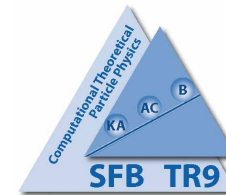


Towards the $O(\alpha_s^3)$ Contributions to the Heavy Flavor Wilson Coefficients of $F_2(x, Q^2)$ for $Q^2 \gg m^2$ at general N

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- Introduction
- The Method
- Results in $O(N_F T_F^2 C_{A,F})$
- First Contributions $\propto T_F^2$
- Conclusions

[Based on: “The $O(\alpha_s^3)$ Massive Operator Matrix Elements of $O(n_f)$ for the Structure Function $F_2(x, Q^2)$ and Transversity”, Nucl.Phys. B844 (2011) 26-54, arXiv:1008.3347]

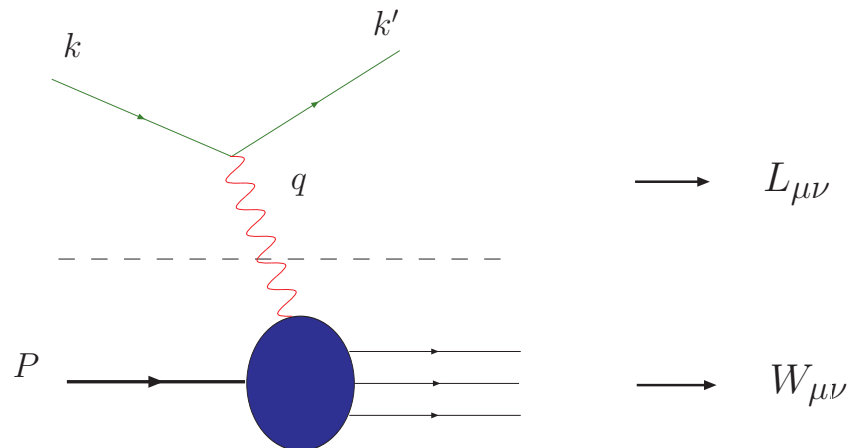
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February 2011

1. Introduction

Deep-Inelastic Scattering (DIS):



$$Q^2 := -q^2, \quad x := \frac{Q^2}{2pq} \quad \text{Bjorken-}x$$

$$\nu := \frac{Pq}{M},$$

$$\frac{d\sigma}{dQ^2 dx} \sim W_{\mu\nu} L^{\mu\nu}$$

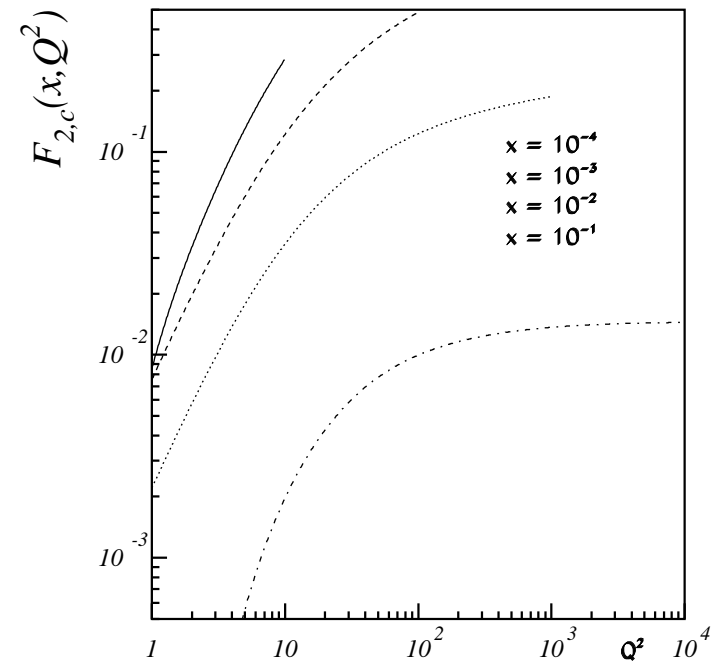
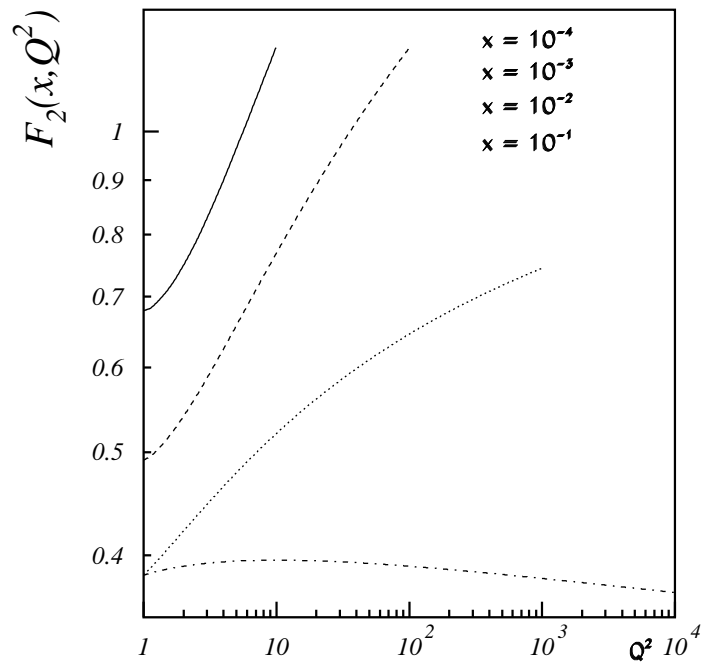
$$W_{\mu\nu}(q, P, s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle$$

$$\text{unpol.} \left\{ \begin{aligned} &= \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) \end{aligned} \right.$$

$$\text{pol.} \left\{ \begin{aligned} &-\frac{M}{2Pq} \varepsilon_{\mu\nu\alpha\beta} q^\alpha \left[s^\beta g_1(x, Q^2) + \left(s^\beta - \frac{sq}{Pq} p^\beta \right) g_2(x, Q^2) \right]. \end{aligned} \right.$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions



LO charm contributions: PDFs from [Alekhin, Melnikov, Petriello, 2006.]

→ different scaling violations,

→ massive contributions at lower values of x are of order 20%-35%.

Hence for the prediction of cross sections at the LHC the precise knowledge of all PDFs is needed.

- in the asymptotic region F_L is known for general values of N to NNLO [Blümlein, De Freitas, van Neerven, Klein, 2006.]
- F_2 for n_f massless and one heavy quark flavor:

$$\begin{aligned}
F_{(2,L)}^{Q\bar{Q}}(x, n_f + 1, Q^2, m^2) &= \sum_{k=1}^{n_f} e_k^2 \left\{ L_{q,(2,L)}^{\text{NS}} \left(x, n_f + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \left[f_k(x, \mu^2, n_f) + f_{\bar{k}}(x, \mu^2, n_f) \right] \right. \\
&+ \frac{1}{n_f} \left[L_{q,(2,L)}^{\text{PS}} \left(x, n_f + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, n_f) + L_{g,(2,L)}^{\text{S}} \left(x, n_f + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, n_f) \right] \left. \right\} \\
&+ e_Q^2 \left[H_{q,(2,L)}^{\text{PS}} \left(x, n_f + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, n_f) + H_{g,(2,L)}^{\text{S}} \left(x, n_f + 1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, n_f) \right]
\end{aligned}$$

- \otimes denotes the Mellin convolution $[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2)$,
- The asymptotic representation for $F_2(x, Q^2)$ becomes effective at $Q^2 \geq 10 \cdot m^2$

- In this limit the massive Wilson coefficients up to $O(a_s^3)$ read

$$\begin{aligned}
L_{q,(2,L)}^{\text{NS}}(n_f + 1) &= a_s^2 \left[A_{qq,Q}^{(2),\text{NS}}(n_f + 1) \delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(n_f) \right] \\
&+ a_s^3 \left[A_{qq,Q}^{(3),\text{NS}}(n_f + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(n_f + 1) C_{q,(2,L)}^{(1),\text{NS}}(n_f + 1) + \hat{C}_{q,(2,L)}^{(3),\text{NS}}(n_f) \right] \\
L_{q,(2,L)}^{\text{PS}}(n_f + 1) &= a_s^3 \left[A_{qq,Q}^{(3),\text{PS}}(n_f + 1) \delta_2 + A_{gg,Q}^{(2)}(n_f) n_f \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) + n_f \hat{C}_{q,(2,L)}^{(3),\text{PS}}(n_f) \right] \\
L_{g,(2,L)}^{\text{S}}(n_f + 1) &= a_s^2 A_{gg,Q}^{(1)}(n_f + 1) n_f \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) + a_s^3 \left[A_{gg,Q}^{(3)}(n_f + 1) \delta_2 \right. \\
&+ A_{gg,Q}^{(1)}(n_f + 1) n_f \tilde{C}_{g,(2,L)}^{(2)}(n_f + 1) + A_{gg,Q}^{(2)}(n_f + 1) n_f \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) \\
&+ A_{Qg}^{(1)}(n_f + 1) n_f \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(n_f + 1) + n_f \hat{C}_{g,(2,L)}^{(3)}(n_f) \left. \right], \\
H_{q,(2,L)}^{\text{PS}}(n_f + 1) &= a_s^2 \left[A_{Qq}^{(2),\text{PS}}(n_f + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(n_f + 1) \right] + a_s^3 \left[A_{Qq}^{(3),\text{PS}}(n_f + 1) \delta_2 \right. \\
&+ \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(n_f + 1) + A_{gg,Q}^{(2)}(n_f + 1) \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) \\
&+ A_{Qq}^{(2),\text{PS}}(n_f + 1) C_{q,(2,L)}^{(1),\text{NS}}(n_f + 1) \left. \right], \\
H_{g,(2,L)}^{\text{S}}(n_f + 1) &= a_s \left[A_{Qg}^{(1)}(n_f + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) \right] + a_s^2 \left[A_{Qg}^{(2)}(n_f + 1) \delta_2 \right. \\
&+ A_{Qg}^{(1)}(n_f + 1) C_{q,(2,L)}^{(1),\text{NS}}(n_f + 1) + A_{gg,Q}^{(1)}(n_f + 1) \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) \\
&+ \tilde{C}_{g,(2,L)}^{(2)}(n_f + 1) \left. \right] + a_s^3 \left[A_{Qg}^{(3)}(n_f + 1) \delta_2 + A_{Qg}^{(2)}(n_f + 1) C_{q,(2,L)}^{(1),\text{NS}}(n_f + 1) \right. \\
&+ A_{gg,Q}^{(2)}(n_f + 1) \tilde{C}_{g,(2,L)}^{(1)}(n_f + 1) + A_{Qg}^{(1)}(n_f + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(n_f + 1) \right. \\
&+ \left. \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(n_f + 1) \right\} + A_{gg,Q}^{(1)}(n_f + 1) \tilde{C}_{g,(2,L)}^{(2)}(n_f + 1) + \tilde{C}_{g,(2,L)}^{(3)}(n_f + 1) \left. \right]
\end{aligned}$$

