RG-2 flow, mass and entropy

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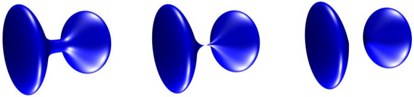

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Overview

The Riemannian Penrose Inequality

Let (Σ, h_{ab}, K_{ab}) be a initial data set with mass M_{ADM}. Let A be the area of the outermost apparent horizon in Σ

Penrose Inequality

$$M_{ADM} \geq \sqrt{\frac{A}{16\pi}}$$

It implies the positive mass theorem.

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(Huiskein and Ilmanen, 1997) Weak formulation and single black hole.
(Bray, 2001) Multiple black holes.

- A purely a geometrical fact.
- (Σ, h) needs to have nonnegative scalar curvature / obey the dominant energy condition.

An overview to the proof

- Let (Σ, h_{ab}, K_{ab}) a foliation of spacetime
- Introduce λ such that the surfaces $\lambda = cte$ are nested topological two spheres in Σ. The innermost sphere corresponds to a point.
- A normal vector to S is defined $n_a = D_a\lambda$. Define $\chi^a = un^a$ and $\chi^a D_a\lambda = 1$
- If $v^2 = n_a n^a$ we have $\chi^a D_a\lambda = uv$

Consider

$$C(\lambda) := \int_{S\in\Sigma} (2R_s^2 - k^2)dA, \quad M_H = \frac{A^{1/2}}{64\pi^{3/2}} C(\lambda)$$

where $k = D_a n^a$

Variation of Compactness

$$\frac{dC(\lambda)}{d\lambda} = \int \left(2k\tilde{D}^a\tilde{D}_a u + ukk^{ab}k_{ab} - ukR_s + ukR\right) dA$$

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Inverse mean curvature flow

If $uk = 1$

$$\frac{dC(\lambda)}{d\lambda} \geq -\frac{1}{2}C(\lambda)$$

When S is a point $C(\lambda) = 0$ then $C(\lambda) \geq 0$ for all τ

When $\lambda \to \infty$ then S expands to a sphere in the infinity and M_H becomes the M_{ADM}

Geometric flows

- Geometric evolution equations. What is the long behaviour of the flow?
- Partial Differential Equations + Riemannian Geometry → Geometric Analysis .

Extrinsic Flows

Curve shortening flow Mullins, '56
Harmonic map heat flow Eells, Sampson,'64
Mean Curvature flow Brakke,'78
Willmore flow Willmore,60's

Intrinsic Flows

Ricci flow Hamilton,'81
Calabi flow Calabi,'??
Yamabe flow Hamilton,'81

Given a 1-parameter family of metrics $g(\lambda)$ on a Riemann manifold M^n, defined on a "time" interval $I \in \mathbb{R}$, the Ricci flow is defined

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij}$$

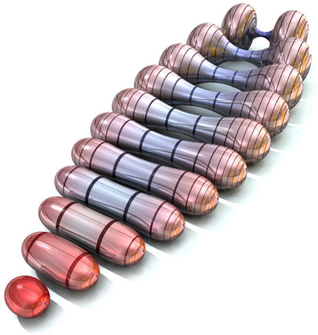
- It is not parabolic.
- Define Ricci-DeTurck flow:

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i$$
$$g(0) = g_0$$

where the 1-form $W = W(\lambda)$ is given by

$$W_j = g_{jk} g^{pq} \left(\Gamma^k_{pq} - \Gamma^k_{pq} \right)$$

The new flow is parabolic.



- With the evolution equation for the metric we can calculate the evolution for all related quantities:

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\Gamma^k_{ij} &= -g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right) \\
\frac{\partial}{\partial\lambda} R &= \Delta R + 2|Rc|^2 \\
\frac{\partial}{\partial\lambda} R_{ij} &= \Delta R_{ij} + 2R_{kijl}R_{kl} - 2R_{ik}R_{jk} \\
\frac{\partial}{\partial\lambda} R_{ijkl} &= \Delta R_{ijkl} + 2\left(B_{ijkl} - B_{ikjl} - B_{iljk}\right) \\
&\quad -\left(R_{ip}R_{pjkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp}\right)
\end{aligned}$$

where

$$B_{ijkl} = -R_{pijq}R_{qlkp}$$

The entropy formula for the Ricci flow and its geometric applications.
Grisha Perelman 2008

"The interplay of statistical physics and (pseudo)-riemannian geometry occurs in the subject of Black Hole Thermodynamics, developed by Hawking et al. Unfortunately, this subject is beyond my understanding at the moment."

