One loop tensor reduction program PJFRY

V. Yundin

in collaboration with J. Fleischer and T. Riemann

1–4 February 2011, Valencia

LHCphenonet

- Introduction
- Reduction algorithm
- Numerical package
- Conclusions

Collider experiments demand multi-leg NLO calculations.

Tensor reduction enters this calculations

1. as a basis element of traditional Feynman diagram approach

   many $2 \to 3$ and $2 \to 4$ NLO calculations have been done using this method

   [Binoth, Bredenstein, Denner, Dittmaier, Pozzorini, Roth, Wieders]

2. in alternative evaluation path for problematic points in OPP

   recently proposed tensorial reconstruction at the integrand level

   [Heinrich, Ossola, Reiter, Tramontano 2010]

Simple Passarino-Veltman reduction breaks down for multi-leg kinematics when Gram determinants become small or zero

Scalar integrals: *No problems here*

- QCDLoop/FF $(n \leq 4)$ [Ellis, Zanderighi 2007; van Oldenborgh 1990]
  dim-reg, real masses
- OneLOop $(n \leq 4)$ [van Hameren 2010]
  dim-reg, complex masses

Tensor integrals:

- LoopTools/FF $(n \leq 5,\, R \leq 4)$ [Hahn 2006; van Oldenborgh 1990]
  no $1/\epsilon^2$, no $R{=}5$, unstable for small Gram determinants
- Golem95 $(n \leq 6)$ [Binoth, Guillet, Heinrich, Pilon, Reiter 2008]
  massless is OK, massive is unstable for small Gram determinants *(work in progress)*
- private codes by various groups

Goal:

- *stable and fast public implementation of tensor reduction*
- *suitable for any physically relevant kinematics*

n-point 1-loop tensor integral of rank R:

$$I_n^{\mu_1\ldots\mu_R} = \int \frac{d^dk}{i\pi^{d/2}} \frac{k^{\mu_1}\ldots k^{\mu_R}}{((k-q_1)^2 - m_1^2 + i\epsilon)\cdots((k-q_n)^2 - m_n^2 + i\epsilon)}$$

where $q_1 = p_1,\ q_2 = p_1 + p_2,\ q_3 = p_1 + p_2 + p_3,\ \ldots$

Modified Cayley determinant and Gram determinant:

$$()_n = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \cdots & Y_{nn} \end{vmatrix}$$

$$Y_{ik} = -(q_i - q_k)^2 + m_i^2 + m_k^2$$

Signed minor: [Melrose 1965]

$$\begin{pmatrix} i_1\ i_2 \cdots i_m \\ k_1\ k_2 \cdots k_m \end{pmatrix}_n$$

$$G_{n-1,ik} = 2\, q_i\, q_k$$

where $i, k = 1, \ldots, n-1$

$$()_n|_{q_n=0} = -|G_{n-1}|$$

Tensor form-factors (rank 3 example):

$$I_n^{\mu_1\mu_2\mu_3} = \sum_{\substack{i,j,k=1 \\ i\le j\le k}}^{n-1} q_i{}^{[\mu_1} q_j{}^{\mu_2} q_k{}^{\mu_3]} F_{ijk}^{(n)} + \sum_{i=1}^{n-1} g^{[\mu_1\mu_2} q_i{}^{\mu_3]} F_{00i}^{(n)}$$

Standard naming convention:

$$F_{\dots}^{(1)} = A_{\dots},\ F_{\dots}^{(2)} = B_{\dots},\ F_{\dots}^{(3)} = C_{\dots},\ F_{\dots}^{(4)} = D_{\dots},\ F_{\dots}^{(5)} = E_{\dots},\ \textit{etc}$$

One step of recursive PV-algorithm (n-point rank R):

1. Contract with p_i^μ and $g^{\mu\nu}$ and cancel propagators
2. Invert the system of linear equations
3. The result is $(n-1)$-point and rank $(R-1)$ functions

Reducing tensor rank introduces inverse Gram determinant
(5 point example, rank $R \to R-1$):

$$I_5^{\mu_1\cdots\mu_{R-1}\mu_R} = \sum_{i=1}^{5} \frac{q_i^{\mu_R}}{|G_4|}\left(\binom{0}{i}_5 I_5^{\mu_1\cdots\mu_{R-1}} - \sum_{s=1}^{5} \binom{s}{i}_5 I_4^{\mu_1\cdots\mu_{R-1},s} \right)$$

Explicit expression for **massless** $|G_4|$, with $s_{ik} = (p_i + p_k)^2$:

$$|G_4| = -s_{12}^2(s_{15}-s_{23})^2 - \left(s_{23}s_{34} + (s_{15}-s_{34})s_{45}\right)^2 +$$
$$+2s_{12}\Big(s_{23}s_{34}(s_{23}-s_{45}) + s_{15}^2 s_{45} - s_{15}\big(s_{34}s_{45} + s_{23}(s_{34}+s_{45})\big)\Big)$$