2. Status of Heavy Flavor Contributions to DIS Structure Functions

Leading Order: [Witten, 1976; Babcock, Sivers, 1978; Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Next-to-Leading Order : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996] via IBP

$(Q^2 \gg m^2)$ [Bierenbaum, Blümlein, Klein, 2007] via $_pF_q$'s, more compact results

NLO fast Mellin space implementation: [Alekhin, Blümlein 2003]

NNLO, $Q^2 \gg m^2$: contribs. to F_L for all N: [Blümlein, De Freitas, van Neerven, Klein 2006]

contributions to F_2 ($N = 2 \dots 10(14)$): [Bierenbaum, Blümlein, Klein 2009]

contributions to transversity ($N = 1 \dots 13$): [Blümlein, Klein, Tödli 2009]

\implies Moment-reference

all $O(\alpha_s^3) \times \ln^k \left(\frac{Q^2}{m^2} \right)$ terms for massive OMEs for general N:

[Bierenbaum, Blümlein, Klein 2010]

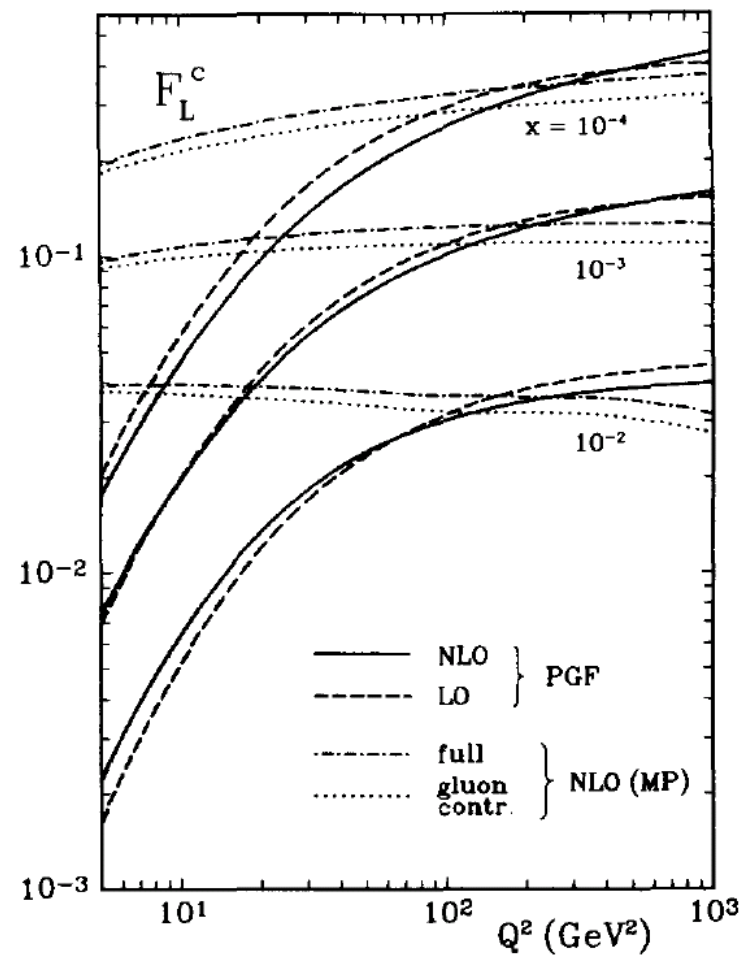
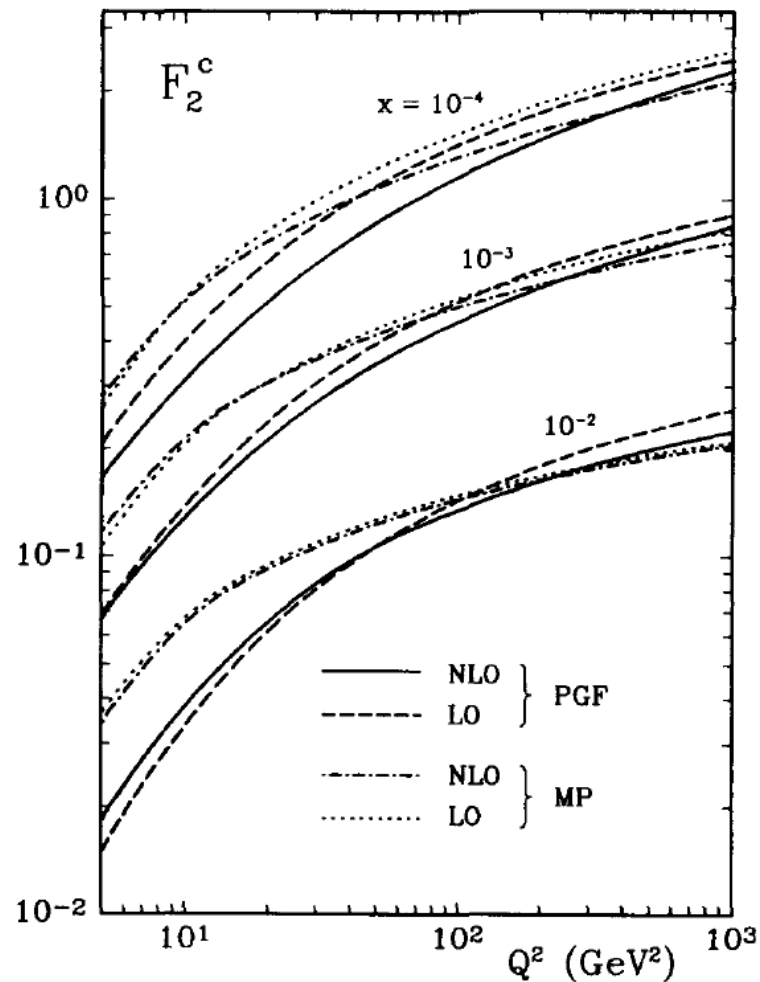
Computation of all 3-loop ladder graphs: [Blümlein, Hasselhuhn, Klein, Schneider]

Goal: Calculate the 3-loop massive Wilson coefficients for $F_2(x, Q^2)$ in the region $Q^2 \gtrsim 10m^2$ for general values of N.

Trading Final- for Initial States

- There are no **genuine** heavy flavor parton densities.
 - ⇒ Power corrections in Wilson coefficients.
 - ⇒ Heavy quarks are not produced **collinearly**
 - ⇒ Their finite mass implies a **finite & short lifetime**.
- Do **heavy** quarks become **light** ? - will be discussed later.
- Simplifying matters with a method à la **Fermi-Williams-Weizsäcker**.
- Produce a **Heavy Quark** of large enough lifetime.
 - ⇒ Its further fate decouples (factorizes) from its past in time.
- **Heavy Quark Initial State**.
 - ⇒ Inapplicable to power corrections !
 - ⇒ Choose a physical quantity and kinematic region for which power corrections are fairly unimportant.
- Opportunities:
 - ⇒ Study of decoupling regimes in QCD.
 - ⇒ Safe one loop for certain Standard Model and “BSM” processes.

Wilson Coefficient vs Massless Parton Approach



[Glück, Reya, Stratmann, 1994]

- Even at high values of Q^2 and W^2 the massless charm approach becomes not effective; it can even lead to misleading results.

FFNS:

- Fixed order perturbation theory and Fixed number of light partons in the proton.
- The heavy quarks are produced extrinsically only.
- The large logarithmic terms in the heavy quark coefficient functions entirely determine the charm component of the structure function for large values of Q^2 .

VFNS:

- Define a threshold above which the heavy quark is treated as light, thereby obtaining a parton density.
- Phenomenological interesting to remove the mass singular terms from the asymptotic heavy quark coefficient functions and absorb them into parton densities.
- Heavy Flavor initial state parton densities for the LHC. E.g. for $c \bar{s} \rightarrow W^+$.

The VFNS is derived from the FFNS directly. New parton density appears corresponding to the heavy quark, which is now treated as light (massless). \implies Relations between parton densities for n_f and $n_f + 1$ flavors.

