Computer Algebra, sums and special functions

Jakob Ablinger

SFB F050 Algorithmic and Enumerative Combinatorics
Research Institute for Symbolic Computation
Johannes Kepler University Linz

Two Mathematica Packages

- **MultiIntegrate** allows to compute multi-dimensional integrals over hyperexponential integrands in terms of (generalized) harmonic sums. This package uses variations and extensions of the multivariate Alkmkvist-Zeilberger algorithm.

- **HarmonicSums** allows to deal with nested sums as well as iterated integrals in an algorithmic fashion.

Note that for some of the functionalities both packages rely on the **Sigma** package by Carsten Schneider.

Nested Sums

We introduce a general class of nested sums which we define recursively by

$$\mathsf{S}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), n\right) = \sum_{\tau_1=1}^{n} f_1(\tau_1)\mathsf{S}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right)$$

with the special case

$$\mathsf{S}\left(n\right) = 1.$$

We introduce a general class of nested sums which we define recursively by

$$\mathsf{S}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), n\right) = \sum_{\tau_1=1}^{n} f_1(\tau_1)\mathsf{S}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right)$$

with the special case

$$\mathsf{S}\left(n\right) = 1.$$

Considering the letters

$$\frac{(\pm 1)^\tau}{\tau^c} \text{ for } c \in \mathbb{N}$$

leads to harmonic sums.

We introduce a general class of nested sums which we define recursively by

$$\mathsf{S}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), n\right) = \sum_{\tau_1=1}^{n} f_1(\tau_1)\mathsf{S}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right)$$

with the special case

$$\mathsf{S}\left(n\right) = 1.$$

Typical letters that we consider are:

$$\frac{(\pm 1)^\tau}{\tau}, \frac{1}{(2\tau+1)^3}, \frac{x^\tau}{(a\tau+b)^c}\binom{2\tau}{\tau}^d \text{ for } a, c \in \mathbb{N},\ b \in \mathbb{N}_0,\ x \in \mathbb{K}^*(\eta),\ d \in \mathbb{Z}$$

We introduce a general class of nested sums which we define recursively by

$$\mathsf{S}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), n\right) = \sum_{\tau_1=1}^{n} f_1(\tau_1)\mathsf{S}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right)$$

with the special case

$$\mathsf{S}\left(n\right) = 1.$$

Typical letters that we consider are:

$$\frac{(\pm 1)^\tau}{\tau}, \frac{1}{(2\tau+1)^3}, \frac{x^\tau}{(a\tau+b)^c}\binom{2\tau}{\tau}^d \text{ for } a, c \in \mathbb{N},\ b \in \mathbb{N}_0,\ x \in \mathbb{K}^*(\eta),\ d \in \mathbb{Z}$$

- quasi shuffle (stuffle) algebra

We introduce a general class of nested sums which we define recursively by

$$\mathsf{S}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), n\right) = \sum_{\tau_1=1}^{n} f_1(\tau_1)\mathsf{S}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right)$$

with the special case

$$\mathsf{S}\left(n\right) = 1.$$

Typical letters that we consider are:

$$\frac{(\pm 1)^\tau}{\tau}, \frac{1}{(2\tau+1)^3}, \frac{x^\tau}{(a\tau+b)^c}\binom{2\tau}{\tau}^d \text{ for } a, c \in \mathbb{N},\ b \in \mathbb{N}_0,\ x \in \mathbb{K}^*(\eta),\ d \in \mathbb{Z}$$

- quasi shuffle (stuffle) algebra
- shift:

$$\begin{aligned}\mathsf{S}\left(f_1(\tau), \cdots, f_k(\tau), n+1\right) \quad = \quad & \mathsf{S}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), n\right) \\ & + f_1(n+1)\mathsf{S}\left(f_2(\tau), \cdots, f_k(\tau), n+1\right)\end{aligned}$$

Iterated Integrals

We introduce a general class of iterated integrals which we define recursively by

$$\mathsf{G}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), x\right) = \int_0^x f_1(\tau_1)\mathsf{G}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right) d\tau_1$$

with the special cases

$$\mathsf{G}\left(x\right) = 1$$

and

$$\mathsf{G}\Big(\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau}}_{k \text{ times}}, x\Big) = \frac{1}{k!}\log(x)^k.$$

We introduce a general class of iterated integrals which we define recursively by

$$\mathsf{G}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), x\right) = \int_0^x f_1(\tau_1)\mathsf{G}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right) d\tau_1$$

with the special cases

$$\mathsf{G}\left(x\right) = 1$$

and

$$\mathsf{G}\Big(\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau}}_{k \text{ times}}, x\Big) = \frac{1}{k!}\log(x)^k.$$

Considering the letters

$$\frac{1}{\tau}, \frac{1}{1+\tau}, \frac{1}{1-\tau}$$

leads to harmonic polylogarithms.

