



FEYNMAN'S TREE THEOREM & LOOP-TREE DUALITIES

in collaboration with

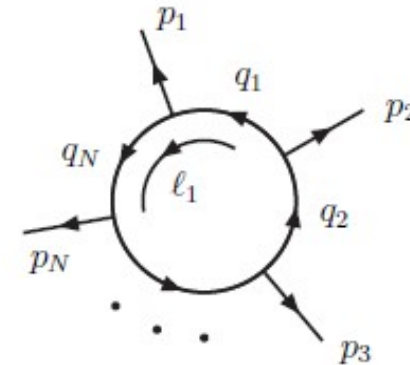
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Feynman's Tree Theorem

A general one loop N-leg diagram



$$L^{(1)}(p_1, p_2, \dots, p_N) = \int_{l_1} \prod_{i=1}^N G_F(q_i) \quad q_i = l_1 + p_{1,i}, \quad i \in \alpha_1 = \{1, 2, \dots, N\}$$

with the Feynman propagator

$$G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} \quad q_{i,0} = \pm \sqrt{\mathbf{q}_i^2 - m_i^2 - i0}$$

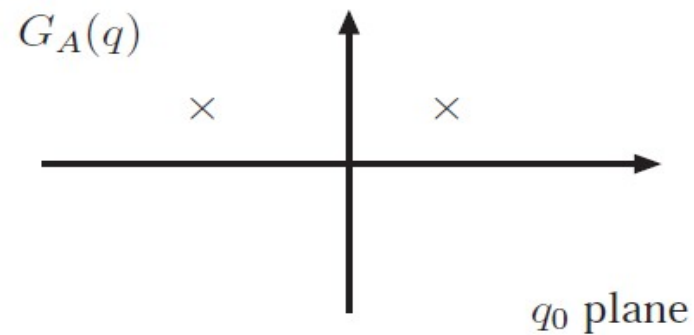
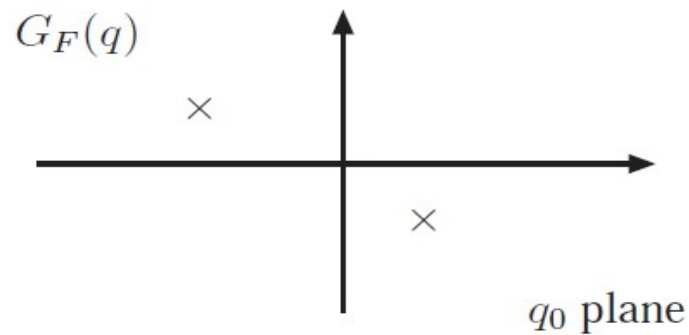
Notation

$$\int_{l_i} \dots = -i \int \frac{d^d l_i}{(2\pi)^d} \dots$$

$$\tilde{\delta}(q_i) \equiv 2\pi i \delta_+(q_i^2 - m_i^2)$$

$$\int \frac{d^d q_i}{(2\pi)^{d-1}} \delta_+(q_i^2 - m_i^2) \dots = \int_{q_i} \tilde{\delta}(q_i) \dots$$

Feynman's Tree Theorem



We also introduce the advanced propagator

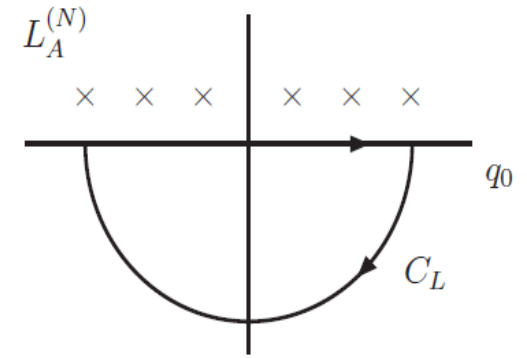
$$G_A(q_i) = \frac{1}{q_i^2 - m_i^2 - i q_{i,0}} \quad q_{i,0} \simeq \pm \sqrt{\mathbf{q}_i^2 - m_i^2} + i0$$

Relation between the two: $G_A(q_i) = G_F(q_i) + \tilde{\delta}(q_i)$

Feynman's Tree Theorem

We start with a modified scalar integral

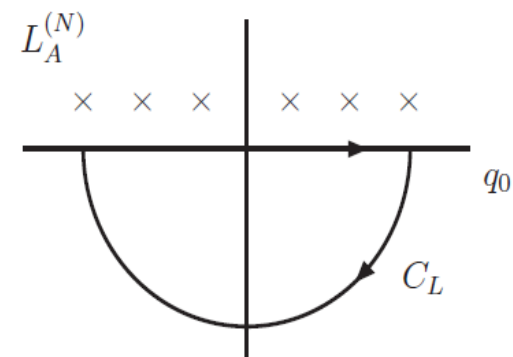
$$L_A^{(1)}(p_1, p_2, \dots, p_N) = \int_{l_1} \prod_{i=1}^N G_A(q_i) = 0$$