The RGE does not leave room for other scenarios.

$$\begin{aligned}
f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) &= A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \left[f_k(n_f, \mu^2) + f_{\bar{k}}(n_f, \mu^2) \right] \\
&+ \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) \\
&+ \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) \\
f_{Q+\bar{Q}}(n_f + 1, \mu^2) &= \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) . \\
G(n_f + 1, \mu^2) &= A_{gq,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes \Sigma(n_f, \mu^2) + A_{gg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \otimes G(n_f, \mu^2) . \\
\Sigma(n_f + 1, \mu^2) &= \sum_{k=1}^{n_f+1} \left[f_k(n_f + 1, \mu^2) + f_{\bar{k}}(n_f + 1, \mu^2) \right] \\
&= \left[A_{qq,Q}^{\text{NS}}\left(n_f, \frac{\mu^2}{m^2}\right) + n_f \tilde{A}_{qq,Q}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qq}^{\text{PS}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \\
&\otimes \Sigma(n_f, \mu^2) \\
&+ \left[n_f \tilde{A}_{qg,Q}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) + \tilde{A}_{Qg}^{\text{S}}\left(n_f, \frac{\mu^2}{m^2}\right) \right] \otimes G(n_f, \mu^2)
\end{aligned}$$

Polarized Case :

$$f_i \rightarrow \Delta f_i, \quad A_{ij} \rightarrow \Delta A_{ij}$$

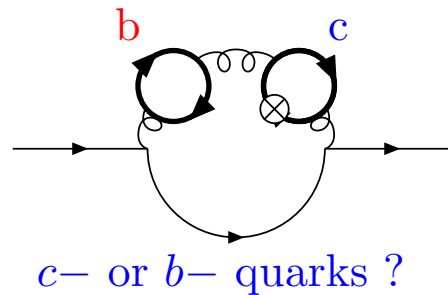
Some Phenomenological Remarks

- Bierenbaum, Blümlein, Klein January 2009 (PLB): correct NLO representation; shortcomings in Laenen et al. 1993, Buza et al. 1996 corrected.
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- Question to the Experts :

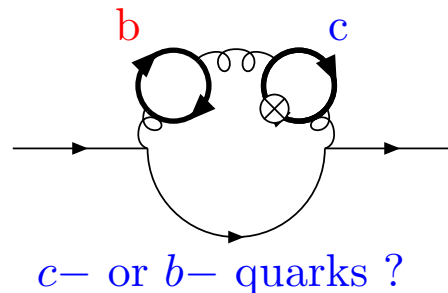
To which flavor distribution do the 3-loop diagrams of the following type belong in the GMVFNS?



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- Answer from the Expert :

To **none** of both. End of the GMVNS, despite of universal terms!

\implies GMVNS requires: $\ln(m_c^2/m_b^2) = 0.40 \approx 0$.

Renormalization

$$\hat{A}_{ij} = \delta_{ij} + \sum_{k=0}^{\infty} \hat{a}_s^k \hat{A}_{ij}^{(k)}$$

- Mass renormalization (on-mass shell scheme)
- Charge renormalization

MOM scheme \rightarrow $\overline{\text{MS}}$ scheme ($D = 4 + \varepsilon$): **NEW!** Terms were forgotten even at NLO.

- Renormalization of ultraviolet singularities
 \implies are absorbed into Z -factors given in terms of anomalous dimensions γ_{ij} .
- Factorization of collinear singularities
 \implies are factored into Γ -factors Γ_{NS} , $\Gamma_{ij,S}$ and $\Gamma_{qq,PS}$.
 For massless quarks it would hold: $\Gamma = Z^{-1}$.
 Here: Γ -matrices apply to parts of the diagrams with massless lines only .

Generic formula for operator renormalization and mass factorization:

$$A_{ij} = Z_{il}^{-1} \hat{A}_{lm} \Gamma_{mj}^{-1}$$

\implies $O(\varepsilon)$ -terms of the 2-loop OMEs are needed for renormalization at 3-loops.

• e.g. A_{Qg}^3 :

$$\begin{aligned}
\hat{A}_{Qg}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2} \left[\frac{\hat{\gamma}_{qg}^{(0)}}{6\varepsilon^3} \left((n_f + 1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)} \left[\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} - 6\beta_0 - 8\beta_{0,Q} \right] + 8\beta_0^2 \right. \right. \\
& + 28\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2 + \gamma_{gg}^{(0)} \left[\gamma_{gg}^{(0)} + 6\beta_0 + 14\beta_{0,Q} \right] \left. \right) + \frac{1}{6\varepsilon^2} \left(\hat{\gamma}_{qg}^{(1)} \left[2\gamma_{qq}^{(0)} - 2\gamma_{gg}^{(0)} \right. \right. \\
& - 8\beta_0 - 10\beta_{0,Q} \left. \right] + \hat{\gamma}_{qg}^{(0)} \left[\hat{\gamma}_{qq}^{(1),PS} \{1 - 2n_f\} + \gamma_{qq}^{(1),NS} + \hat{\gamma}_{qq}^{(1),NS} + 2\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} - 2\beta_1 \right. \\
& - 2\beta_{1,Q} \left. \right] + 6\delta m_1^{(-1)} \hat{\gamma}_{qg}^{(0)} \left[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 3\beta_0 + 5\beta_{0,Q} \right] \left. \right) + \frac{1}{\varepsilon} \left(\frac{\hat{\gamma}_{qg}^{(2)}}{3} - n_f \frac{\hat{\gamma}_{qg}^{(2)}}{3} \right. \\
& + \hat{\gamma}_{qg}^{(0)} \left[a_{gg,Q}^{(2)} - n_f a_{Qq}^{(2),PS} \right] + a_{Qg}^{(2)} \left[\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 4\beta_0 - 4\beta_{0,Q} \right] + \frac{\hat{\gamma}_{qg}^{(0)} \zeta_2}{16} \left[\gamma_{gg}^{(0)} \left\{ 2\gamma_{qq}^{(0)} \right. \right. \\
& - \gamma_{gg}^{(0)} - 6\beta_0 + 2\beta_{0,Q} \left. \right\} - (n_f + 1)\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{qq}^{(0)} \left\{ -\gamma_{qq}^{(0)} + 6\beta_0 \right\} - 8\beta_0^2 \\
& + 4\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2 \left. \right] + \frac{\delta m_1^{(-1)}}{2} \left[-2\hat{\gamma}_{qg}^{(1)} + 3\delta m_1^{(-1)}\hat{\gamma}_{qg}^{(0)} + 2\delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)} \right] \\
& + \delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)} \left[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 2\beta_0 + 4\beta_{0,Q} \right] - \delta m_2^{(-1)}\hat{\gamma}_{qg}^{(0)} \left. \right) + a_{Qg}^{(3)} \left. \right]
\end{aligned}$$

- Renormalized expression for $A_{Qg}^{(3)}$:

$$\begin{aligned}
A_{Qg}^{(3),\overline{\text{MS}}} &= \frac{\hat{\gamma}_{qg}^{(0)}}{48} \left\{ (n_f + 1)\gamma_{gg}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{gg}^{(0)} \left(\gamma_{gg}^{(0)} - 2\gamma_{qq}^{(0)} + 6\beta_0 + 14\beta_{0,Q} \right) + \gamma_{qq}^{(0)} \left(\gamma_{qq}^{(0)} \right. \right. \\
&\quad \left. \left. - 6\beta_0 - 8\beta_{0,Q} \right) + 8\beta_0^2 + 28\beta_{0,Q}\beta_0 + 24\beta_{0,Q}^2 \right\} \ln^3\left(\frac{m^2}{\mu^2}\right) + \frac{1}{8} \left\{ \hat{\gamma}_{qg}^{(1)} \left(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} \right. \right. \\
&\quad \left. \left. - 4\beta_0 - 6\beta_{0,Q} \right) + \hat{\gamma}_{qg}^{(0)} \left(\hat{\gamma}_{gg}^{(1)} - \gamma_{gg}^{(1)} + (1 - n_f)\hat{\gamma}_{qq}^{(1),\text{PS}} + \gamma_{qq}^{(1),\text{NS}} + \hat{\gamma}_{qq}^{(1),\text{NS}} - 2\beta_1 \right. \right. \\
&\quad \left. \left. - 2\beta_{1,Q} \right) \right\} \ln^2\left(\frac{m^2}{\mu^2}\right) + \left\{ \frac{\hat{\gamma}_{qg}^{(2)}}{2} - n_f \frac{\hat{\gamma}_{qg}^{(2)}}{2} + \frac{a_{Qg}^{(2)}}{2} \left(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 4\beta_0 - 4\beta_{0,Q} \right) \right. \\
&\quad \left. + \frac{\hat{\gamma}_{qg}^{(0)}}{2} \left(a_{gg,Q}^{(2)} - n_f a_{Qq}^{(2),\text{PS}} \right) + \frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16} \left(-(n_f + 1)\gamma_{gg}^{(0)}\hat{\gamma}_{qg}^{(0)} + \gamma_{gg}^{(0)} \left[2\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 6\beta_0 \right. \right. \right. \\
&\quad \left. \left. \left. - 6\beta_{0,Q} \right] - 4\beta_0[2\beta_0 + 3\beta_{0,Q}] + \gamma_{qq}^{(0)} \left[-\gamma_{qq}^{(0)} + 6\beta_0 + 4\beta_{0,Q} \right] \right) \right\} \ln\left(\frac{m^2}{\mu^2}\right) + \bar{a}_{Qg}^{(2)} \left(\gamma_{gg}^{(0)} \right. \\
&\quad \left. - \gamma_{qq}^{(0)} + 4\beta_0 + 4\beta_{0,Q} \right) + \hat{\gamma}_{qg}^{(0)} \left(n_f \bar{a}_{Qq}^{(2),\text{PS}} - \bar{a}_{gg,Q}^{(2)} \right) + \frac{\hat{\gamma}_{qg}^{(0)}\zeta_3}{48} \left((n_f + 1)\gamma_{gg}^{(0)}\hat{\gamma}_{qg}^{(0)} \right. \\
&\quad \left. + \gamma_{gg}^{(0)} \left[\gamma_{gg}^{(0)} - 2\gamma_{qq}^{(0)} + 6\beta_0 - 2\beta_{0,Q} \right] + \gamma_{qq}^{(0)} \left[\gamma_{qq}^{(0)} - 6\beta_0 \right] + 8\beta_0^2 - 4\beta_0\beta_{0,Q} \right. \\
&\quad \left. - 24\beta_{0,Q}^2 \right) + \frac{\hat{\gamma}_{qg}^{(1)}\beta_{0,Q}\zeta_2}{8} + \frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16} \left(\gamma_{gg}^{(1)} - \hat{\gamma}_{qq}^{(1),\text{NS}} - \gamma_{qq}^{(1),\text{NS}} - \hat{\gamma}_{qq}^{(1),\text{PS}} + 2\beta_1 \right. \\
&\quad \left. + 2\beta_{1,Q} \right) + \frac{\delta m_1^{(-1)}}{8} \left(16a_{Qg}^{(2)} + \hat{\gamma}_{qg}^{(0)} \left[-24\delta m_1^{(0)} - 8\delta m_1^{(1)} - \zeta_2\beta_0 - 9\zeta_2\beta_{0,Q} \right] \right) \\
&\quad + \frac{\delta m_1^{(0)}}{2} \left(2\hat{\gamma}_{qg}^{(1)} - \delta m_1^{(0)}\hat{\gamma}_{qg}^{(0)} \right) + \delta m_1^{(1)}\hat{\gamma}_{qg}^{(0)} \left(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 2\beta_0 - 4\beta_{0,Q} \right) \\
&\quad + \delta m_2^{(0)}\hat{\gamma}_{qg}^{(0)} + a_{Qg}^{(3)}.
\end{aligned}$$