Reducing further $I_4^{\mu_1\cdots\mu_{R-1},s}$ gives five $|G_3|$ in the denominators:

$$|G_3(s,t)| = 2st(s+t)$$

$|G_3(s_{12},s_{23})|$, $|G_3(s_{23},s_{34})|$, $|G_3(s_{34},s_{45})|$, $|G_3(s_{45},s_{15})|$, $|G_3(s_{15},s_{12})|$

Form-factors can be expressed in terms of scalar integrals in higher dimensions [Davydychev 1991, Fleischer 2000] (rank 3 box example)

$$D_{00i} = \frac{1}{2} I_{4,i}^{[d+]^2}, \quad D_{ijk} = -n_{[ijk]} I_{4,[ijk]}^{[d+]^3}$$

$$I_{n,ij\ldots}^{[d+]^l,st\ldots} = \int^{[d+]^l} \prod_{r=1}^{n} \frac{1}{((k-q_r)^2 - m_r^2)^{1+\delta_{ri}+\delta_{rj}+\cdots-\delta_{rs}-\delta_{rt}-\cdots}}$$

$[d+]^l = 4 - 2\epsilon + 2l$ – dimension shifting operator
lower indices – raise powers of propagators
upper indices – scratch out propagators *(also reduce n)*
n_{ijk} – combinatorial factor

All $()_5 = |G_4|$ can be eliminated by using

- dimension recurrence relations [Tarasov 1996, Fleischer 1999]
- algebra of signed minors [Melrose 1965]

$$E_i = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0i}{0s}_5 I_4^s$$

$$E_{ij} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]$$

$$E_{ijk} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}$$

All $()_5 = |G_4|$ can be eliminated by using

- dimension recurrence relations [Tarasov 1996, Fleischer 1999]
- algebra of signed minors [Melrose 1965]

$$E_i = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0i}{0s}_5 I_4^s$$

$$E_{ij} = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]$$

$$E_{ijk} = -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \left\{ \left[\binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}$$

Cayley determinant $\neq 0$ inside physical phase-space

Massless example:

$$\binom{0}{0}_5 = -2 s_{12} s_{15} s_{23} s_{34} s_{45} \qquad \binom{0}{0}_4 = s^2 t^2$$

Reduction of $()_4 = I_{4,\ldots}^{[d+]^l}$ will give $|G_3|$ in the denominator

$$I_{4,i}^{[d+]^l} = -\frac{\binom{0}{i}_4}{()_4} I_4^{[d+]^{l-1}} + \sum_{t=1}^{4} \frac{\binom{t}{i}_4}{()_4} I_3^{[d+]^{l-1},t}$$

$$\nu_{ij} I_{4,ij}^{[d+]^l} = -\frac{\binom{0}{j}_4}{()_4} I_{4,i}^{[d+]^{l-1}} + \sum_{t=1}^{4} \frac{\binom{t}{j}_4}{()_4} I_{3,i}^{[d+]^{l-1},t} + \frac{\binom{i}{j}_4}{()_4} I_4^{[d+]^{l-1}}$$

Dimension recursions + signed minor algebra

$$I_{4,i}^{[d+]^l} = \frac{1}{\binom{0}{0}_4}\left[-\binom{0}{i}_4 I_4^{[d+]^l} + \sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+]^{l-1},t}\right]$$

$$\nu_{ij} I_{4,ij}^{[d+]^l} = (d+2l-6)\frac{\binom{0}{j}_4}{\binom{0}{0}_4^2}\left[\binom{0}{i}_4 (d+2l-5) I_4^{[d+]^l} - \sum_{t=1}^{4}\binom{0t}{0i}_4 I_3^{[d+]^{l-1},t}\right]$$

$$+ \frac{1}{\binom{0}{0}_4}\left[\binom{0i}{0j}_4 I_4^{[d+]^{l-1}} + \sum_{t=1}^{4}\binom{0t}{0j}_4 I_{3,i}^{[d+]^{l-1}}\right]$$

Basis integrals ($l = 0, 1, 2, 3, 4$):

$I_4^{[d+]^l}$ — scalar boxes in $4+2l$ dimensions

$I_3^{[d+]^l}$ — scalar triangles in $4+2l$ dimensions

$I_2^{[d+]^l}$ — scalar bubbles in $4+2l$ dimensions

See [Fleischer, Riemann 2010] for the detailed explanation

- Compact analytic formulae for tensor coefficients
- Leading Gram determinants $|G_4|$ are eliminated
- Sub-leading Gram determinants $|G_3|$ are isolated within four scalar $(d+2l)$ dimensional boxes ($l = 1, 2, 3, 4$)
- Preserved d-dimensional structure

Scalar 1-loop integrals in $4 - 2\epsilon$ dimensions

- Coefficients of ϵ-expansion are known analytically
- several programs available (OneLOop, QCDLoop)

Scalar 1-loop integrals in higher dimensions

1. Analytic evaluation in terms of PolyLogs
   - Requires a lot of work (especially in complex mass case)
   - Likely to suffer stability problems in small Gram region
2. Numeric integration of Feynman parameter integrals
   - Also a lot of work (but less than fully analytic)
   - No stability problems, but slower than algebraic approach
3. **Hybrid:** series expansion in small $|G_3|$ region
   dimensional recurrence everywhere else
   - Effective reuse of existing building blocks
   - Slower only in small $|G_3|$ region