Feynman's Tree Theorem

We start with a modified scalar integral

$$L_A^{(1)}(p_1, p_2, \dots, p_N) = \int_{l_1} \prod_{i=1}^N G_A(q_i) = 0$$

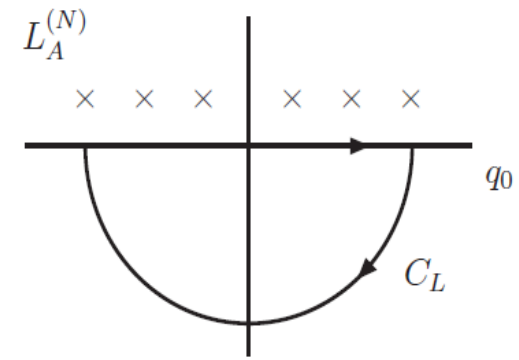


$$G_A(q_i) = G_F(q_i) + \tilde{\delta}(q_i)$$

Feynman's Tree Theorem

We start with a modified scalar integral

$$L_A^{(1)}(p_1, p_2, \dots, p_N) = \int_{l_1} \prod_{i=1}^N G_A(q_i) = 0$$



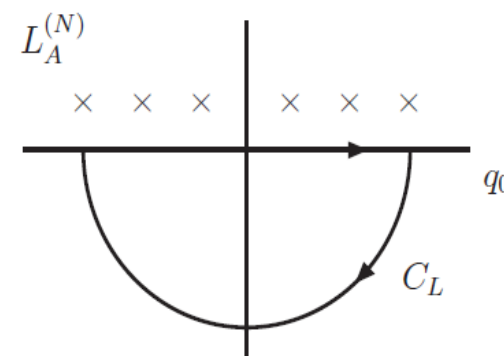
$$L^{(1)}(p_1, p_2, \dots, p_N) + L_{1-cut}^{(1)}(p_1, p_2, \dots, p_N) + \dots + L_{N-cut}^{(1)}(p_1, p_2, \dots, p_N) = 0$$

$$L_{m-cut}^{(N)}(p_1, p_2, \dots, p_N) = \int_{l_i} \{ \tilde{\delta}(q_1) \cdots \tilde{\delta}(q_m) G_F(q_{m+1}) \cdots G_F(q_N) + perms. \}$$

Feynman's Tree Theorem

We start with a modified scalar integral

$$L_A^{(1)}(p_1, p_2, \dots, p_N) = \int_{l_1} \prod_{i=1}^N G_A(q_i) = 0$$



$$L^{(1)}(p_1, p_2, \dots, p_N) + L_{1-cut}^{(1)}(p_1, p_2, \dots, p_N) + \dots + L_{N-cut}^{(1)}(p_1, p_2, \dots, p_N) = 0$$

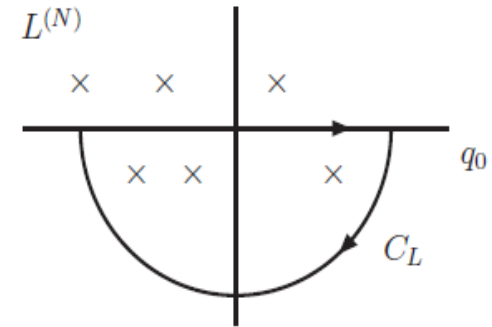
$$L^{(1)}(p_1, p_2, \dots, p_N) = -L_{1-cut}^{(1)}(p_1, p_2, \dots, p_N) - \dots - L_{N-cut}^{(1)}(p_1, p_2, \dots, p_N)$$

FEYNMAN'S TREE THEOREM

Duality relation at one loop

We apply directly the residue theorem

$$L^{(1)}(\{p_m\}) = -2\pi i \int_{l_1} \sum \text{Res}_{\Im q_{i,0} < 0} \left[\prod_{j=1}^N G_F(q_j) \right]$$



$$\text{Res}_{\Im q_{i,0} < 0} \frac{1}{q_i^2 - m_i^2 + i0} = \int dl_{1,0} \delta_+(q_i^2 - m_i^2) \left[\prod_{j \neq i} G_F(q_j) \right]_{(i\text{-th pole})} = \prod_{j \neq i} \frac{1}{q_j^2 - m_j^2 - i0 \eta(q_j - q_i)}$$

