NNLO Jet Cross Sections via Local Subtraction.

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LHCPhenoNet Kick-off meeting

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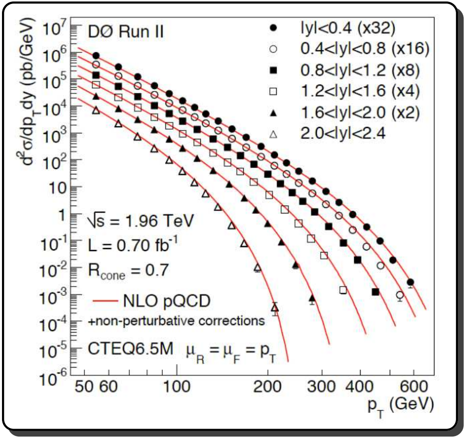
DESY

Motivation

Why jets at NNLO?

Jets are essential analysis tools at LHC: good understanding is needed

- **Status at Tevatron: looks good. . .**



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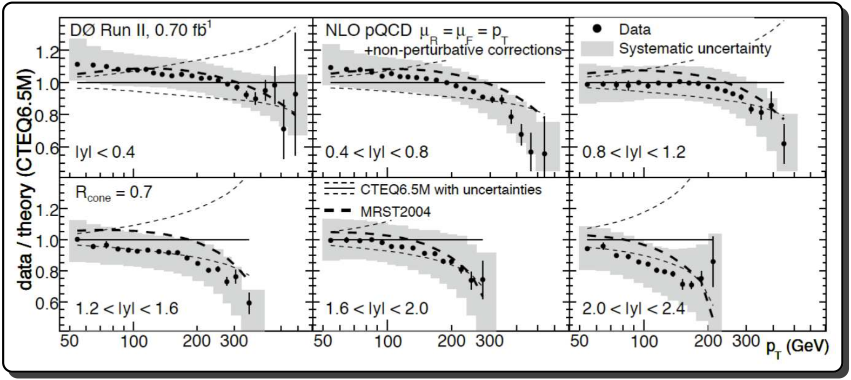

Why jets at NNLO?

Jets are essential analysis tools at LHC: good understanding is needed

- **Status at Tevatron: looks good...but have a closer look!**
- **Energy scale uncertainty $\approx$ 10% warrants precision physics**
- **Precision predictions for 'standard candles':**
  inclusive jet, $V+$ jet,...
- **Missing piece for precise determination of PDFs**
- **NLO is effectively LO:**
  energy distribution inside jets, jet $p_\perp$ asymmetry,...

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Less sophisticated answer: matrix elements are known but not yet used

The problem

Consider the NNLO correction to a generic m-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$
- **Tree MEs with $m+2$-parton kinematics**
- **kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration**
- **no explicit ϵ poles**

Real-virtual

- $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1}$
- **One-loop MEs with $m+1$-parton kinematics**
- **kin. singularities as one parton unresolved: up to $\mathrm{O}(\epsilon^{-2})$ poles from PS integration**
- **explicit ϵ poles up to $\mathrm{O}(\epsilon^{-2})$**

Doubly-virtual

- $\mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$
- **One- and two-loop MEs with m-parton kinematics**
- **kin. singularities screened by jet function: PS integration finite**
- **explicit ϵ poles up to $\mathrm{O}(\epsilon^{-4})$**

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THE KLN THEOREM

Infrared singularities cancel between real and virtual quantum corrections at the same order in perturbation theory, for sufficiently inclusive (IR safe) observables.

HOWEVER

How to make this cancellation explicit, so that the various contributions can be computed numerically?

Approaches

Sector decomposition

Binoth, Heinrich;
Anastasiou, Melnikov, Petriello

- ✔ first method to yield physical cross sections
- ✔ cancellation of divergences fully numerical
- ✘ cancellation of poles also, and depends on jet function
- ✘ can it handle complicated final states? (Czakon: yes)

