Fermionic Perturbations in Hybrid Loop Quantum Cosmology

Santiago Prado Loy

Instituto de Estructura de la Materia-CSIC

30 Oct 2018

Based on PRD.98.063535

Motivation and Introduction

Motivation

-Divergences in QFT in Curved Spacetimes
-Singularities in General Relativity

Possible resolution

LQG $\rightarrow$ LQC $\rightarrow$ Hybrid Loop Quantum Cosmology $\xrightarrow{\textit{physical case}}$ Fermions

- Our model will be that of an inflationary flat FRW cosmology and a Dirac field treated as perturbations
- We describe the system in a canonical fashion, so that constraints naturally arise.
- We truncate the action up to quadratic terms in the perturbations
- We see how the perturbations have changed the Hamiltonian constraint, the only one that matters.

Procedure

- -> Define canonical variables (choosing a Fock representation)
- -> Express the constraint operator à la Dirac.
- -> Apply a Born-Oppenheimer approximation
- -> Obtain a Schrödinger equation

Properties

-Unitarily implementable evolution
-**Finite backreaction**

Creation and Annihilation Varia-bles

- Homogeneous sector: Flat FRW cosmology (with compact spatial sections, isomorphic to T^3) (α, π_α) +Inflaton $(\phi, \pi_\phi) \to H_{|0}$

-Perturbative sector: Dirac field with mass $M \to \tilde{H}_D$

$$\Psi = \begin{pmatrix} \varphi^A \\ \bar{\chi}_{A'} \end{pmatrix} \xrightarrow{decompose} \begin{pmatrix} x^+_{\vec{k}},\ \bar{y}^+_{\vec{k}} \\ x^-_{\vec{k}},\ \bar{y}^-_{\vec{k}} \end{pmatrix}$$

Since the choice of variables can depend on the geometry in many ways, and lead to unequivalent Fock representations, we consider the general choice

$$\begin{aligned} a_{\vec{k}}^{\pm} &= f_1^{\vec{k},\pm}(\alpha,\pi_\alpha)x_{\vec{k}} + f_2^{\vec{k},\pm}(\alpha,\pi_\alpha)\bar{y}_{-\vec{k}}, \\ \bar{b}_{\vec{k}}^{\pm} &= g_1^{\vec{k},\pm}(\alpha,\pi_\alpha)x_{\vec{k}} + g_2^{\vec{k},\pm}(\alpha,\pi_\alpha)\bar{y}_{-\vec{k}}, \end{aligned}$$

And to maintain the standard anticommutative relations,

$$\begin{aligned} g_1^{\vec{k},\pm} &= e^{iJ_{\vec{k}}^{\pm}}\bar{f}_2^{\vec{k},\pm}, \qquad g_2^{\vec{k},\pm} = -e^{iJ_{\vec{k}}^{\pm}}\bar{f}_1^{\vec{k},\pm}, \\ f_2^{\vec{k},\pm} &= e^{iF_2^{\vec{k},\pm}}\sqrt{1-\left|f_1^{\vec{k},\pm}\right|^2}. \end{aligned}$$

Contribution to Hamiltonian constraint

Which leads to a contribution to the Hamiltonian constraint

$$\tilde{H}_{\vec{k}} = \sum_{\pm} \Bigg[h_D^{\vec{k}} \left(\bar{a}_{\vec{k}}^{\pm} a_{\vec{k}}^{\pm} - a_{\vec{k}}^{\pm} \bar{a}_{\vec{k}}^{\pm} + \bar{b}_{\vec{k}}^{\pm} b_{\vec{k}}^{(x,y)} - b_{\vec{k}}^{\pm} \bar{b}_{\vec{k}}^{\pm} \right) + h_J^{\vec{k}} \left(\bar{b}_{\vec{k}}^{\pm} b_{\vec{k}}^{\pm} - b_{\vec{k}}^{\pm} \bar{b}_{\vec{k}}^{(x,y)} \right) + h_I^{\vec{k}} a_{\vec{k}}^{\pm} b_{\vec{k}}^{\pm} - \bar{h}_I^{\vec{k}} \bar{a}_{\vec{k}}^{\pm} \bar{b}_{\vec{k}}^{\pm} \Bigg],$$

where the interaction term is

$$\bar{h}_I^{\vec{k}} = e^{-iJ_{\vec{k}}^{(x,y)}} \left\{ i \left(f_2^{\vec{k}} \partial f_1^{\vec{k}} - f_1^{\vec{k}} \partial f_2^{\vec{k}} \right) + 2e^{-\alpha} \omega_k f_1^{\vec{k}} f_2^{\vec{k}} + \tilde{M} \left[\left(f_1^{\vec{k}} \right)^2 - \left(f_2^{\vec{k}} \right)^2 \right] \right\}$$

$$\partial = e^{-3\alpha} \pi_\alpha \partial_\alpha + 8\pi V(\phi) e^{3\alpha} \partial_{\pi_\alpha}.$$

Hybrid Quantization

$$[g_{\mu\nu}] \xrightarrow{LQG} [\{A_i^a, E_a^i\} + \mathcal{H}, \mathcal{G}_a, \mathcal{H}_i] \xrightarrow{LQC} [\{v, b\} + \mathcal{H}]$$

v is proportional to the physical volume of the spatial sections $V = 2\pi\gamma\sqrt{\Delta_g}|v|$ and b is related to the Hubble parameter.

