N^2 (N+1)(2-N)_j} + \frac{\sum_{i=1}^j \frac{(2-N)_i}{(-i+N-1)^2 (i+1)!}}{(N+1)(2-N)_j} + \frac{(-1)^{j+N} (-j-2)(-j+N-2)!}{(j-N+1)(N+1)^2 N!} \right) (j+1)! - \frac{1}{(N+1)^2 (-j+N-1)} \right)$$

||

$$\frac{-N^2 - N - 1}{N^2 (N+1)^3} + \frac{(-1)^N (N^2 + N + 1)}{N^2 (N+1)^3} - \frac{2S_{-2}(N)}{N+1} + \frac{S_1(N)}{(N+1)^2} + \frac{S_2(N)}{-N-1}$$

Note: $S_a(N) = \sum_{i=1}^N \frac{\text{sign}(a)^i}{i^{|a|}}$, $a \in \mathbb{Z} \setminus \{0\}$.

A typical sum

$$\sum_{j=0}^{N-2} \sum_{s=1}^{j+1} \sum_{r=0}^{N+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+N+s-2}{r} (N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

A typical sum

$$\sum_{j=0}^{N-2} \sum_{s=1}^{j+1} \sum_{r=0}^{N+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+N+s-2}{r} (N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!}$$

$$= \frac{(2N^2 + 6N + 5) S_{-2}(N)^2}{2(N+1)(N+2)} + S_{-2,-1,2}(N) + S_{-2,1,-2}(N)$$

$$+ \dots$$

where, e.g.,

$$S_{-2,1,-2}(N) = \sum_{i=1}^N \frac{(-1)^i \sum_{j=1}^i \frac{\sum_{k=1}^j (-1)^k}{k^2}}{i^2}$$

Vermaseren 98; Blümlein/Kurth 99

A typical sum

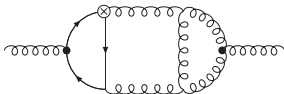
$$\begin{aligned}
& \sum_{j=0}^{N-2} \sum_{s=1}^{j+1} \sum_{r=0}^{N+s-j-2} \sum_{\sigma=0}^{\infty} \frac{-2(-1)^{s+r} \binom{j+1}{s} \binom{-j+N+s-2}{r} (N-j)!(s-1)!\sigma! S_1(r+2)}{(N-r)(r+1)(r+2)(-j+N+\sigma+1)(-j+N+\sigma+2)(-j+N+s+\sigma)!} \\
&= \frac{(2N^2 + 6N + 5) S_{-2}(N)^2}{2(N+1)(N+2)} + S_{-2,-1,2}(N) + S_{-2,1,-2}(N) \\
&+ \dots - S_{2,1,1,1}(-1, 2, \frac{1}{2}, -1; N) + S_{2,1,1,1}(1, \frac{1}{2}, 1, 2; N) \\
&+ \dots
\end{aligned}$$

where, e.g.,

145 S -sums occur

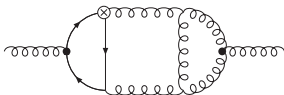
$$S_{2,1,1,1}(1, \frac{1}{2}, 1, 2; N) = \sum_{i=1}^N \frac{\sum_{j=1}^i \frac{\left(\frac{1}{2}\right)^j \sum_{k=1}^j \frac{\sum_{l=1}^k \frac{2^l}{l}}{k}}{j}}{i^2}$$

S. Moch, P. Uwer, S. Weinzierl 02



Sigma.m

Around 1000 sums are calculated containing in total 533 S -sums



Sigma.m

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J. Ablinger's HarmonicSum.m

After elimination the following sums remain:

$$S_{-4}(N), S_{-3}(N), S_{-2}(N), S_1(N), S_2(N), S_3(N), S_4(N), S_{-3,1}(N), \\ S_{-2,1}(N), S_{2,-2}(N), S_{2,1}(N), S_{3,1}(N), S_{-2,1,1}(N), S_{2,1,1}(N)$$

For 3-loop ladder graphs we dealt (so far) with up to 6-fold sums. E.g.,

$$\sum_{l=2}^N \sum_{j=2}^l \sum_{k=1}^j \sum_{r=0}^{l-k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{2(-1)^{j+k+l+r} (k-1)! \binom{j}{k} \binom{l}{j} \binom{l-k}{r} \binom{N}{l}}{(N-1)N(k+m+n+r+2)(k+m)!}$$

$$\frac{(k+m-1)!(N-j)!(l+r-2)!(n+r+1)! (k+m+n+r-1)!}{(-j+N+2)!(k+r-1)!(l+n+r-1)! (k+m+n+r+1)!}$$

$$= \frac{1}{N(N+1)(N+2)} \left(2((3-2^{N+3}) - (-1)^N) \zeta(3) \right.$$

$$+ \frac{1}{6} S_1(N)^3 + \frac{4(2N+3)}{(N+1)^2(N+2)} S_1(N) + \frac{8(2N+3)}{(N+1)^3(N+2)}$$

$$- \frac{-56 - 40N - 3N^2 + 2S_1(N) + 3NS_1(N) + N^2 S_1(N)}{2(1+N)(2+N)} S_2(N)$$

$$+ \frac{(16+12N+N^2)}{2(1+N)(2+N)} S_1(N)^2 + \frac{1}{3}(-3N-17) S_3(N)$$

$$- (-1)^N S_{-3}(N) + (-N-3) S_{2,1}(N) - 2(-1)^N S_{-2,1}(N)$$

$$\left. + 2^{N+4} S_{1,2}(\frac{1}{2}, 1, N) + 2^{N+3} S_{1,1,1}(\frac{1}{2}, 1, 1, N) \right)$$

and ...