$q_\perp$ subtraction

Catani, Grazzini;
Cieri, Ferrera, de Florian

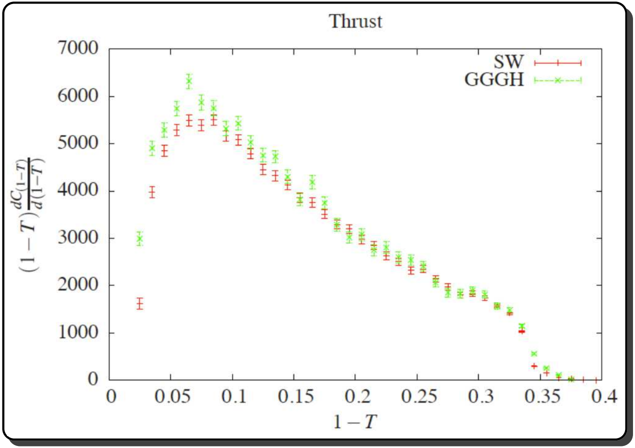
- ✔ exploits universal behavior of $q_\perp$ distribution at small $q_\perp$
- ✔ efficient and fully exclusive calculation
- ✘ limited scope: applicable only to production of colorless final states in hadron collisions

Antenna subtraction

Gehrmann, Gehrmann-De Ridder,
Glover; Weinzierl

- ✔ successfully applied to $e^+e^- \to 2, 3j$
- ✔ analytic integration of antennae over unresolved phase space is understood
- ✘ counterterms are nonlocal
- ✘ treatment of color is implicit
- ✘ cannot cut factorized PS

Is the agreement between antenna implementations satisfactory? (Weinzierl)



Goal

Devise a subtraction scheme with:

- **fully local counterterms, taking account of all color and spin correlations (efficiency, mathematical rigor)**
- **explicit expressions, including color (color space notation is used)**
- **very algorithmic construction (valid at any order in perturbation theory)**
- **option to constrain subtraction near singular regions (efficiency, important check)**

Subtraction at NNLO

G.S., Z. Trócsányi, V. Del Duca hep-ph/0502226, hep-ph/0609042
G.S., Z. Trócsányi, hep-ph/0609041, hep-ph/0609043
Z. Nagy, G.S., Z. Trócsányi, hep-ph/0702273

Structure

Rewrite the NNLO correction as

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

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$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

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In a rigorous mathematical sense, the cancellation of both kinematical singularities and ϵ-poles must be local. I.e. the counterterm must have the following general properties

- **must match the singularity structure of (doubly-) real emissions pointwise, in d dimensions**
- **its integrated form must be combined with the (real- or doubly-) virtual cross section explicitly, before phase space integration; ϵ-poles must cancel point by point**

The construction should be universal (i.e. process and observable independent)

- **to avoid tedious adaptation to every specific problem**
- **the integration of counterterms can be performed once and for all**
- **the IR limits of QCD (squared) matrix elements are universal, so a general construction should be possible**

Different specific choices of counterterms correspond to different IR subtraction schemes.

Defining a subtraction scheme

The following three problems must be addressed

1. **Matching of limits to avoid multiple subtraction in overlapping singular regions of PS. Easy at NLO: collinear limit + soft limit - collinear limit of soft limit.**

$$\mathbf{A}_1|\mathcal{M}_{m+1}^{(0)}|^2 = \sum_i \left[\sum_{i\neq r} \frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r} \mathbf{C}_{ir}\mathbf{S}_r\right] |\mathcal{M}_{m+1}^{(0)}|^2$$

2. **Extension of IR factorization formulae over full PS using momentum mappings that respect factorization and delicate structure of cancellations in all limits.**

$$\{p\}_{m+1} \xrightarrow{r} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+1}(\{p\}_{m+1}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{1,m}]$$
$$\{p\}_{m+2} \xrightarrow{r,s} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+2}(\{p\}_{m+2}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{2,m}]$$

3. **Integration of the counterterms over the phase space of the unresolved parton(s).**

Specific issues at NNLO

1. Matching is cumbersome if done in a brute force way. However, an efficient solution that works at any order in PT is known.
2. Extension is delicate. E.g. counterterms for singly-unresolved real emission (unintegrated and integrated) must have universal IR limits. This is not guaranteed by QCD factorization.
3. Choosing the counterterms such that integration is (relatively) easy generally conflicts with the delicate cancellation of IR singularities.

NNLO subtraction terms - general features

The counterterms are given completely explicitly for any process without colored particles in the initial state. (The extension to hadronic processes is known explicitly to NLO.)