Fixed moments using MATAD

- three-loop “self-energy” type diagrams with an operator insertion
- **Extension:** additional scale compared to massive propagators: Mellin variable N
- Genuine tensor integrals due to

$$\Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | O_{\mu_1 \dots \mu_n} | p \rangle = \Delta^{\mu_1} \dots \Delta^{\mu_n} \langle p | S \bar{\Psi} \gamma_{\mu_1} D_{\mu_2} \dots D_{\mu_n} \Psi | p \rangle = A(N) \cdot (\Delta p)^N$$

$$D_\mu = \partial_\mu - i g t_a A_\mu^a, \quad \Delta^2 = 0.$$

- Construction of a projector to obtain the desired moment in N [undo Δ -contraction]
- 3-loop OMEs are generated with QGRAF [Nogueira, 1993.]
- Color factors are calculated with [van Ritbergen, Schellekens, Vermaseren, 1998.]
- Translation to suitable input for MATAD [Steinhauser, 2001.]

Tests performed:

- Various 2-loop calculations for $N = 2, 4, 6, \dots$ were repeated
→ agreement with our previous calculation.
- Several non-trivial scalar 3-loop diagrams were calculated using Feynman-parameters for all N
→ agreement with MATAD.

- Number of Diagrams to be calculated:

$$A_{Q(q)q}^{(3),PS} : 132, \quad A_{qq}^{(3),NS} : 128, \quad A_{gq}^{(3)} : 89, \quad A_{Qg}^{(3)} : 1498, \quad A_{gg,Q}^{(3)} : 865. \quad \sim 250 \text{ days CT}$$

- We calculated the terms

$$A_{Qq}^{(3),PS} + A_{qq,Q}^{(3),PS}, \quad A_{qq,Q}^{(3),NS}, \quad A_{gq,Q}^{(3)} : (2-12); \quad A_{Qg}^{(3)}, \quad A_{qq,Q}^{(3)}, \quad A_{gg,Q}^{(3)} : (2-10);$$

for $N = 2, 4, 6, 8, 10, (12)$ using **MATAD** and find **agreement** of the pole terms with the prediction obtained from renormalization.

- An additional check is provided by the sum rule

$$A_{qq,Q}^{(3),NS} \Big|_{N=2} + A_{qq,Q}^{(3),PS} \Big|_{N=2} + A_{Qq}^{(3),PS} \Big|_{N=2} + A_{gq,Q}^{(3)} \Big|_{N=2} = 0,$$

which is fulfilled by our result.

- All terms proportional to ζ_2 cancel in the renormalized result for $A_{Q(q)q}^{(3),NS,PS}, A_{gq,Q}^{(3)}$
- We observe the number

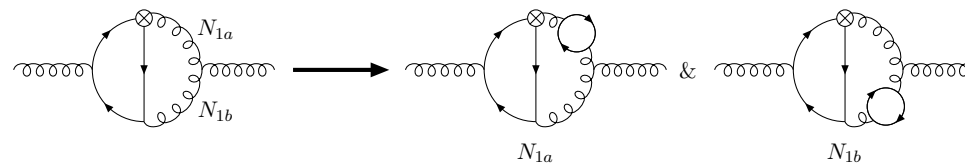
$$\mathbf{B4} = -4\zeta_2 \ln^2 2 + \frac{2}{3} \ln^4 2 - \frac{13}{2} \zeta_4 + 16 \text{Li}_4\left(\frac{1}{2}\right) = -8\sigma_{-3,-1} + \frac{11}{2} \zeta_4.$$

- The term **B4** appears as

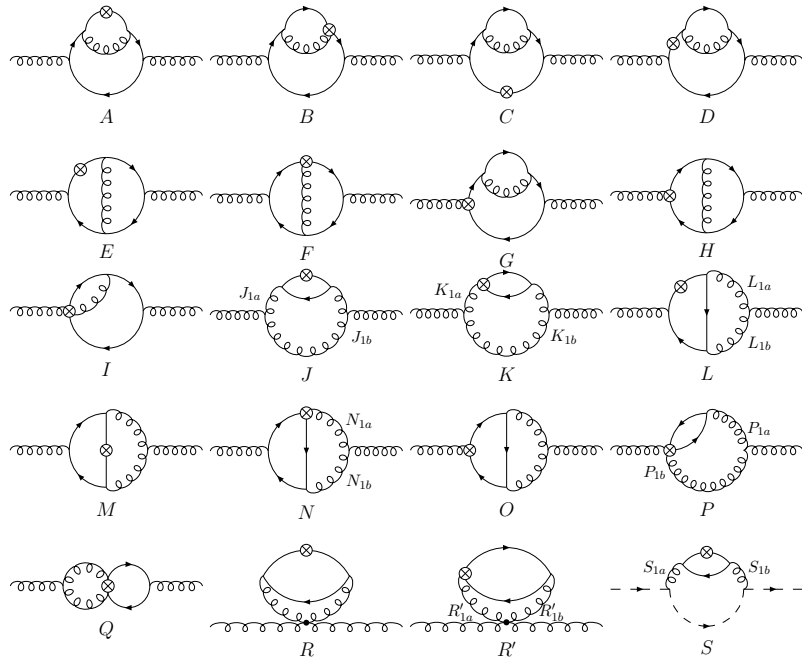
$$\left(C_F - \frac{C_A}{2}\right) \mathbf{B4}.$$

Contributing Diagrams

- 289 diagrams contribute. They are generated using QGRAF with operator insertions [Nogueira, 1991; Bierenbaum, Blümlein, Klein, 2009]
- Due to symmetry reasons, many of them are identical.
- Many diagrams can be generated from 2-loop diagrams by bubble insertions.
- e.g. from diagram N :

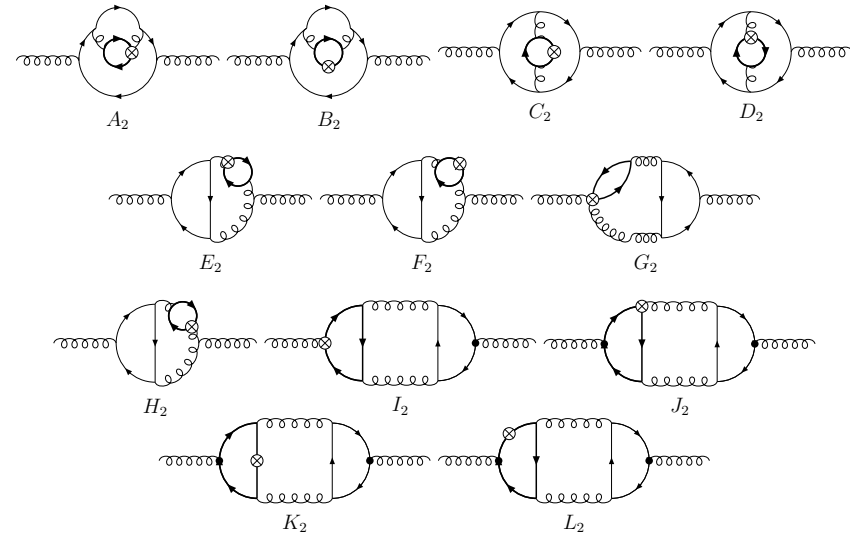


- generating 2-loop diagrams



\Rightarrow contributing diagrams obtained by bubble insertions

- Further diagrams:



- Further diagrams contribute to the other OMEs $A_{qq,Q}, A_{qq,Q}^{\text{PS}}, A_{Qq}^{\text{PS}}, A_{qq,Q}^{\text{NS}}, A_{qq,Q}^{\text{NS,TR}}$

- Typical Feynman parameter integral after momentum integration

$$I_1 = \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_4 \int_0^1 dx_5 x_1^{2+\varepsilon} x_2^{1-\varepsilon/2} x_5^{1-\varepsilon} (1-x_1)^{\varepsilon/2} (1-x_5)^2 (x_4 - x_5 x_4 + x_2 x_5)^N \left(1 - x_5 \left(1 - \frac{1}{1-x_1}\right)\right)^{3/2\varepsilon}$$

- Performing the integral gives a linear combination of sums over B-functions and Hypergeometric ${}_P F_Q$

$$I_1 = \frac{\Gamma(1-\varepsilon) \Gamma(3+\varepsilon)}{6(N+1)} \left\{ \sum_{j=1}^{N+1} \binom{1+N}{j} (-1)^j B(2-\varepsilon+j, 2) B(1+j, 2-\varepsilon/2) {}_3F_2 \left[\begin{matrix} -3/2, \varepsilon, 2, 3+\varepsilon \\ 4+j-\varepsilon, 4 \end{matrix}; 1 \right] \right. \\ \left. + B(3+N-\varepsilon, 2) B(1, 3+N-\varepsilon/2) {}_3F_2 \left[\begin{matrix} -3/2\varepsilon, 2, 3+\varepsilon \\ 5-\varepsilon, 4 \end{matrix}; 1 \right] \right\}$$

- The generalized hypergeometric function ${}_P F_Q$ is defined by

$${}_P F_Q \left[\begin{matrix} a_1, \dots, a_P \\ b_1, \dots, b_Q \end{matrix}; z \right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \dots (a_P)_i}{(b_1)_i \dots (b_Q)_i} \frac{z^i}{\Gamma(i+1)} .$$

- Now: perform a series expansion in ε and evaluate the remaining sums
- Up to 4 (in)finite sums occur, which are computed using [modern summation methods](#) encoded in [SIGMA](#) [C. Schneider, 2007]

Mathematical Structure: Harmonic Sums

- only ζ_2, ζ_3 , harmonic sums $S_{\vec{a}}(N)$ and rational numbers appear
- Harmonic Sums are defined as

$$S_{a_1, \dots, a_m}(N) = \sum_{n_1=1}^N \sum_{n_2=1}^{n_1} \dots \sum_{n_m=1}^{n_{m-1}} \frac{(\text{sign}(a_1))^{n_1}}{n_1^{|a_1|}} \frac{(\text{sign}(a_2))^{n_2}}{n_2^{|a_2|}} \dots \frac{(\text{sign}(a_m))^{n_m}}{n_m^{|a_m|}},$$

$$\text{weight } w = \sum |a_i|, \quad m - \text{depth}$$

[Blümlein, Kurth 1998; Vermaseren 1998]

- complete set of algebraic relations between harmonic sums of the same weight is known \Rightarrow **reduces** number of **independent harmonic sums** with same weight **significantly**, e.g. for $w=3$ from 18 to 8. Further reduction: structural relations. [Blümlein, 2003, 2009]
- harmonic sums of depth 1 can be expressed through polygamma-functions $\psi^{(k)}$

$$S_a(N) = \frac{(-1)^{a-1}}{\Gamma(a)} \psi^{(a-1)}(N+1) + \zeta_a, \quad k \geq 2,$$

- harmonic sums of higher depth can be expressed by Mellin-transforms of **harmonic polylogarithms**

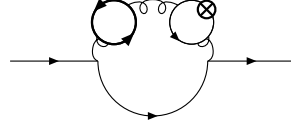
- in intermediary steps: [generalized harmonic sums](#) occur

$$\begin{aligned} \tilde{S}_{m_1, \dots}(x_1, \dots; N) &= \sum_{i_1}^N \frac{x_1^{i_1}}{i_1^{m_1}} \sum_{i_2=1}^{i_1-1} \frac{x_2^{i_2}}{i_2^{m_2}} \tilde{S}_{m_3, \dots}(x_3, \dots; i_2) \\ &\quad + \tilde{S}_{m_1+m_2, m_3, \dots}(x_1 \cdot x_2, x_3, \dots; N) . \end{aligned}$$

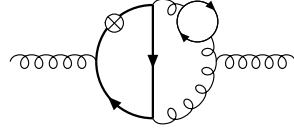
[Moch, Uwer, Weinzierl, 2002]

- can be reduced to [nested harmonic sums](#) for $x_i \in \{-1, 1\}$
- in our case: $x_i \in \{-1/2, 1/2, -2, 2\}$
- algebraic and structural relations have been worked out [\[Ablinger, Blümlein, Schneider, 2010\]](#)

3. The Results for the N_F -contributions



$$\begin{aligned}
A_{qq,Q}^{(3),PS,B} = & C_F T_F^2 N_F \left\{ \left[-\frac{128}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} \right] \frac{1}{\epsilon^3} \right. \\
& + \left[\frac{128}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1 - \frac{64}{27} \frac{16N^4 + 26N^3 - 25N^2 - 11N + 6}{N^3(1+N)^3} \right] \frac{1}{\epsilon^2} \\
& + \left[-\frac{64}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_2 - \frac{64}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1^2 - 16/3 \frac{(2+N)(-1+N)}{N^2(1+N)^2} \zeta_2 \right. \\
& \left. + \frac{64}{27} \frac{(16N^4 + 26N^3 - 25N^2 - 11N + 6)}{N^3(1+N)^3} S_1 - \frac{32}{81} \frac{181N^6 + 447N^5 - 32N^4 - 297N^3 - 92N^2 + 15N - 18}{N^4(1+N)^4} \right] \frac{1}{\epsilon} \\
& + \frac{128}{27} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_3 + \frac{64}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_2 S_1 + \frac{64}{27} \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1^3 + 16/3 \frac{(2+N)(-1+N)}{N^2(1+N)^2} S_1 \zeta_2 \\
& + \frac{112}{9} \frac{(2+N)(-1+N)}{N^2(1+N)^2} \zeta_3 - \frac{32}{27} \frac{(16N^4 + 26N^3 - 25N^2 - 11N + 6)}{N^3(1+N)^3} S_2 - \frac{32}{27} \frac{(16N^4 + 26N^3 - 25N^2 - 11N + 6)}{N^3(1+N)^3} S_1^2 \\
& - \frac{8}{9} \frac{(16N^4 + 26N^3 - 25N^2 - 11N + 6)}{N^3(1+N)^3} \zeta_2 + \frac{32}{81} \frac{(181N^6 + 447N^5 - 32N^4 - 297N^3 - 92N^2 + 15N - 18)}{N^4(1+N)^4} S_1 \\
& \left. - \frac{16}{243} \frac{-3503N^5 + 4927N^6 - 5309N^4 - 929N^3 + 7210N^7 + 54 + 2074N^8 + 231N^2 + 9N}{N^5(1+N)^5} \right\}
\end{aligned}$$



$$\begin{aligned}
A_{Qg}^{(3),11} = & N_F T_F^2 C_A \left\{ \left[\frac{128}{9N} S_1 + \frac{32}{9} \frac{2N^3 + 7N^2 + 6N + 3}{N^2(1+N)^2} \right] \frac{1}{\varepsilon^3} \right. \\
& + \left[-\frac{64}{27} \frac{(5N+14)}{N(1+N)} S_1 - \frac{16}{27} \frac{P_1(N)}{N^3(2+N)(1+N)^3} + \frac{64}{9N} S_1^2 + \frac{128}{9N} S_2 \right] \frac{1}{\varepsilon^2} \\
& + \left[\frac{128}{9N} S_1 S_2 + \frac{416}{27N} S_3 - \frac{32}{27} \frac{(5N+14)}{N(1+N)} S_1^2 + \frac{4}{81} \frac{P_2(N)}{N^4(2+N)^2(1+N)^4} \right. \\
& + \frac{128}{9N} S_{21} + \frac{16}{3N} \zeta_2 S_1 - \frac{16}{27} \frac{(74N^3 + 121N^2 + 38N - 27)}{N^2(1+N)^2} S_2 + \frac{64}{27N} S_1^3 \\
& \left. + \frac{16}{81} \frac{(47N^3 + 13N^2 - 196N - 108)}{N^2(1+N)^2} S_1 + \frac{4}{3} \frac{(2N^3 + 7N^2 + 6N + 3)}{N^2(1+N)^2} \zeta_2 \right] \frac{1}{\varepsilon} \\
& + \frac{8}{81} \frac{P_3(N)}{N^3(2+N)(1+N)^3} S_2 - \frac{2}{9} \frac{P_1(N)}{N^3(2+N)(1+N)^3} \zeta_2 - \frac{8}{9} \frac{(5N+14)}{N(1+N)} S_1 \zeta_2 \\
& - \frac{64}{27} \frac{(5N+14)}{N(1+N)} S_2 S_1 - \frac{8}{81} \frac{(616N^3 + 899N^2 + 202N - 243)}{N^2(1+N)^2} S_3 \\
& - \frac{1}{243} \frac{P_4(N)}{(2+N)^3 N^5(1+N)^5} - \frac{64}{27} \frac{(5N+14)(N)}{N(1+N)} S_{21} - \frac{112}{9N} \zeta_3 S_1 + \frac{8}{3N} \zeta_2 S_1^2 \\
& + \frac{128}{9N} S_1 S_{21} + \frac{16}{3N} \zeta_2 S_2 + \frac{64}{9N} S_1^2 S_2 \\
& + \frac{8}{81} \frac{(47N^3 + 13N^2 - 196N - 108)}{N^2(1+N)^2} S_1^2 - \frac{128}{9N} S_{2,1,1} + \frac{160}{9N} S_2 \\
& + \frac{16}{27N} S_1^4 + \frac{256}{9N} S_4 - \frac{28}{9} \frac{(2N^3 + 7N^2 + 6N + 3)}{N^2(1+N)^2} \zeta_3 - \frac{32}{81} \frac{(5N+14)}{N(1+N)} S_1^3 \\
& \left. + \frac{416}{27N} S_1 S_3 + \frac{64}{3N} S_{3,1} - \frac{4}{243} \frac{(323N^5 - 3972N^4 - 9291N^3 - 4456N^2 - 1080N + 648)}{N^3(1+N)^3} S_1 \right\}
\end{aligned}$$