Dimensional recurrence:

$$\text{let} \quad X = \frac{|G_3|}{\binom{0}{0}} \quad \text{and} \quad Z_4^{(l)} = \sum_{t=1}^{4} \frac{\binom{t}{0}}{\binom{0}{0}} I_3^{[d+]^l,t}$$

$$\text{then} \quad X(d+2l-5) I_4^{[d+]^l} = I_4^{[d+]^{l-1}} - Z_4^{[d+]^{l-1}}$$

- $X \gg 1$: recurse down to 4-dim I_4, I_3, I_2
- $X = 0$: $I_4^{[d+]^l}$ degenerates into $Z_4^{(l)}$
- $X \ll 1$: recurse up, leads to **asymptotic** expansion in X

$$I_4^{[d+]} = Z_4^{(1)} + X(d-1) I_4^{[d+]^2}$$

$$I_4^{[d+]} = Z_4^{(1)} + X(d-1)\left[Z_4^{(2)} + X(d+1) I_4^{[d+]^3}\right]$$

$$I_4^{[d+]^l} = \sum_{n=1}^{N} a_n^{(l)} X^n Z_4^{(l+n)} + \left[a_N^{(l)} X^N I_4^{[d+]^{l+N}}\right], \quad a_n^{(l)} = 2^n \left(\frac{d-3}{2} + l\right)_n$$

PJFry — numerical package

Numerical implementation of described algorithms:
C++ package **PJFry** [in preparation]

- Reduction of **5-point** 1-loop tensor integrals up to **rank 5**
  4- and 3-point tensor integrals come "for free" as a by-product
- No limitations on internal/external **masses combinations**
- Automatic selection of optimal formula for each coefficient
- Small sub-Gram determinants $|G_3|$ are treated by **asymptotic expansion** improved by ϵ-algorithm
- Cache system for tensor coefficients and signed minors
- Interfaces for C, C++, FORTRAN and MATHEMATICA
- Uses QCDLoop or OneLOop for 4-dim scalar integrals
- Will be soon made publicly available

PRELIMINARY (expect improvement)

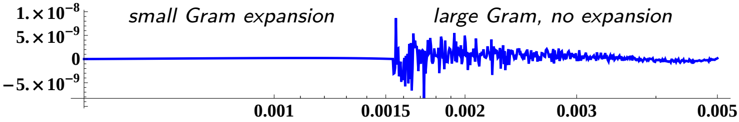
Average time per phase-space point on Core2 2GHz laptop for evaluation of all 81 **rank 5** tensor form-factors: **2 ms**

Accuracy test for 6 kinematic configurations (total $6 \cdot 10^5$ points)

Rank	AVG$[\Delta X/X]$	MAX$[\Delta X/X]$
1	$4 \cdot 10^{-15}$	$8 \cdot 10^{-11}$
2	$7 \cdot 10^{-15}$	$3 \cdot 10^{-10}$
3	$7 \cdot 10^{-12}$	$7 \cdot 10^{-11}$
4	$5 \cdot 10^{-9}$	$8 \cdot 10^{-6}$
5	$5 \cdot 10^{-6}$	$2 \cdot 10^{-3}$

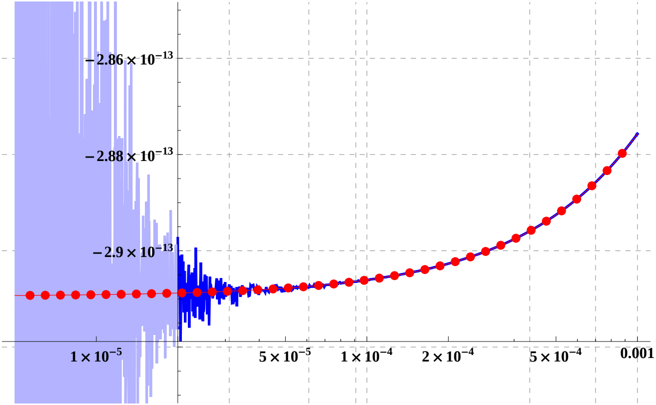
Example:

Relative accuracy of E_{3333} coef. around small $|G_3|$ region



Example: E_{3333} coefficient in small $|G_3|$ region $(x = 0)$

Comparison of Regular and Expansion formulae:



$x{=}0$: $E_{3333}(0, 0, -6{\cdot}10^4(x{+}1), 0, 0, 10^4, -3.5{\cdot}10^4, 2{\cdot}10^4, -4{\cdot}10^4, 1.5{\cdot}10^4, 0, 6550, 0, 0, 8315)$

Summary

- Algebraic tensor reduction eliminates leading Gram's
- Sub-Gram determinants are avoided by asymptotic expansion
- Efficient numerical implementation in C++

Outlook

- Extension to 6 and more legs is simple

  will be implemented in the next version

not covered in this talk:

- Analytic algorithm was used to extract rational terms for $pp \to t\bar{t}$
  Numerical package was used for cross-checks in [Badger, Sattler, VY 2011]

Thank you for your attention!