Based on universal IR limit formulae

- **simple and general procedure for matching of limits using physical gauge**
- **extension based on momentum mappings that can be generalized to any number of unresolved partons**

Fully local in color $\otimes$ spin space

- **no need to consider the color decomposition of real emission matrix elements**
- **azimuthal correlations correctly taken into account in gluon splitting**
- **can check explicitly that the ratio of the sum of counterterms to the real emission cross section tends to unity in any IR limit**

Straightforward to constrain subtractions to near singular regions

- **gain in efficiency**
- **independence of physical results on phase space cut is a strong check**

Integrating the counterterms

G.S., Z. Trócsányi arXiv:0807.0509

U. Aglietti, V. Del Duca, C. Duhr, G.S., Z. Trócsányi arXiv:0807.0514

P. Bolzoni, S.-O. Moch, G.S., Z. Trócsányi arXiv:0905.4390

P. Bolzoni, G.S., Z. Trócsányi arXiv:1011.1909

Integrated counterterms

Counterterm	Types of integrals	Done
$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$	**tree level singly-unresolved**	✔
$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$	**one-loop singly-unresolved**	✔
$\int_1 \big(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\big)^{\mathrm{A}_1}$	**tree level iterated singly-unresolved (1)**	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	**tree level iterated singly-unresolved (2)**	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	**tree level doubly-unresolved**	✘

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$\int_1 \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}$	tree level iterated singly-unresolved (1)	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	**tree level iterated singly-unresolved (2)**	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	tree level doubly-unresolved	✘

Phase space integrals - an example

Example (abelian soft-double soft counterterm)

Among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}}$ we find the abelian soft-double soft counterterm

$$\left(\mathcal{S}_t\mathcal{S}_{rt}^{(0)}\right)^{\mathrm{ab}} = (8\pi\alpha_\mathrm{s}\mu^{2\epsilon})^2 \sum_{i,j,k,l} \frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by an iterated mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathrm{S}_t} \{\hat{p}\}_{m+1} \xrightarrow{\mathrm{S}_{\hat{r}}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\};Q)[\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]$$

Then we must compute

$$\int [\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]\mathcal{S}_t\mathcal{S}_{rt}^{(0)} \equiv \left[\frac{\alpha_\mathrm{s}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i,k,j,l}[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$

where $[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl} \equiv [\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}(p_i,p_k,p_j,p_l,\epsilon,y_0,d_0')$ is a kinematics dependent function.

Example (abelian soft-double soft integral)

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}_t \mathrm{S}_{rt}^{(0)}]_{ikik}$. Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$.

Using angles and energies in some specific Lorentz frame to parametrize the factorized phase space measures, $[\mathrm{d}\widehat{p}_{1,m}]$ and $[\mathrm{d}p_{1,m+1}]$, we find that $[\mathrm{S}_t \mathrm{S}_{rt}^{(0)}]_{ikik}$ is proportional to

$$\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q};\epsilon,y_0,d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)}\frac{B_{y_0}(-2\epsilon,d_0'+1)}{\epsilon}Y_{ik,Q}\int_0^{y_0}\mathrm{d}y\,y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\times\int_{-1}^{1}\mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon}\int_{-1}^{1}\mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon}\left[f(\vartheta,\varphi;0)\right]^{-1}\left[f(\vartheta,\varphi;Y_{ik,Q})\right]^{-1}$$
$$\times\left[Y(y,\vartheta,\varphi;Y_{ik,Q})\right]^{-\epsilon}{}_2F_1\big(-\epsilon,-\epsilon,1-\epsilon,1-Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y)+y\,f(\vartheta,\varphi;0)][2(1-y)+y\,f(\vartheta,\varphi;Y_{ik,Q})]}$$

Example (abelian soft-double soft integral)

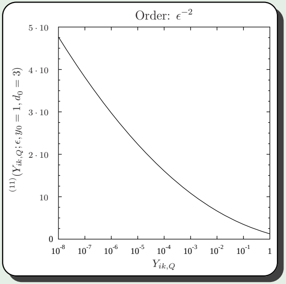
This integral is equal to

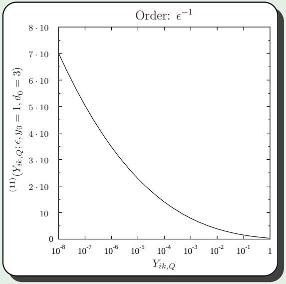
$$\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q};\epsilon,y_0,d_0') = \frac{1}{\epsilon^4} - 2\Big[\ln(Y_{ik,Q}) + \Sigma(y_0,D_0') + \Sigma(y_0,D_0'-1)\Big]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})$$

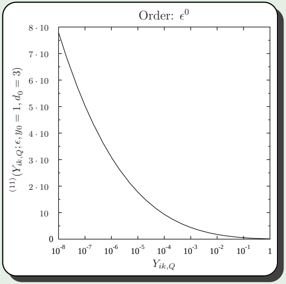
where $D_0' = d_0'|_{\epsilon=0}$ and the dependence on the PS cut parameter, y_0, enters in

$$\Sigma(z,N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

Higher order expansion coefficients can be computed numerically ($y_0 = 1,\ D_0' = 3$)







Spinoff - angular integrals in d dimensions (G.S. arXiv:1101.3557)

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables.