DUAL
PROPAGATOR

$$G_D(q_i; q_j) = \frac{1}{q_j^2 - m_j^2 - i0 \eta(q_j - q_i)}$$

η is a future-like vector

$$\eta_0 \geq 0, \quad \eta^2 \geq 0$$

Duality relation at one loop

$$\begin{aligned}
 & \text{Diagram with loop } q \text{ and external momenta } p_1, p_2, p_3, \dots, p_N \\
 & = - \sum_{i=1}^N \text{Diagram with loop } q \text{ and external momenta } p_{i-1}, p_i, p_{i+1}, \dots, p_N \\
 & \quad \frac{1}{(q + p_i)^2 - i0 \eta p_i}
 \end{aligned}$$

$$L^{(1)}(p_1, p_2, \dots, p_N) = - \sum \int_{l_1} \tilde{\delta}(q_i) \prod_{\substack{j=1 \\ j \neq i}}^N G_D(q_i; q_j)$$

TREE-LOOP DUALITY THEOREM

Duality relation at one loop

Recasts virtual corrections to a form similar to real radiative corrections

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}, q) M^{(m+1)}(\{p_i\}) F^{(m+1)}(\{p_i\})$$



$$\int_{m+q} d\sigma^{DUAL} = \int d\Phi^{(m+1)}(\{p_i\}, q) \tilde{M}^{(m+1)}(\{p_i\}, q) F^{(m+1)}(\{p_i\})$$

Single cut terms



Duality relation at one loop

- Unlike FTT it contains only single cuts at the price of introducing a modified i0 prescription, the dual prescription
- The singularities of the loop diagram appear as singularities of the Dual Integrals
- Feynman graphs are treated in the same way since the duality operation works only on propagators
- The form of the duality theorem closely resembles the form of real radiative corrections

Towards the extension to two loops

Extension to sets of momenta

$$G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i)$$

$$G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{\substack{j \in \alpha_k \\ j \neq i}} G_D(q_i; q_j)$$

$$G_D(\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_N) = \sum_{\beta_N^{(1)} \cup \beta_N^{(2)} = \beta_N} \prod_{i_1 \in \beta_N^{(1)}} G_D(\alpha_{i_1}) \prod_{i_2 \in \beta_N^{(2)}} G_F(\alpha_{i_2})$$

For example: $G_D(\alpha_1 \cup \alpha_2) = G_D(\alpha_1)G_D(\alpha_2) + G_D(\alpha_1)G_F(\alpha_2) + G_F(\alpha_1)G_D(\alpha_2)$

Relation between props: $G_A(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k)$

Duality in compact form: $L^{(1)}(p_1, p_2, \dots, p_N) = -\int_{l_1} G_D(\alpha_1)$

Reverse flow of momenta

$$G_D(-\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(-q_i) \prod_{\substack{j \in \alpha_k \\ j \neq i}} G_D(-q_i; -q_j)$$

$$G_F(-\alpha_k) = G_F(\alpha_k)$$

$$G_A(-\alpha_k) = G_R(\alpha_k)$$

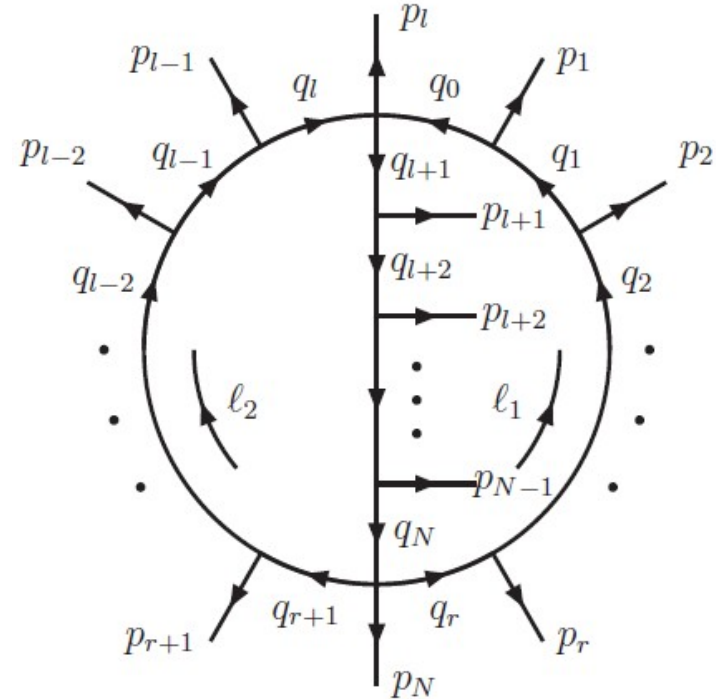
Duality relation at two loops

A general two-loop master diagram

$$q_i = \begin{cases} l_1 + p_{1,i} & , i \in \alpha_1 \\ l_2 + p_{i,l-1} & , i \in \alpha_2 \\ l_1 + l_2 + p_{i,l-1} & , i \in \alpha_3 \end{cases}$$