- Gluonic contributions:

$$\begin{aligned}
\hat{a}_{Qg}^{(3),0} = & n_f T_F^2 C_A \left\{ \frac{16(N^2 + N + 2)}{27N(N+1)(N+2)} \left[108S_{-2,1,1} - 78S_{2,1,1} - 90S_{-3,1} + 72S_{2,-2} - 6S_{3,1} \right. \right. \\
& - 108S_{-2,1}S_1 + 42S_{2,1}S_1 - 6S_{-4} + 90S_{-3}S_1 + 118S_3S_1 + 120S_4 + 18S_{-2}S_2 + 54S_{-2}S_1^2 \\
& \left. \left. + 33S_2S_1^2 + 15S_2^2 + 2S_1^4 + 18S_{-2}\zeta_2 + 9S_2\zeta_2 + 9S_1^2\zeta_2 - 42S_1\zeta_3 \right] \right. \\
& + 32 \frac{5N^4 + 14N^3 + 53N^2 + 82N + 20}{27N(N+1)^2(N+2)^2} \left[6S_{-2,1} - 5S_{-3} - 6S_{-2}S_1 \right] \\
& - \frac{64(5N^4 + 11N^3 + 50N^2 + 85N + 20)}{27N(N+1)^2(N+2)^2} S_{2,1} - \frac{16(40N^4 + 151N^3 + 544N^2 + 779N + 214)}{27N(N+1)^2(N+2)^2} S_2S_1 \\
& - \frac{32(65N^6 + 429N^5 + 1155N^4 + 725N^3 + 370N^2 + 496N + 648)}{81(N-1)N^2(N+1)^2(N+2)^2} S_3 \\
& - \frac{16(20N^4 + 107N^3 + 344N^2 + 439N + 134)}{81N(N+1)^2(N+2)^2} S_1^3 + \frac{Q_1(N)}{81(N-1)N^3(N+1)^3(N+2)^3} S_2 \\
& + \frac{32(47N^6 + 278N^5 + 1257N^4 + 2552N^3 + 1794N^2 + 284N + 448)}{81N(N+1)^3(N+2)^3} S_{-2} \\
& + \frac{8(22N^6 + 271N^5 + 2355N^4 + 6430N^3 + 6816N^2 + 3172N + 1256)}{81N(N+1)^3(N+2)^3} S_1^2 \\
& + \frac{Q_2(N)}{243(N-1)N^2(N+1)^4(N+2)^4} S_1 + \frac{448(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& - \frac{16(5N^4 + 20N^3 + 59N^2 + 76N + 20)}{9N(N+1)^2(N+2)^2} S_1\zeta_2 - \frac{Q_3(N)}{9(N-1)N^3(N+1)^3(N+2)^3} \zeta_2 \\
& \left. - \frac{Q_4(N)}{243(N-1)N^5(N+1)^5(N+2)^5} \right\}
\end{aligned}$$

$$\begin{aligned}
& +n_f T_F^2 C_F \left\{ \frac{16(N^2 + N + 2)}{27N(N+1)(N+2)} \left[144S_{2,1,1} - 72S_{3,1} - 72S_{2,1}S_1 + 48S_4 - 16S_3S_1 \right. \right. \\
& - 24S_2^2 - 12S_2S_1^2 - 2S_1^4 - 9S_1^2\zeta_2 + 42S_1\zeta_3 \left. \right] + 32 \frac{10N^3 + 49N^2 + 83N + 24}{81N^2(N+1)(N+2)} \left[3S_2S_1 + S_1^3 \right] \\
& - \frac{128(N^2 - 3N - 2)}{3N^2(N+1)(N+2)} S_{2,1} - \frac{Q_5(N)}{81(N-1)N^3(N+1)^3(N+2)^2} S_3 \\
& + \frac{Q_6(N)}{27(N-1)N^4(N+1)^4(N+2)^3} S_2 - \frac{32(10N^4 + 185N^3 + 789N^2 + 521N + 141)}{81N^2(N+1)^2(N+2)} S_1^2 \\
& - \frac{16(230N^5 - 924N^4 - 5165N^3 - 7454N^2 - 10217N - 2670)}{243N^2(N+1)^3(N+2)} S_1 \\
& + \frac{16(5N^3 + 11N^2 + 28N + 12)}{9N^2(N+1)(N+2)} S_1\zeta_2 - \frac{Q_7(N)}{9(N-1)N^3(N+1)^3(N+2)^2} \zeta_3 \\
& \left. + \frac{Q_8(N)}{9(N-1)N^4(N+1)^4(N+2)^3} \zeta_2 + \frac{Q_9(N)}{243(N-1)N^6(N+1)^6(N+2)^5} \right\}
\end{aligned}$$

$$\begin{aligned}
a_{qg,Q}^{(3),0} = & n_f T_F^2 \left\{ C_F \left[\frac{N^2 + N + 2}{N(N+1)(N+2)} \left[-\frac{56}{9} S_4 + \frac{32}{27} S_3 S_1 + \frac{8}{9} S_2 S_1^2 + \frac{4}{9} S_2^2 + \frac{4}{27} S_1^4 + \frac{256}{9} S_1 \zeta_3 \right] \right. \right. \\
& - \frac{16(10N^3 + 13N^2 + 29N + 6)}{81N^2(1+N)(2+N)} [S_1^3 + 3S_2 S_1] + \frac{32(5N^3 - 16N^2 + N - 6)}{81N^2(1+N)(2+N)} S_3 \\
& + \frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)}{27N^2(1+N)^2(2+N)} S_2 \\
& + \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)}{81N^2(1+N)^2(2+N)} S_1^2 - \frac{R_4(N)}{243N^2(1+N)^3(2+N)} S_1 \\
& \left. - \frac{64(N^2 + N + 2)R_5(N)}{9(N-1)N^3(1+N)^3(2+N)^2} \zeta_3 + \frac{R_6(N)}{243(N-1)N^6(1+N)^6(2+N)^5} \right] \\
& + C_A \left[\frac{N^2 + N + 2}{N(N+1)(N+2)} \left[-\frac{56}{9} S_4 - \frac{128}{9} S_{-4} + \frac{160}{27} S_3 S_1 - \frac{4}{9} S_2^2 + \frac{8}{9} S_2 S_1^2 \right. \right. \\
& - \frac{4}{27} S_1^4 - \frac{64}{9} S_{2,1} S_1 - \frac{128}{9} S_{3,1} + \frac{64}{9} S_{2,1,1} - \frac{256}{9} \zeta_3 S_1 \left. \right] \\
& + \frac{32(5N^4 + 20N^3 + 41N^2 + 49N + 20)}{81N(1+N)^2(2+N)^2} [S_1^3 + 12S_{2,1} - 3S_2 S_1] \\
& + \frac{64}{81} \frac{(5N^4 + 38N^3 + 59N^2 + 31N + 20)}{N(1+N)^2(2+N)^2} S_3 + \frac{128}{27} \frac{(5N^2 + 8N + 10)}{N(1+N)(2+N)} S_{-3} \\
& + \frac{512}{9} \frac{(N^2 + N + 1)(N^2 + N + 2)}{(N-1)N^2(1+N)^2(2+N)^2} \zeta_3 - \frac{16R_7(N)}{81N(1+N)^3(2+N)^3} S_2 \\
& - \frac{32(121N^3 + 293N^2 + 414N + 224)}{81N(1+N)^2(2+N)} S_{-2} - \frac{R_8(N)}{81N(1+N)^3(2+N)^3} S_1^2 \\
& \left. + \frac{16R_9(N)}{243(N-1)N^2(1+N)^4(2+N)^4} S_1 + \frac{8R_{10}(N)}{243(N-1)N^5(1+N)^5(2+N)^5} \right] \left. \right\}
\end{aligned}$$