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where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \ldots, j_n, 1 - j - \epsilon), \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2 - j - 2\epsilon)$$

and $\mathbf{L_s} = L_{s_1} \times \ldots L_{s_N}$, where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

Spinoff - angular integrals in d dimensions (G.S. arXiv:1101.3557)

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where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1 \cdots -1 \end{array}\right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. B is zero, while the $n \times N$ dimensional matrix M has the following block form:

$$\mathbf{M}_{n\times N} = \left[\ \mathbf{m}_{n\times n} \;\middle|\; \mathbf{m}_{n\times(n-1)} \;\middle|\; \cdots \;\middle|\; \mathbf{m}_{n\times 1}\ \right] \quad \textbf{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1 \cdots 1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]$$

Spinoff - angular integrals in d dimensions (G.S. arXiv:1101.3557)

Consider the d dimensional angular integral with n denominators

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon} \frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)}$$
$$\times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left[\prod_{k=1}^{n}\prod_{l=k}^{n} \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}} \Gamma(-z_{kl})\, (v_{kl})^{z_{kl}}\right] \left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right] \Gamma(1-j-\epsilon-z)\,.$$

where

$$z = \sum_{k=1}^{n}\sum_{l=k}^{n} z_{kl}\,, \qquad \textbf{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl}\,.$$

Phase space integrals - methods

Several different methods to compute the integrals have been explored

- **use of IBPs to reduce to master integrals + solution of MIs by differential equations**
- **use of MB representations to extract pole structure + summation of nested series**
- **use of sector decomposition**

Phase space integrals - methods

Method	Analytical	Numerical
IBP	✔ Singly-unresolved integrals ✘ Bottleneck is the proliferation of denominators	✔ By evaluating the analytic expressions ✘ No numbers without full analytical results
MB	✔ Iterated singly-unresolved integrals ✘ Bottleneck is the evaluation of sums	✔ Direct numerical evaluation of MB integrals possible ✔ Fast and accurate
SD	✔ Easy to automate ✘ Only in principle, except for lowest order poles	✘ Numerical behavior is generally worse than MB method (speed, accuracy)

Analytical vs. numerical

AS A MATTER OF PRINCIPLE

- **The rigorous proof of cancellation of IR poles requires the poles in integrated counterterms in analytical form.**
- **Analytical forms are fast and accurate compared to numerical ones.**

HOWEVER

- **Analytical results show (in all cases where they are available) that the integrated counterterms are smooth functions of the kinematic variables.**

HENCE

- **Numerical forms of the integrated counterterms are sufficient for practical purposes. Final results can be conveniently given by interpolating tables computed once and for all or approximating functions. Hence, an efficient implementation is possible even in cases where the full analytical calculation is not feasible or practical (e.g. finite parts of integrated counterterms).**

Results

Structure of the integrated counterterm

After summing over unresolved flavors, the integrated iterated singly-unresolved counterterm is a product of an insertion operator times the Born cross section

$$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon)$$

The insertion operator has the following structure in color $\otimes$ flavor space

$$\mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \left[\mathrm{C}_{12,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{C}_{12,f_i f_k}^{(0)} \mathbf{T}_k^2\right] \mathbf{T}_i^2$$
$$+ \sum_{j,l} \left[\mathrm{S}_{12}^{(0),(j,l)} C_{\mathrm{A}} + \sum_i \mathrm{CS}_{12,f_i}^{(0),(j,l)} \mathbf{T}_i^2\right] \mathbf{T}_j \mathbf{T}_l$$
$$+ \sum_{i,k,j,l} \mathrm{S}_{12}^{(0),(i,k)(j,l)} \{\mathbf{T}_i \mathbf{T}_k, \mathbf{T}_j \mathbf{T}_l\} \Bigg\}$$

The kinematical functions $\mathrm{C}_{12,f_i}^{(0)}$, $\mathrm{C}_{12,f_i f_k}^{(0)}$, $\mathrm{S}_{12}^{(0),(j,l)}$, $\mathrm{CS}_{12,f_i}^{(0),(j,l)}$ and $\mathrm{S}_{12}^{(0),(i,k)(j,l)}$, which appear in $\mathbf{I}_{12}^{(0)}(\{p\}_m;\epsilon)$, have poles in ϵ up to $\mathrm{O}(\epsilon^{-4})$, and also depend on PS cut parameters.