$$\alpha_1 = \{0, 1, \dots, r\} \quad \alpha_2 = \{r+1, r+2, \dots, l\}$$

$$\alpha_3 = \{l+1, l+2, \dots, N\}$$



Application of duality to one of the integration momenta

$$\int_{l_i} G_F(\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_N) = - \int_{l_i} G_D(\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_N)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$


$$G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) = G_F(\alpha_1) G_F(\alpha_2) G_F(\alpha_3)$$

Duality relation at two loops

$$\int_{l_1} G_F(\alpha_1 \cup \alpha_3) = - \int_{l_1} G_D(\alpha_1 \cup \alpha_3)$$

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets which depend on loop momentum l_1

Assignment to sets

$$q_i = \begin{cases} l_1 + p_{1,i} & , i \in \alpha_1 \\ l_2 + p_{i,l-1} & , i \in \alpha_2 \\ l_1 + l_2 + p_{i,l-1} & , i \in \alpha_3 \end{cases}$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Expand the dual propagator of the union

$$G_D(\alpha_1 \cup \alpha_3) = G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3)$$

Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$

Expand the dual propagator of the union

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} \{ G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3) \} G_F(\alpha_2)$$

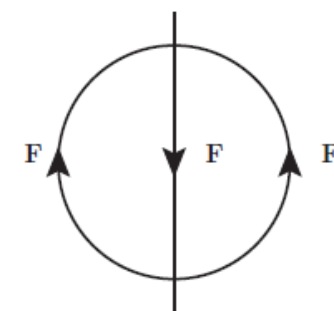
Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$



Expand the dual propagator of the union

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} \{ G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3) \} G_F(\alpha_2)$$

$$\int G_F(\alpha_2) G_F(\alpha_3) = - \int G_D(\alpha_2 \cup \alpha_3)$$

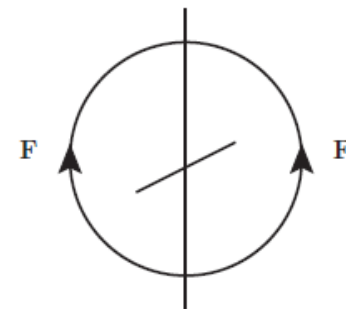
Duality relation at two loops

Starting point

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3)$$

Apply to all sets with depend on loop momentum l_1

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} G_D(\alpha_1 \cup \alpha_3) G_F(\alpha_2)$$



Expand the dual propagator of the union

$$L^{(2)}(p_1, p_2, \dots, p_N) = - \int_{l_1} \int_{l_2} \{ G_D(\alpha_1) G_D(\alpha_3) + G_D(\alpha_1) G_F(\alpha_3) + G_F(\alpha_1) G_D(\alpha_3) \} G_F(\alpha_2)$$

Combine to a dual propagator

$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} \{ -G_D(\alpha_1) G_F(\alpha_2) G_D(\alpha_3) + G_D(\alpha_1) G_D(\alpha_1 \cup \alpha_3) + G_D(\alpha_3) G_D(-\alpha_1 \cup \alpha_2) \}$$

Duality relation at two loops

Expand the remaining dual propagators of the union of sets

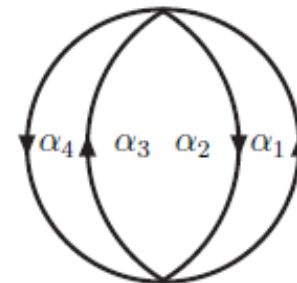
$$L^{(2)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} \left\{ G_D(\alpha_1) G_D(\alpha_2) G_F(\alpha_3) + G_D(-\alpha_1) G_F(\alpha_2) G_D(\alpha_3) \right. \\ \left. + G^*(\alpha_1) G_D(\alpha_2) G_D(\alpha_3) \right\}$$

$$G^*(\alpha_k) = G_F(\alpha_k) + G_D(\alpha_k) + G_D(-\alpha_k)$$

- The $i0$ prescription depends on external momenta only
- The duality relation contains triple cuts as well