Non-linear sigma models

- Let (M, g) be a Riemannian manifold and let Σ be a two dimensional Riemannian surface with metric γ . We define a map $\phi : \Sigma \to M$ and the functional

$$S(\phi) = \int \frac{1}{\alpha} g_{ij}(\phi(x)) \partial^{\mu} \phi^{i}(x) \partial^{\nu} \phi^{j}(x) \gamma_{\mu\nu} dx$$

with $\alpha > 0$.

- Non-linear sigma models: (Σ, γ) is called the worldsheet and (M, g) is the target space.
- Quantizing the action $\Rightarrow$ renormalization group

The RG-2 flow

The second order approximation in λ to the RG flow is

$$\underbrace{\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij}}_{Ricci\ flow} - \frac{\alpha}{2} R_{iklm} R_{j}^{\ klm}$$

Properties

- Curvature small $\Rightarrow$ The Ricci flow approximates full renormalization group flow.
- The RG-2 flow is *not* parabolic.
- In 3-dimensions the flow is weakly parabolic only when

$$1 + \alpha K_{ij} > 0$$

for all sectional curvatures K_{ij}.(There is a similar result in n-dimensions).
- Mathematical Problem $\Rightarrow$ There is no gradient formulation for the RG-2 flow (Carfora and Guenther 2018) "Scale invariant" (α is not a fixed constant) RG-2 flow. Flow coupled to a backwards Fokker-Plack equation.

Initial value formulation for general relativity

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- $g_{\mu\nu} \Rightarrow h_{ab} \Rightarrow \gamma_{ij}$

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- After a foliation: the extrinsic curvature k_{ab} of Σ , the Ricci scalar ${}_3R$, the matter current J^b, the matter density ρ, have to satisfy:

$$\nabla_b(k^{ab} - h^{ab}k) = 8\pi J^b$$

$${}_3R + k^2 - k_{ab}k^{ab} = 16\pi\rho$$

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- The flow must preserve the non-negativity.
- The scalar curvature under RG-2 flow satisfies

$$\frac{dR}{d\lambda} = \Delta R + 2|Rc|^2 + \alpha\left(\frac{1}{2}\Delta|Rm|^2 - \frac{1}{2}\nabla_a\nabla_b Rm^2_{ab} - R_a^{\ s} R_{sb} R^{ab} + 2R|Rc|^2 - \frac{R^3}{2}\right)$$

Evolving quantities

Let S be a closed surface on M, R_s is the scalar curvature on S, and κ is the trace of the extrinsic curvature.

The Hawking Mass

$$M_H(S) = \frac{\sqrt{A(S)}}{64\pi^{3/2}} C(S)$$

where

$$A(S) = \int_S dA = \int_S d^2x \sqrt{\gamma}$$

and

$$C(S) = \int_S dA(2R_s - \kappa^2)$$

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The **a** form of the metric:

$$ds^2 = a(r,\lambda)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

The **b** form of the metric

$$ds^2 = dr^2 + b(r,\lambda)(d\theta^2 + \sin^2(\theta)d\phi^2)$$

(Samuel, Roy Chowdhury, 2008) Ricci flow case

- We took the metric in the **b** form.
- We assume the existence of an apparent horizon in $r = r_a$.