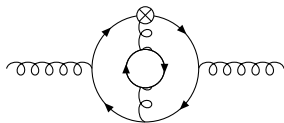
$$\begin{aligned}
& \sum_{j=0}^{N-3} \sum_{k=0}^j \sum_{l=0}^k \sum_{q=0}^{-j+N-3} \sum_{s=1}^{-l+N-q-3} \sum_{r=0}^{-l+N-q-s-3} \\
& \frac{\binom{j+1}{k+1} \binom{k}{l} \binom{N-1}{j+2} \binom{-j+N-3}{q} \binom{-l+N-q-3}{s} \binom{-l+N-q-s-3}{r} r! (-l+N-q-r-s-3)! (s-1)!}{(-l+N-q-2)!} \\
& \left(\frac{2 \frac{(-1)^{-j+k-l+N-q-3} (2S_1(-j+N-1) - S_1(-j+N-2))}{-j+N-1} - \frac{(-1)^{-j+k-l+N-q-3} S_1(k)}{-j+N-1}}{(N-q-r-s-2)(q+s+1)} \right. \\
& \left. - \frac{(-1)^{-j+k-l+N-q-3} (S_1(-l+N-q-2) + S_1(-l+N-q-r-s-3) - 2S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)} \right. \\
& \left. + \frac{2(-1)^{-j+k-l+N-q-3} (S_1(s-1) - S_1(r+s))}{(-j+N-1)(N-q-r-s-2)(q+s+1)} \right)
\end{aligned}$$

= polynomial expression in terms of 49 harmonic sums and S -sums

Example 2: 3-Loop All N-Results for the N_f Contributions

Joint work with J. Ablinger (RISC), J. Blümlein (DESY),
F. Wißbrock (DESY), S. Klein (RWTH)

E.g., for the diagram



768 sums are simplified.

Simple example:

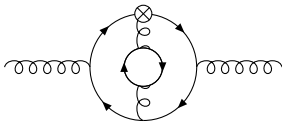
$$\begin{aligned}
& \sum_{j=1}^{N-2} \frac{j(j+1)(j+2)(N-j)(j-1)!^2(-j+N-1)!^2}{-j+N-1} \\
&= \frac{(-N^3 - 5N^2 - 4N + 6)(N!)^2}{(N-1)^2 N^2} \\
&+ \frac{3(N!)^2(N^3 + 6N^2 + 11N + 6)}{2(N-1)N(2N+1)\binom{2N}{N}} \sum_{i=1}^N \frac{\binom{2i}{i}}{i}.
\end{aligned}$$

Not expressible in terms of harmonic sums or S -sums!

The final expression is given in terms of 703 indefinite nested sums and products. Typical examples are:

$$\sum_{i=1}^N \frac{\sum_{j=1}^i \frac{\binom{2j}{j}}{j}}{\binom{2i}{i}}, \quad \sum_{i=1}^N \frac{S_1(i) \sum_{j=1}^i \frac{\binom{2j}{j}}{j}}{\binom{2i}{i}};$$

Sigma finds all algebraic relations among them. We get:



$$\begin{aligned}
& - \frac{20S(1,N)^4}{27(N+1)(N+2)} + \frac{32(6N^3+61N^2-21N+24)S_1(N)^3}{81N^2(N+1)(N+2)} - \frac{16(48N^5+746N^4+2697N^3+2746N^2+1104N+240)S_1(N)^2}{81N^2(N+1)^2(N+2)^2} \\
& + \frac{32(264N^7+4046N^6+21591N^5+52844N^4+74856N^3+66812N^2+30576N+2640)S_1(N)}{243N^2(N+1)^3(N+2)^3} \\
& - \frac{4(48N^2+101N+96)S_2(N)^2}{9N(N+1)(N+2)} - \frac{32(363N^7+6758N^6+41285N^5+121235N^4+190235N^3+150758N^2+46964N+2904)}{243N(N+1)^4(N+2)^3} \\
& + \left(- \frac{40S_1(N)^2}{9(N+1)(N+2)} + \frac{32(6N^3+61N^2-21N+24)S_1(N)}{27N^2(N+1)(N+2)} \right. \\
& \left. - \frac{16(124N^5+198N^4-2387N^3-6162N^2-3632N-480)}{81N^2(N+1)^2(N+2)^2} \right) S_2(N) + \\
& + \left(- \frac{32(9N^3-623N^2+894N+276)}{81N^2(N+1)(N+2)} - \frac{160S_1(N)}{27(N+1)(N+2)} \right) S_3(N) - \frac{8(56N^2+169N+112)S_4(N)}{9N(N+1)(N+2)} \\
& + \left(\frac{64S_1(N)}{3(N+1)(N+2)} - \frac{128(N^3+9N^2-10N-6)}{9N^2(N+1)(N+2)} \right) S_{2,1}(N) + \frac{64S_{3,1}(N)}{3(N+1)(N+2)} + \frac{64(3N^2+7N+6)}{3N(N+1)(N+2)} S_{2,1,1}(N) \\
& + \zeta(2) \left(\frac{8S_1(N)^2}{3(N+1)(N+2)} + \frac{16(3N^3-N^2+30N+12)S_1(N)}{9N^2(N+1)(N+2)} - \frac{16(3N^3+2N^2+17N+6)}{9N(N+1)^2(N+2)} - \frac{8(4N^2+9N+8)S_2(N)}{3N(N+1)(N+2)} \right) + \\
& + \zeta(3) \left(\frac{448}{9(N+1)(N+2)} - \frac{448S_1(N)}{9(N+1)(N+2)} \right)
\end{aligned}$$

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- ▶ J. Ablinger's harmonic sum package facilitates this task (J. Blümlein 2009; J. Ablinger, J. Blümlein, C.S. 2011)
 - ▶ Algebraic relations of harmonic sums (up to weight 8)
(covering also differentiation/half integer relations)

Weight	Number of							
	<i>All</i>	N_A	N_D	N_H	N_{AD}	N_{AH}	N_{DH}	N_{ADH}
1	2	2	2	1	2	1	1	1
2	6	3	4	4	1	2	3	1
3	18	8	12	14	5	6	10	4
4	54	18	36	46	10	15	32	9
5	162	48	108	146	30	42	100	27
6	486	116	324	454	68	107	308	65
7	1458	312	972	1394	196	294	940	187
8	4374	810	2916	4246	498	780	2852	486

Algebraic relations for S -sums: up to weight 6 (so far)

Concluding remarks

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- ▶ J. Ablinger's harmonic sum package facilitates this task (J. Blümlein 2009; J. Ablinger, J. Blümlein, C.S. 2011)
 - ▶ asymptotic expansion of harmonic sums

$$\begin{aligned}
 S(3, 2, 1, n) = & \frac{143}{210} \zeta_2^3 + 3 \zeta_3^2 + \left(\frac{-1}{n^2} + \frac{1}{n^3} - \frac{1}{2n^4} + \frac{1}{6n^6} - \frac{1}{6n^8} + \frac{3}{10n^{10}} - \frac{5}{6n^{12}} + \frac{691}{210n^{14}} - \frac{35}{2n^{16}} + \frac{3617}{30n^{18}} - \frac{43867}{42n^{20}} \right) \zeta_3 \\
 & + \left(\frac{1}{3n^3} - \frac{5}{8n^4} + \frac{37}{60n^5} - \frac{7}{24n^6} - \frac{37}{420n^7} + \frac{13}{80n^8} + \frac{118}{945n^9} - \frac{11}{42n^{10}} - \frac{785}{2772n^{11}} + \frac{169}{240n^{12}} \right. \\
 & + \left. \frac{42376}{45045n^{13}} - \frac{725}{264n^{14}} - \frac{1664693}{386100n^{15}} + \frac{105723}{7280n^{16}} + \frac{19976612}{765765n^{17}} - \frac{399}{4n^{18}} - \frac{58623353743}{290990700n^{19}} + \frac{3519341}{4080n^{20}} \right) (\log(n) + \gamma) \\
 & + \frac{4}{9n^3} - \frac{15}{32n^4} + \frac{107}{1800n^5} + \frac{17}{48n^6} - \frac{3341}{9800n^7} - \frac{583}{9600n^8} + \frac{2464639}{9525600n^9} + \frac{24449}{141120n^{10}} - \frac{20380229}{42688800n^{11}} \\
 & - \frac{10399}{17280n^{12}} + \frac{60101665187}{43286443200n^{13}} + \frac{643811}{232320n^{14}} - \frac{3703037408669}{649296648000n^{15}} - \frac{37263089359}{2248646400n^{16}} \\
 & + \frac{585708761937371}{18764673127200n^{17}} + \frac{77481991}{617760n^{18}} - \frac{14828831197152090581}{67740469989192000n^{19}} - \frac{102648938023}{87393600n^{20}} + O\left(\frac{1}{n^{21}}\right)
 \end{aligned}$$

Needed for limits (see first example) and for analytic continuation of harmonic sums

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- ▶ Alternative summation methods are available: e.g.,
(J. Blümlein, M. Kauers, S. Klein, C.S; 2009)
(J. Blümlein, S. Klein, C.S., F. Stan; 2010)