Three Loops and beyond

Duality can be readily applied to three loops



$$L_{basket}^{(3)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} \int_{l_3} G_D(\alpha_1 \cup \alpha_2) G_D(\alpha_3 \cup \alpha_4)$$

$$L_{basket}^{(3)}(p_1, p_2, \dots, p_N) = \int_{l_1} \int_{l_2} \int_{l_3} \left\{ G_D(\alpha_2, \alpha_3, -\alpha_4) G_F(\alpha_1) + G_D(\alpha_1, \alpha_3, -\alpha_4) G_F(\alpha_2) \right. \\ \left. + G_D(-\alpha_1, \alpha_2, \alpha_4) G_F(\alpha_3) + G_D(-\alpha_1, \alpha_2, \alpha_3) G_F(\alpha_2) \right. \\ \left. + G_D(-\alpha_1, \alpha_2, \alpha_3, \alpha_4) + G_D(\alpha_1, \alpha_2, \alpha_3, -\alpha_4) + G_D(-\alpha_1, \alpha_2, \alpha_3, -\alpha_4) \right\}$$

Notation

$$G_D(\alpha_1, \dots, \alpha_N) = \prod_i G_D(\alpha_i)$$

Summary and conclusions

- One Loop Integrals are written as linear combinations of N single cut phase space integrals
- We rederived and extended the one loop Duality Theorem to two- and three-loops and beyond
- Number of cuts equals the number of loops, so that a loop diagram is opened up to a tree diagram
- This is a systematic procedure with the potential to treat numerically higher loop corrections
- Treatment of singularities of the Dual Integrals in one and two loops is in progress
- Treatment of higher order poles, relevant for higher order corrections is under investigation

Extra Slides

Example: Massless sunrise 2-Loop 2-point function

The scalar sunrise integral

$$L^{(2)} = \int_{l_1} \int_{l_2} \{ \tilde{\delta}(q_1) \tilde{\delta}(q_2) G_F(q_3) + \tilde{\delta}(-q_1) G_F(q_2) \tilde{\delta}(q_3) + G^*(q_1) \tilde{\delta}(q_2) \tilde{\delta}(q_3) \}$$

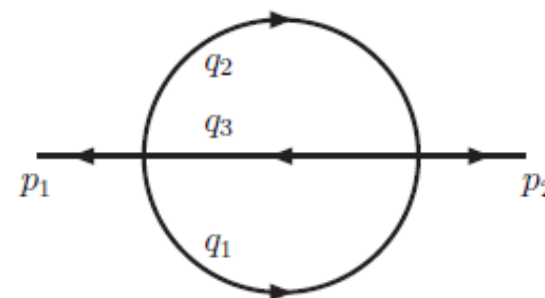
We integrate first over l_1

$$\int_{l_1} \tilde{\delta}(l_1) G_F(l_1 + p) = d_\Gamma [p^2 + i0]^{-\epsilon} [1 + \theta(p^2) \theta(-p_0) (e^{2i\pi\epsilon} - 1)]$$

$$\int_{l_1} \tilde{\delta}(l_1) \tilde{\delta}(l_1 + p) = d_\Gamma [p^2 + i0]^{-\epsilon} \theta(-p^2) (e^{2i\pi\epsilon} - 1)$$

$$d_\Gamma = -\frac{c_\Gamma}{2} \frac{1}{\epsilon(1-2\epsilon)} \frac{1}{\cos(\pi\epsilon)}$$

$$c_\Gamma = \frac{\Gamma(1+\epsilon) \Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon} \Gamma(1-2\epsilon)}$$



Example: Massless sunrise 2-Loop 2-point function

One final integration over l_2

$$d_\Gamma \int_{l_2} \tilde{\delta}(l_2) [(l_2 + p)^{-\epsilon}] = -G_2 \frac{\sin(\pi \epsilon)}{\sin(3\pi \epsilon)} e^{-2i\pi \epsilon} [-p^2 - i0]^{1-2\epsilon} [1 + \theta(p^2)\theta(-p_0)(e^{2i\pi \epsilon} - 1)]$$

$$d_\Gamma \int_{l_2} \tilde{\delta}(l_2) [(l_2 + p)^{-\epsilon}] \theta((l_2 + p)^2) \theta((l_2 + p)_0) = G_2 \frac{\sin(\pi \epsilon)}{\sin(3\pi \epsilon)} [-p^2 - i0]^{1-2\epsilon} [\theta(-p^2) - \theta(p^2)\theta(p_0) e^{-2i\pi \epsilon}]$$

$$G_2 = \frac{\Gamma(-1+2\epsilon)\Gamma^3(1-\epsilon)}{(4\pi)^{4-2\epsilon}\Gamma(3-3\epsilon)}$$

$$L^{(2)}(p) = -G_2 (-p^2 - i0)^{1-2\epsilon}$$