Area of a apparent horizon under RG-2 flow

$$\frac{dA_a}{d\lambda} < -4\pi(1+\alpha b^2)$$

Area of a sphere under RG-2 flow

$$\frac{dA}{d\lambda} \leqslant -16\pi^{3/2}\frac{M_H}{\sqrt{A}} - \frac{\alpha}{2}\left(\frac{b'^2}{2b} - b''\right)^2$$

Area of a surface S under RG-2 flow

$$\frac{dA}{d\lambda} \leq -\frac{1}{4}C(S) - \frac{\alpha}{4}C_{RR_S}(S)$$

where

$$C_{RR_S} = \int_S \sqrt{\gamma} d^2x R(R_S + \frac{1}{4}K^2).$$

Remember the Hawking Mass

$$M_H(S) = \frac{\sqrt{A(S)}}{64\pi^{3/2}} C(S)$$

We need:

$$\begin{aligned} \frac{dC}{d\lambda} &= \int dAK \left\{ h^{ab} n^c \nabla_c \left(2R_{ab} + \frac{\alpha}{2} Rm^2_{ab} \right) + 2\nabla_a \left(2R^{ab} + \frac{\alpha}{2} Rm^{2\;ab} n_b \right) \right. \\ & \left. -n^a \nabla_a \left(2R^{cd} - \frac{\alpha}{2} Rm^{2\;cd} n_c n_d \right) \right\} \\ & + \int dAK^2 \left\{ R + n^a n^b R_{ab} + \frac{\alpha}{4} \left(R_m^{2\;cd} n_c n_d + R_m^2 \right) \right\}, \end{aligned}$$

RG-2 flow in 2 dimensions

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- At "time" ($\lambda = 0$) we chose a metric of the Schwarzschild type:

$$ds^2 = f(x)dt^2 + \frac{1}{f(x)} dx^2,$$

  $f(x)$ is positive everywhere but becomes zero when evaluated at the "horizon".
- Define the metric for $\lambda > 0$:

$$ds^2 = f(x,\lambda)dt^2 + \frac{e^{2\phi(x,\lambda)}}{f(x,\lambda)} dx^2,$$

- The RG-2 flow becomes

$$\frac{\partial f}{\partial \lambda} = -e^{-2\phi}(-\partial_{xx} f + \partial_x \phi \partial_x f) f - \frac{\alpha}{4} e^{-4\phi}(-\partial_{xx} f + \partial_x \phi \partial_x f)^2 f$$

$$\frac{\partial e^{2\phi}}{\partial \lambda} = -2(-\partial_{xx} f + \partial_x \phi \partial_x f) - \frac{\alpha}{2} e^{-2\phi}(-\partial_{xx} f + \partial_x \phi \partial_x f)^2$$

- A solution

$$\phi = \ln\left(\frac{f(x,\lambda)}{f(x,0)}\right).$$

- Metric in Schwarzschild like form

$$ds^2 = f(r,\lambda)dt^2 + \frac{1}{f(r,\lambda)}dr^2.$$

- The RG-2 flow becomes

$$\frac{\partial f}{\partial \lambda} = f\partial_{rr}f + \frac{\alpha}{4}f(\partial_{rr}f)^2 - (\partial_r f)^2 + q_o\partial_r f$$

here $q_o = f(x,0)|_{x=x_h}$.

- An asymptotically flat metric will remain asymptotically flat after evolution under the RG-2 flow.

Evolution of scalar curvature

$$\frac{\partial R}{\partial \lambda} = (1+\frac{\alpha}{2}R)fR'' + \frac{\alpha}{2}|R'|^2 + \left(1+\frac{\alpha}{4}\right)R^2 + q_o R'$$

Some results

Curvature

R remains positive everywhere including the point $r = 0$.

Non-existence of horizons

$$1 - \frac{\alpha}{2} R > 0$$

Entanglement entropy of 2d a black hole

$$\frac{\partial S(\lambda)}{\partial \lambda} = -\frac{c}{12} \left(R(0,\lambda) - \alpha \frac{q_o}{2} \int_0^{L_r} \frac{(R(0,\lambda))^2}{f} \right)$$

Final Remarks

- Mathematical properties of some flows and how are related to Physics.
- We have calculated how the area of an apparent horizon evolves under RG-2 flow.
- We have calculated how the area of an arbitrary closed surface evolves under RG-2 flow.
- Clues about gradient formulation of the RG-2 flow.
- We have found conditions for the existence of horizons
- 2d black hole entanglement entropy is strictly decreasing under RG-2 flow.
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