The insertion operator – examples

Example ($e^+e^- \to 2j$)

The Born matrix element is $|\mathcal{M}_2^{(0)}(1_q, 2_{\bar{q}})|^2$. Color and kinematics is trivial

$$\mathbf{T}_1^2 = \mathbf{T}_2^2 = -\mathbf{T}_1\mathbf{T}_2 = C_{\rm F}\,, \qquad y_{12} = \frac{2p_1\cdot p_2}{Q^2} = 1$$

We find the insertion operator

$$\mathbf{I}_{12}^{(0)}(p_1, p_2; \epsilon) = \left[\frac{\alpha_{\rm s}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2\left\{\frac{2C_{\rm F}(3C_{\rm F}-C_{\rm A})}{\epsilon^4} + \frac{C_{\rm F}}{6}\Big[20C_{\rm A} + 81C_{\rm F} - 4T_{\rm R}n_{\rm f}\right.$$
$$\left. + 12(3C_{\rm A} - 2C_{\rm F})\Sigma(y_0, D_0') + 12(2C_{\rm A} - C_{\rm F})\Sigma(y_0, D_0' - 1)\Big]\frac{1}{\epsilon^3} + {\rm O}(\epsilon^{-2})\right\}$$

Notice the dependence on the factorized PS cut parameters y_0 and D_0' through

$$\Sigma(z, N) = \ln z - \sum_{k=1}^{N}\frac{1-(1-z)^k}{k}$$

which should cancel between the various integrated counterterms in the full doubly-virtual contribution.

Example ($e^+e^- \to 2j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,2j}^{(\text{Col},i)} + \text{O}(\epsilon^1)$$

Kinematical dependence would enter through $y_{12} = 2p_1 \cdot p_2/Q^2$, but $y_{12} = 1$, hence no PS dependence

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	$\frac{76}{3}$	32.09	−87.90	−554.5
$C_\text{A}\, C_\text{F}$	−2	$-\frac{27}{2}$	−52.40	−150.7	−339.5
$C_\text{F}\, T_\text{R}\, n_\text{f}$	0	−1	−6.332	−17.65	1.013

The PS cut parameters are $\alpha_0 = y_0 = 1$, $d_0 = d_0' = 3$.

Example ($e^+e^- \to 3j$)

The Born matrix element is $|\mathcal{M}_3^{(0)}(1_q, 2_{\bar{q}}, 3_g)|^2$. Color is still trivial

$$\mathbf{T}_1^2 = \mathbf{T}_2^2 = C_\mathrm{F}\,, \quad \mathbf{T}_3^2 = C_\mathrm{A}\,, \quad \mathbf{T}_1\mathbf{T}_2 = \frac{C_\mathrm{A} - 2C_\mathrm{F}}{2}\,, \quad \mathbf{T}_1\mathbf{T}_3 = \mathbf{T}_2\mathbf{T}_3 = -\frac{C_\mathrm{A}}{2}$$

We find the insertion operator

$$\begin{aligned}\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ &\frac{C_\mathrm{A}^2 + 2C_\mathrm{A}C_\mathrm{F} + 6C_\mathrm{F}^2}{\epsilon^4} + \Bigg[\frac{11C_\mathrm{A}^2}{2} + \frac{50C_\mathrm{A}C_\mathrm{F}}{3} \\ &+ 12C_\mathrm{F}^2 - \frac{C_\mathrm{A}T_\mathrm{R}n_\mathrm{f}}{3} - \frac{C_\mathrm{A}^2 T_\mathrm{R} n_\mathrm{f}}{C_\mathrm{F}} - 4C_\mathrm{F}T_\mathrm{R}n_\mathrm{f} + \left(\frac{5C_\mathrm{A}^2}{2} - C_\mathrm{A}C_\mathrm{F} - 8C_\mathrm{F}^2\right)\ln y_{12} \\ &- \frac{C_\mathrm{A}(5C_\mathrm{A} + 8C_\mathrm{F})}{2}(\ln y_{13} + \ln y_{23}) + (C_\mathrm{A}^2 + 6C_\mathrm{A}2C_\mathrm{F} - 4C_\mathrm{F}^2)\Sigma(y_0, D_0') \\ &+ 4C_\mathrm{F}(C_\mathrm{A} - C_\mathrm{F})\Sigma(y_0, D_0' - 1)\Bigg]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})\Bigg\}\end{aligned}$$

Again depends on PS cut parameters through $\Sigma(y_0, D_0' - 1)$ and $\Sigma(y_0, D_0')$.