We create the basis of states on the gravitational Hilbert space,$\mathcal{H}_{\text{kin}}^{\text{grav}}$, with v, so that $\hat{v}$ acts as $\hat{v}|v\rangle = v|v\rangle$, and so that $e^{i\hat{b}/2}$ are translation operators

$$e^{\pm i\hat{b}/2}|v\rangle = |v \pm 1\rangle.$$

-We call the kinematical space of the inflaton $\mathcal{H}_{\text{kin}}^{\text{matt}}$, for which we choose $L^2(\mathbb{R}, d\phi)$.
-The Hamiltonian constraint of the unperturbed system is then

$$\hat{H}_{|0} = \frac{1}{2}(\hat{\pi}_\phi^2 - \hat{\mathcal{H}}_0^{(2)}(\hat{v}, \hat{b}))$$

And the total Hilbert space is

$$\mathcal{H} = \mathcal{H}_{\text{kin}}^{\text{matt}} \otimes \mathcal{H}_{\text{kin}}^{\text{grav}} \otimes \mathcal{F}_D,$$

where $\mathcal{F}_D$ is the associated Fock space, with states $|\mathcal{N}_D\rangle$. In this way, the total Hamiltonian constraint is

$$\hat{H} = \frac{1}{2}[\hat{\pi}_\phi^2 - \hat{\mathcal{H}}_0^{(2)} - (l_0 \widehat{V^{2/3} e^{\tilde{\alpha}} \tilde{H}_D})].$$

Born-Oppenheimer approximation

$$\Xi = \Gamma(V,\phi)\psi_D(\mathcal{N}_D,\phi)$$

Where we also assume that Γ's evolution in ϕ is unitary and generated by a positive operator.

$$-i\partial_\phi \Gamma(V,\phi) = \hat{\tilde{\mathcal{H}}}_0 \Gamma(V,\phi).$$

And ignoring geometry transitions mediated by the action of the quantum Hamilitonian, we obtain the Schrödinger equation

$$i\partial_\phi \psi_D(\mathcal{N}_D,\phi) = \frac{l_0 \langle \widehat{V^{2/3}e^{\tilde{\alpha}}\tilde{H}_D} \rangle_\Gamma - C_D^{(\Gamma)}(\phi)}{\langle \hat{\tilde{\mathcal{H}}}_0 \rangle_\Gamma} \psi_D(\mathcal{N}_D,\phi) \equiv \mathcal{H}_D^{(\Gamma)}(\phi)\psi_D(\mathcal{N}_D,\phi),$$

Where the $C_D^{(\Gamma)}$ term codifies the backreaction:

$$C_D^{(\Gamma)} = \langle (\hat{\tilde{\mathcal{H}}}_0)^2 - \hat{\mathcal{H}}_0^{(2)} \rangle_\Gamma.$$

We make a change of variables $d\eta_\Gamma = \frac{l_0\langle\hat{V}^{2/3}\rangle_\Gamma}{\langle\hat{\tilde{\mathcal{H}}}_0\rangle_\Gamma} d\phi$, so that we get the Heisenberg equations in $\eta_\Gamma = \eta$:

$$\begin{aligned} d_{\eta_\Gamma}\hat{a}^{\pm}_{\vec{k}}(\eta,\eta_0) &= -iF^{(\Gamma)}_{\vec{k}}\hat{a}^{\pm}_{\vec{k}}(\eta,\eta_0) + G^{(\Gamma)}_{\vec{k}}\hat{b}^{\pm\dagger}_{\vec{k}}(\eta,\eta_0), \\ d_{\eta_\Gamma}\hat{b}^{\pm\dagger}_{\vec{k}}(\eta,\eta_0) &= i\left(F^{(\Gamma)}_{\vec{k}} + \tilde{J}^{(\Gamma)}_{\vec{k}}\right)\hat{b}^{\pm\dagger}_{\vec{k}}(\eta,\eta_0) - \bar{G}^{(\Gamma)}_{\vec{k}}\hat{a}^{(x,y)}_{\vec{k}}(\eta,\eta_0), \end{aligned}$$

with, $G^{(\Gamma)}_{\vec{k}} = \frac{i\langle\widehat{h^{\vec{k}}_i V^{2/3}}\rangle_\Gamma}{\langle\hat{V}^{2/3}\rangle_\Gamma}$. The Heisenberg equations can be integrated to obtain

$$\begin{aligned} \hat{a}^{\pm}_{\vec{k}}(\eta,\eta_0) &= \alpha_{\vec{k}}(\eta,\eta_0)\hat{a}^{\pm}_{\vec{k}}(\eta_0) + \beta_{\vec{k}}(\eta,\eta_0)\hat{b}^{\pm\dagger}_{\vec{k}}(\eta_0), \\ \hat{b}^{\pm\dagger}_{\vec{k}}(\eta,\eta_0) &= -e^{i\int_{\eta_0}^{\eta} d\eta_\Gamma \tilde{J}^{(\Gamma)}_{\vec{k}}}\bar{\beta}_{\vec{k}}(\eta,\eta_0)\hat{a}^{\pm}_{\vec{k}}(\eta_0) + e^{i\int_{\eta_0}^{\eta} d\eta_\Gamma \tilde{J}^{(\Gamma)}_{\vec{k}}}\bar{\alpha}_{\vec{k}}(\eta,\eta_0)\hat{b}^{\pm\dagger}_{\vec{k}}(\eta_0). \end{aligned}$$