(complete OME)

$$\begin{aligned}
\gamma_{qg}^{(2)} = & \frac{n_f^2 T_F^2}{(N+1)(N+2)} \left\{ C_A \left[(N^2 + N + 2) \left(\frac{128}{3N} S_{2,1} + \frac{128}{3N} S_{-3} + \frac{64}{9N} S_3 + \frac{32}{9N} S_1^3 \right. \right. \right. \\
& \left. \left. - \frac{32}{3N} S_2 S_1 \right) - \frac{128(5N^2 + 8N + 10)}{9N} S_{-2} - \frac{64(5N^4 + 26N^3 + 47N^2 + 43N + 20)}{9N(N+1)(N+2)} S_2 \right. \\
& \left. - \frac{64(5N^4 + 20N^3 + 41N^2 + 49N + 20)}{9N(N+1)(N+2)} S_1^2 + \frac{64P_1(N)}{27N(N+1)^2(N+2)^2} S_1 \right. \\
& \left. + \frac{16P_2(N)}{27(N-1)N^4(N+1)^3(N+2)^3} \right] \\
& + C_F \left[\frac{32}{9} \frac{N^2 + N + 2}{N} \{ 10S_3 - S_1^3 - 3S_1 S_2 \} \right. \\
& + \frac{32(5N^2 + 3N + 2)}{3N^2} S_2 + \frac{32(10N^3 + 13N^2 + 29N + 6)}{9N^2} S_1^2 \\
& \left. \left. - \frac{32(47N^4 + 145N^3 + 426N^2 + 412N + 120)}{27N^2(N+1)} S_1 + \frac{4P_3(N)}{27(N-1)N^5(N+1)^4(N+2)^3} \right] \right\}
\end{aligned}$$

agreement with [Moch, Vermaseren, Vogt 2004]

- Flavor non-singlet contributions:

Vector current

$$\hat{\gamma}_{qq}^{(2),NS} = C_F T_F^2 N_F \left\{ -\frac{256}{27} S_1 - \frac{1280}{27} S_2 + \frac{256}{9} S_3 + \frac{16}{27} \frac{(51 N^6 + 153 N^5 + 57 N^4 + 35 N^3 + 96 N^2 + 16 N - 24)}{N^3 (1 + N)^3} \right\}$$

agreement with [Gracey 1993; Moch, Vermaseren, Vogt 2004]

$$a_{qq,Q}^{(3),NS} = C_F T_F^2 N_F \left\{ -\frac{55552}{729} S_1 + \frac{448}{27} \zeta_3 S_1 - \frac{160}{27} \zeta_2 S_1 + \frac{640}{27} S_2 + \frac{32}{9} \zeta_2 S_2 - \frac{320}{81} S_3 + \frac{64}{27} S_4 \right. \\ \left. + \frac{2}{729} \frac{P_1(N)}{N^4 (1 + N)^4} - \frac{112}{27} \frac{(3 N^2 + 3 N + 2)}{N (1 + N)} \zeta_3 + \frac{4}{27} \frac{(3 N^4 + 6 N^3 + 47 N^2 + 20 N - 12)}{N^2 (1 + N)^2} \zeta_2 \right\}$$

Transversity

$$\hat{\gamma}_{qq}^{(2),TR} = C_F T_F^2 N_F \left\{ -\frac{256}{27} S_1 - \frac{1280}{27} S_2 + \frac{256}{9} S_3 + \frac{16}{9} \frac{(17 N^2 + 17 N - 8)}{N (1 + N)} \right\}$$

agreement with [Gracey 2003]

$$a_{Qq}^{(3),TR} = C_F T_F^2 N_F \left\{ -\frac{55552}{729} S_1 + \frac{448}{27} \zeta_3 S_1 - \frac{160}{27} \zeta_2 S_1 + \frac{640}{27} S_2 + \frac{32}{9} \zeta_2 S_2 - \frac{320}{81} S_3 + \frac{64}{27} S_4 \right. \\ \left. + \frac{2}{243} \frac{(3917 N^4 + 7834 N^3 + 4157 N^2 - 48 N - 144)}{N^2 (1 + N)^2} - \frac{112}{9} \zeta_3 + \frac{4}{9} \zeta_2 \right\}$$

- Flavor pure singlet contributions:

$$\hat{\gamma}_{qq}^{(2),PS} = C_F T_F^2 N_F \left\{ -\frac{64}{3} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} (S_1^2 + S_2) + \frac{128}{9} \frac{(68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48) S_1}{N^3 (1 + N)^3 (2 + N)^2 (-1 + N)} \right. \\ \left. - \frac{128}{27} \frac{P_1(N)}{(-1 + N) N^4 (1 + N)^4 (2 + N)^3} \right\}$$

$$P_1(N) = 1353 N^7 + 1200 N^8 - 317 N^6 - 1689 N^5 - 2103 N^4 - 2672 N^3 + 144 - 48 N - 1496 N^2 + 392 N^9 + 52 N^{10}$$

agreement with [\[Moch, Vermaseren, Vogt 2004; \(Blümlein, Kauers, Klein, Schneider, 2009\)\]](#)

$$a_{qq,Q}^{(3),PS} = C_F T_F^2 N_F \left\{ \frac{128}{27} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_1^3 - \frac{64}{27} \frac{(266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24)}{N^3 (-1 + N) (2 + N)^2 (1 + N)^3} S_1^2 \right. \\ + \frac{128}{9} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_1 S_2 + \frac{64}{81} \frac{P_3(N)}{(-1 + N) N^4 (1 + N)^4 (2 + N)^3} + \frac{32}{3} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} \zeta_2 S_1 \\ - \frac{64}{27} \frac{(266 N^4 + 181 N^5 + 269 N^3 + 230 N^2 + 74 N^6 + 16 N^7 + 44 N - 24)}{N^3 (-1 + N) (2 + N)^2 (1 + N)^3} S_2 + \frac{256}{27} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_3 \\ \left. - \frac{32}{243} \frac{P_4(N)}{N^5 (-1 + N) (2 + N)^4 (1 + N)^5} - \frac{16}{9} \frac{P_5(N)}{N^3 (-1 + N) (2 + N)^2 (1 + N)^3} \zeta_2 + \frac{224}{9} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} \zeta_3 \right\}$$

(complete OME)

$$a_{Qq}^{(3),PS} = C_F T_F^2 N_F \left\{ -\frac{16}{27} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_1^3 + \frac{16}{27} \frac{(68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48)}{N^3 (1 + N)^3 (2 + N)^2 (-1 + N)} S_1^2 \right. \\ - \frac{208}{9} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_1 S_2 - \frac{32}{81} \frac{P_6(N)}{(-1 + N) N^4 (1 + N)^4 (2 + N)^3} S_1 - \frac{16}{3} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} \zeta_2 S_1 \\ + \frac{208}{27} \frac{68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48}{N^3 (1 + N)^3 (2 + N)^2 (-1 + N)} S_2 - \frac{1760}{27} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} S_3 \\ + \frac{32}{243} \frac{P_7(N)}{N^5 (1 + N)^5 (2 + N)^4 (-1 + N)} + \frac{224}{9} \frac{(N^2 + N + 2)^2}{(-1 + N) N^2 (1 + N)^2 (2 + N)} \zeta_3 \\ \left. + \frac{16}{9} \frac{(68 N^5 + 37 N^6 + 8 N^7 - 11 N^4 - 86 N^3 - 56 N^2 - 104 N - 48)}{N^3 (1 + N)^3 (2 + N)^2 (-1 + N)} \zeta_2 \right\}$$

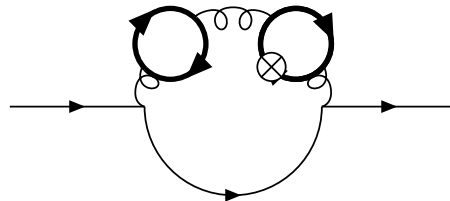
Mellin moments for the PS-, NS- and Transversity-contributions to the anomalous dimensions and constant terms a_{ij} (prefactor $T_F^2 N_F$ taken out)

agreement with [\[Bierenbaum, Blümlein, Klein, 2009; Blümlein, Klein, Tödli, 2009\]](#)

N	2	8
$\hat{\gamma}_{qq}^{(2)}$	$\frac{16928}{243}C_A - \frac{2768}{243}C_F$	$-\frac{20758849082}{2755620000}C_A + \frac{15806595692962}{1620304560000}C_F$
$a_{Qg}^{(3)}$	$\left(\frac{-6706}{2187} - \frac{616}{81}\zeta_3 - \frac{250}{81}\zeta_2\right)C_A$ $+ \left(\frac{158}{243} + \frac{896}{81}\zeta_3 + \frac{40}{9}\zeta_2\right)C_F$	$\left(\frac{24718362393463}{1322697600000} - \frac{125356}{18225}\zeta_3 + \frac{2118187}{2916000}\zeta_2\right)C_A$ $+ \left(-\frac{291376419801571603}{32665339929600000} + \frac{887741}{174960}\zeta_3 - \frac{139731073}{1143072000}\zeta_2\right)C_F$
$a_{qg,Q}^{(3)}$	$\left(\frac{83204}{2187} - \frac{616}{81}\zeta_3 + \frac{290}{81}\zeta_2\right)C_A$ $+ \left(-\frac{5000}{243} + \frac{896}{81}\zeta_3 - \frac{4}{3}\zeta_2\right)C_F$	$\left(\frac{157327027056457}{3968092800000} - \frac{125356}{18225}\zeta_3 + \frac{7917377}{2268000}\zeta_2\right)C_A$ $+ \left(-\frac{201046808090490443}{10888446643200000} + \frac{887741}{174960}\zeta_3 - \frac{3712611349}{3429216000}\zeta_2\right)C_F$
$\hat{\gamma}_{qq}^{(2),PS}$	$-\frac{10048}{243}C_F$	$-\frac{13131081443}{6751269000}C_F$
$a_{Qq}^{(3),PS}$	$\left(-\frac{76408}{2187} - \frac{112}{81}\zeta_2 + \frac{896}{81}\zeta_3\right)C_F$	$\left(-\frac{16194572439593}{15122842560000} - \frac{343781}{14288400}\zeta_2 + \frac{1369}{3645}\zeta_3\right)C_F$
$a_{qq,Q}^{(3),PS}$	$\left(-\frac{100096}{2187} - \frac{256}{81}\zeta_2 + \frac{896}{81}\zeta_3\right)C_F$	$\left(-\frac{20110404913057}{27221116608000} + \frac{135077}{4762800}\zeta_2 + \frac{1369}{3645}\zeta_3\right)C_F$
$\gamma_{qq}^{(2),NS}$	$-\frac{3584}{243}C_F$	$-\frac{38920977797}{1125211500}C_F$
$a_{qq,Q}^{(3),NS}$	$\left(-\frac{100096}{2187} - \frac{256}{81}\zeta_2 + \frac{896}{81}\zeta_3\right)C_F$	$\left(-\frac{4763338626853463}{34026395760000} - \frac{36241943}{3572100}\zeta_2 + \frac{39532}{1215}\zeta_3\right)C_F$
$\gamma_{qq}^{(2),TR}$	$-\frac{368}{27}C_F$	$-\frac{711801943}{20837250}C_F$
$a_{qq,Q}^{(3),TR}$	$\left(-\frac{4390}{81} - 4\zeta_2 + \frac{112}{9}\zeta_3\right)C_F$	$\left(-\frac{29573247248999}{210039480000} - \frac{2030251}{198450}\zeta_2 + \frac{4408}{135}\zeta_3\right)C_F$