Example ($e^+e^- \to 3j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_s}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0}\sum_{\text{color}} \frac{\text{Col}}{\epsilon^i}\mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1,p_2,p_3) + \text{O}(\epsilon^1)$$

Kinematical dependence enters through $y_{ij} = 2p_i \cdot p_j/Q^2$, $i,j = 1,2,3$. **E.g. choose**

$$y_{12} = 0.333333, \qquad y_{13} = 0.333333, \qquad y_{23} = 0.333333$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	34.12	82.98	34.59	-543.8
$C_A C_F$	2	9.721	1.209	-142.2	-696.6
C_A^2	1	6.497	12.80	15.87	-47.92
$C_F T_R n_f$	0	$-\frac{13}{3}$	-32.40	-127.9	-355.2
$C_A T_R n_f$	0	$-\frac{3}{2}$	-12.01	-46.90	-104.1

The PS cut parameters are $\alpha_0 = y_0 = 1$, $d_0 = d_0' = 3$.

Example ($e^+e^- \to 3j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1,p_2,p_3) + \text{O}(\epsilon^1)$$

Kinematical dependence enters through $y_{ij} = 2p_i \cdot p_j/Q^2$**,** $i,j = 1,2,3$**. E.g. choose**

$$y_{12} = 0.238667, \qquad y_{13} = 0.758153, \qquad y_{23} = 0.003180$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	36.79	106.0	120.6	−431.0
$C_\text{A}\, C_\text{F}$	2	25.38	143.6	537.3	1505
C_A^2	1	15.24	119.5	660.5	2902
$C_\text{F}\, T_\text{R}\, n_\text{f}$	0	$-\frac{13}{3}$	−31.30	−121.7	−346.0
$C_\text{A}\, T_\text{R}\, n_\text{f}$	0	$-\frac{3}{2}$	−17.72	−109.1	−470.9

The PS cut parameters are $\alpha_0 = y_0 = 1$**,** $d_0 = d_0' = 3$**.**

Example ($e^+e^- \to 3j$)

Higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \mathrm{O}(\epsilon^1)$$

Kinematical dependence enters through $y_{ij} = 2p_i \cdot p_j / Q^2$**,** $i, j = 1, 2, 3$**. E.g. choose**

$$y_{12} = 0.937044, \qquad y_{13} = 0.024207, \qquad y_{23} = 0.038749$$

Col	$\mathrm{O}(\epsilon^{-4})$	$\mathrm{O}(\epsilon^{-3})$	$\mathrm{O}(\epsilon^{-2})$	$\mathrm{O}(\epsilon^{-1})$	$\mathrm{O}(\epsilon^{0})$
C_F^2	6	25.85	34.59	−84.25	−566.8
$C_\mathrm{A} C_\mathrm{F}$	2	27.79	136.8	330.6	46.20
C_A^2	1	21.02	195.4	1174	5355
$C_\mathrm{F} T_\mathrm{R} n_\mathrm{f}$	0	$-\frac{13}{3}$	−57.59	−405.2	−2120
$C_\mathrm{A} T_\mathrm{R} n_\mathrm{f}$	0	$-\frac{3}{2}$	−24.07	−194.7	−1083

The PS cut parameters are $\alpha_0 = y_0 = 1$**,** $d_0 = d_0' = 3$**.**

Conclusions

Conclusions

- ✔ We have set up a general subtraction scheme for computing NNLO jet cross sections, for processes with no colored particles in the initial state.
- ✔ For hadron initiated processes, our scheme is fully worked out at NLO.
- ✔ We have investigated various methods to compute the integrated counterterms.
- ✔ We used the method of MB representations to perform the integration of the iterated singly-unresolved counterterm, discussed in this talk. Sector decomposition was used to provide independent checks.
- ✔ The integration of all but the doubly-unresolved counterterm is finished.
- ✘ The integration of the doubly-unresolved counterterm is feasible with our methods, and is work in progress.