Unitarity and backreaction

In order for the evolution to be a unitary Bogoliubov transformation, the $\beta_{\vec{k}}$ coefficients must be square-summable for all η. If we wish this to be true, plus a vacuum state invariant to the spatial sections isometries and a standard convention of the notion of particles and antiparticles, then:

$$f_1^{\vec{k}} = e^{iF_2^{\vec{k}}} \frac{Me^{\alpha}}{2\omega_k} + \theta_{\vec{k}}, \quad \sum_{\vec{k}} |\theta_{\vec{k}}|^2 < \infty$$

So that the coefficients grow asymptotically like

$$\beta_{\vec{k}}(\eta, \eta_0) = \mathcal{O}(\theta_{\vec{k}}^{(x,y)})$$

And the interaction part of the Hamiltonian has an asymptotic order

$$h_I = \mathcal{O}(\omega_k \theta_{\vec{k}}). \qquad (1)$$

We can create an operator such that

$$\begin{aligned}\hat{U}_B^{-1}\hat{a}_{\vec{k}}^{\pm}\hat{U}_B &= \alpha_{\vec{k}}(\eta,\eta_0)\hat{a}_{\vec{k}}^{\pm} + \beta_{\vec{k}}(\eta,\eta_0)\hat{b}_{\vec{k}}^{\pm\dagger},\\ \hat{U}_B^{-1}\hat{b}_{\vec{k}}^{\pm\dagger}\hat{U}_B &= -e^{i\int_{\eta_0}^{\eta} d\eta_\Gamma \tilde{J}_{\vec{k}}^{(\Gamma)}}\bar{\beta}_{\vec{k}}(\eta,\eta_0)\hat{a}_{\vec{k}}^{\pm}(\eta_0) + e^{i\int_{\eta_0}^{\eta} d\eta_\Gamma \tilde{J}_{\vec{k}}^{(\Gamma)}}\bar{\alpha}_{\vec{k}}(\eta,\eta_0)\hat{b}_{\vec{k}}^{\pm\dagger}(\eta_0),\end{aligned}$$

Which we can use to evolve our vacuum state. With this evolved vacuum we can solve for the term proportional to the backreaction in the Schrödinger equation. The backreaction is then

$$C_D^{(\Gamma)}(\phi) = l_0\langle\hat{V}^{2/3}\rangle_\Gamma \sum_{\vec{k},\pm}\left[\Im(G_{\vec{k}}^{(\Gamma)} e^{i\int_{\eta_0}^{\eta} d\eta_\Gamma \tilde{J}_{\vec{k}}^{(\Gamma)}}\bar{\Delta}_{\vec{k}}) - d_{\eta_\Gamma} c_{\vec{k}}^{\pm}\right].$$

In the asymptotic limit of big ω_k

$$\mathcal{O}\left(\Im(G_{\vec{k}}^{(\Gamma)} e^{i\int_{\eta_0}^{\eta} d\eta_\Gamma \tilde{j}_{\vec{k}}^{(\Gamma)}} \bar{\Delta}_{\vec{k}})\right) = \mathcal{O}\left((G_{\vec{k}}^{(\Gamma)})^2\right)$$

Which, in essence, fixes $\theta_{\vec{k}}$ to be

$$\theta_{\vec{k}}^{\pm} = -i\frac{\tilde{M}e^{-\alpha}}{4\omega_k^2}\pi_\alpha e^{iF_2^{\vec{k},\pm}} + \iota_{\vec{k}}^{\pm}, \quad \sum_{\vec{k}\in\tilde{\mathbb{Z}}_1^3} \omega_k |\iota_{\vec{k}}^{\pm}|^2 < \infty,$$

And the interaction part of the Hamiltonian has an asymptotic order

$$h_I = \mathcal{O}(\omega_k \iota_{\vec{k}}). \tag{2}$$

Conclusions

- Summarizing, we used the available freedom in hybrid LQC to separate the homogeneous background and the inhomogeneous perturbations to find a formalism in which the fundamental tool (the Hamiltonian constraint) is well defined.
- To avoid divergences, we restricted our choice of creation and annihilation variables, so that the order of the interaction term was lower.
- We can conclude with a future perspective to this research line, which would be to further restrict our choice, decreasing the asymptotic order of the interaction part Hamiltonian constraint until we get rid of it.
- With this choice, the description of the fermionic degrees of freedom would be optimally adapted to that of the cosmological system.