4. The T_F^2 contributions

Two massive lines with $m_1^2 = m_2^2$



- Problem: Feynman-Parameter integrals can not be mapped directly onto hypergeometric functions:

$$I = \int_0^1 dx \int_0^1 dy \int_0^1 dz x^{\alpha_1} (1-x)^{\beta_1} y^{\alpha_2} (1-y)^{\beta_2} z^{\alpha_3} (1-z)^{\beta_3} \left(\frac{z}{x(1-x)} + \frac{1-z}{y(1-y)} \right)^\gamma$$

- \rightarrow use Mellin-Barnes representation at the momentum level
- we obtain integrals like

$$I_1 = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \frac{\Gamma(-s) \Gamma(s - \frac{\varepsilon}{2}) \Gamma(s - \frac{3}{2}\varepsilon) \Gamma(\varepsilon - 1 - s) \Gamma^2(2 - s + \frac{\varepsilon}{2}) \Gamma(3 + s - \varepsilon) \Gamma(s + N - \varepsilon)}{\Gamma(4 - 2s + \varepsilon) \Gamma(3 + 2s + N - 2\varepsilon)}$$

- these contour integrals can be mapped on a linear combination of hypergeometric ${}_pF_q$'s containing also half-integer values, e.g.:

$$I_2 = {}_5F_4 \left[\begin{matrix} -\frac{1}{2}\varepsilon, -\frac{3}{2}\varepsilon, 3 - \varepsilon, N - \varepsilon, -\frac{3}{2} - \frac{1}{2}\varepsilon \\ 2 + \frac{N}{2} - \varepsilon, \frac{3}{2} + \frac{N}{2} - \varepsilon, 2 - \varepsilon, -1 - \frac{1}{2}\varepsilon \end{matrix}; 1 \right]$$

- \rightarrow expand in ε
- perform infinite sums, new classes of sums contribute, e.g.:

$$\sum_{n=2}^{\infty} \frac{1}{n} S_1(N + 2n), \quad \sum_{n=2}^{\infty} \frac{1}{2n - 3} S_1(N + 2n), \quad \sum_{n=2}^{\infty} \frac{1}{n} \frac{\Gamma(2n)\Gamma(n + N)}{\Gamma(n)\Gamma(2n + N)} S_1(n)$$

- \rightarrow potential for new mathematical structures in the results

First Results

- Flavor non-singlet contributions:

Vector current

$$\hat{\gamma}_{qq}^{(2),NS} = C_F T_F^2 \left(\frac{128S_3}{9} - \frac{640S_2}{27} - \frac{128S_1}{27} + \frac{8(51N^6 + 153N^5 + 57N^4 + 35N^3 + 96N^2 + 16N - 24)}{27N^3(N+1)^3} \right)$$

agreement with [Gracey 1993; Moch, Vermaseren, Vogt 2004]

$$\hat{a}_{qq,Q}^{(3),NS} = T_F^2 C_F \left\{ \frac{128}{27} S_4 - \frac{1024}{27} \zeta_3 S_1 + \frac{64}{9} \zeta_2 S_2 + \frac{256(3N^2 + 3N + 2)}{27N(N+1)} \zeta_3 - \frac{320}{27} \zeta_2 S_1 - \frac{640}{81} S_3 \right. \\ \left. + \frac{8(3N^4 + 6N^3 + 47N^2 + 20N - 12)}{27N^2(N+1)^2} \zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 \right. \\ \left. - \frac{4(417N^8 + 1668N^7 - 4822N^6 - 12384N^5 - 6507N^4 + 740N^3 + 216N^2 + 144N + 432)}{729N^4(N+1)^4} \right\}$$

Transversity

agreement with [Gracey 2003]

$$\hat{\gamma}_{qq}^{(2),TR} = C_F T_F^2 \left\{ \frac{128S_3}{9} - \frac{640S_2}{27} - \frac{128S_1}{27} + \frac{8(17N^2 + 17N - 8)}{9N(N+1)} \right\}$$

$$a_{qq,Q}^{(3),TR} = C_F T_F^2 \left\{ \frac{128}{27} S_4 - \frac{1024}{27} \zeta_3 S_1 + \frac{64}{9} S_2 \zeta_2 + \frac{256}{9} \zeta_3 - \frac{320}{27} S_1 \zeta_2 - \frac{640}{81} S_3 \right. \\ \left. + \frac{8}{9} \zeta_2 + \frac{1856}{81} S_2 - \frac{19424}{729} S_1 - \frac{4(139N^4 + 278N^3 - 101N^2 + 48N + 144)}{243N^2(N+1)^2} \right\}$$

First Results

- Pure singlet contributions:

$$\hat{a}_{Qq}^{(3),PS} = \frac{T_F^2 C_F}{(N-1)N^2(N+1)^2(2+N)} \left\{ (N^2 + N + 2)^2 \left(\frac{32}{27} S_1^3 - \frac{512}{27} S_3 + \frac{128}{3} S_{2,1} \right. \right. \\ \left. \left. - \frac{1024}{9} \zeta_3 - \frac{160}{9} S_2 S_1 + \frac{32}{3} \zeta_2 S_1 \right) - \frac{32 P_1(N)}{9N(N+2)} \zeta_2 \right. \\ \left. + \frac{32 P_2(N)}{27N(N+2)(N+3)(N+4)(N+5)} S_2 \right. \\ \left. - \frac{32 P_3(N)}{27N(N+1)(N+2)(N+3)(N+4)(N+5)} S_1^2 \right. \\ \left. + \frac{64 P_4(N)}{81N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)} S_1 \right. \\ \left. - \frac{64 P_5(N)}{243N^3(N+1)^2(N+2)^3(N+3)(N+4)(N+5)} \right\},$$

with

$$\begin{aligned} P_1(N) &= 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24, \\ P_2(N) &= 40N^9 + 625N^8 + 3284N^7 + 5392N^6 - 7014N^5 - 33693N^4 - 47454N^3 \\ &\quad - 46100N^2 - 26280N + 7200, \\ P_3(N) &= 8N^{10} + 133N^9 + 1095N^8 + 5724N^7 + 18410N^6 + 34749N^5 + 40683N^4 \\ &\quad + 37370N^3 + 22748N^2 - 3960N - 7200, \\ P_4(N) &= 52N^{13} + 746N^{12} + 4658N^{11} + 20431N^{10} + 79990N^9 + 251778N^8 + 553796N^7 \\ &\quad + 837697N^6 + 886552N^5 + 599060N^4 + 155864N^3 - 82368N^2 - 76896N \\ &\quad - 17280, \\ P_5(N) &= 293N^{15} + 4670N^{14} + 32280N^{13} + 145948N^{12} + 559575N^{11} + 1871440N^{10} \\ &\quad + 4877344N^9 + 9333994N^8 + 12958212N^7 + 12693884N^6 + 8472792N^5 \\ &\quad + 4514336N^4 + 3109248N^3 + 2192832N^2 + 1026432N + 207360. \end{aligned}$$

5. Conclusion

- We computed the $O(\alpha_s^3)N_F T_F^2$ contributions to all the OMEs A_{ij} which contribute to the nucleonic structure function $F_2(x, Q^2)$ and transversity for general values of the Mellin variable N .
- This computations constitute first complete expressions for one color factor to the heavy flavor Wilson Coefficients for $F_2(x, Q^2)$ at $O(\alpha_s^3)$. The Wilson Coefficients $L_{qq,Q}^{\text{PS}}$ and $L_{qg,Q}^{\text{S}}$ are now completely known.
- Along with the computation of the massive OMEs we obtained the corresponding parts of the 3-loop anomalous dimensions and confirmed analytically results given in the literature, partly for the first time.
- The method used provides most compact results underlining the strength of the approach relying on the use of generalized hypergeometric and related functions combined with modern summation methods.
- First results (NS, PS) have been obtained for the $O(\alpha_s^3)T_F^2$ terms resulting from the graphs with two massive lines.