We introduce a general class of iterated integrals which we define recursively by

$$\mathsf{G}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), x\right) = \int_0^x f_1(\tau_1)\mathsf{G}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right) d\tau_1$$

with the special cases

$$\mathsf{G}\left(x\right) = 1$$

and

$$\mathsf{G}\Big(\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau}}_{k \text{ times}}, x\Big) = \frac{1}{k!}\log(x)^k.$$

Typical letters that we consider are:

$$\frac{1}{\tau}, \frac{1}{1+\tau}, \frac{1}{\sqrt{\tau}\sqrt{1-\tau}}, \frac{1}{\sqrt{\tau}\sqrt{1+\tau}}, \sqrt{\tau}\sqrt{1-\tau}, \sqrt{1-\eta-\tau}\sqrt{\tau}$$

We introduce a general class of iterated integrals which we define recursively by

$$\mathsf{G}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), x\right) = \int_0^x f_1(\tau_1)\mathsf{G}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right) d\tau_1$$

with the special cases

$$\mathsf{G}\left(x\right) = 1$$

and

$$\mathsf{G}\Big(\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau}}_{k \text{ times}}, x\Big) = \frac{1}{k!}\log(x)^k.$$

Typical letters that we consider are:

$$\frac{1}{\tau}, \frac{1}{1+\tau}, \frac{1}{\sqrt{\tau}\sqrt{1-\tau}}, \frac{1}{\sqrt{\tau}\sqrt{1+\tau}}, \sqrt{\tau}\sqrt{1-\tau}, \sqrt{1-\eta-\tau}\sqrt{\tau}$$

- shuffle algebra

We introduce a general class of iterated integrals which we define recursively by

$$\mathsf{G}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), x\right) = \int_0^x f_1(\tau_1)\mathsf{G}\left(f_2(\tau), \cdots, f_k(\tau), \tau_1\right) d\tau_1$$

with the special cases

$$\mathsf{G}\left(x\right) = 1$$

and

$$\mathsf{G}\Big(\underbrace{\frac{1}{\tau}, \frac{1}{\tau}, \ldots, \frac{1}{\tau}}_{k \text{ times}}, x\Big) = \frac{1}{k!}\log(x)^k.$$

Typical letters that we consider are:

$$\frac{1}{\tau}, \frac{1}{1+\tau}, \frac{1}{\sqrt{\tau}\sqrt{1-\tau}}, \frac{1}{\sqrt{\tau}\sqrt{1+\tau}}, \sqrt{\tau}\sqrt{1-\tau}, \sqrt{1-\eta-\tau}\sqrt{\tau}$$

- shuffle algebra
- differentiation:

$$\frac{d}{dx}\mathsf{G}\left(f_1(\tau), f_2(\tau), \cdots, f_k(\tau), x\right) = f_1(x)\mathsf{G}\left(f_2(\tau), \cdots, f_k(\tau), x\right)$$

MultiIntegrate

We use the Almkvist Zeilberger algorithm to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(n; x_1, \ldots, x_d)$ is a hyperexponential function i.e.,

$$F(n; x_1, \ldots, x_d) = q(n; x_1, \ldots, x_d) \cdot e^{\frac{a(x_1, \ldots, x_d)}{b(x_1, \ldots, x_d)}} \cdot \left(\prod_{p=1}^{P} S_p(x_1, \ldots, x_d)^{\alpha_p} \right) \cdot \left(\frac{s(x_1, \ldots, x_d)}{t(x_1, \ldots, x_d)} \right)^n,$$

with

$$a(x_1, \ldots, x_d), b(x_1, \ldots, x_d), s(x_1, \ldots, x_d), t(x_1, \ldots, x_d), q(n; x_1, \ldots, x_d) \in \mathbb{K}[x_1, \ldots, x_d]$$

and $S_p(x_1, \ldots, x_d) \in \mathbb{K}[x_1, \ldots, x_d], \alpha_p \in \mathbb{K}$ for $1 \leq p \leq P$. With e.g., $\mathbb{K} = \mathbb{Q}(\varepsilon)$.

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(n; x_1, \ldots, x_d)$ is a hyperexponential function is:

- compute a recurrence for the integrand $F(n; x_1, \ldots, x_d)$

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(n; x_1, \ldots, x_d)$ is a hyperexponential function is:

- compute a recurrence for the integrand $F(n; x_1, \ldots, x_d)$
- use the reccurrence for the integrand to derive a recurrence for the integral

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

where $F(n; x_1, \ldots, x_d)$ is a hyperexponential function is:

- compute a recurrence for the integrand $F(n; x_1, \ldots, x_d)$
- use the reccurrence for the integrand to derive a recurrence for the integral
- solve the recurrence

$$F(n;x_1,\ldots,x_d) = q(n;x_1,\ldots,x_d)\cdot e^{\frac{a(x_1,\ldots,x_d)}{b(x_1,\ldots,x_d)}}\cdot\left(\prod_{p=1}^{P} S_p(x_1,\ldots,x_d)^{\alpha_p}\right)\cdot\left(\frac{s(x_1,\ldots,x_d)}{t(x_1,\ldots,x_d)}\right)^n,$$

where

$a(x_1,\ldots,x_d), b(x_1,\ldots,x_d), s(x_1,\ldots,x_d), t(x_1,\ldots,x_d), q(n;x_1,\ldots,x_d) \in \mathbb{K}[x_1,\ldots,x_d]$
and $S_p(x_1,\ldots,x_d) \in \mathbb{K}[x_1,\ldots,x_d], \alpha_p \in \mathbb{K}$ for $1 \le p \le P$. With e.g., $\mathbb{K} = \mathbb{Q}(\varepsilon)$.

$$F(n;x_1,\ldots,x_d) = q(n;x_1,\ldots,x_d)\cdot e^{\frac{a(x_1,\ldots,x_d)}{b(x_1,\ldots,x_d)}}\cdot\left(\prod_{p=1}^{P} S_p(x_1,\ldots,x_d)^{\alpha_p}\right)\cdot\left(\frac{s(x_1,\ldots,x_d)}{t(x_1,\ldots,x_d)}\right)^n,$$

where

$$a(x_1,\ldots,x_d), b(x_1,\ldots,x_d), s(x_1,\ldots,x_d), t(x_1,\ldots,x_d), q(n;x_1,\ldots,x_d) \in \mathbb{K}[x_1,\ldots,x_d]$$
and $S_p(x_1,\ldots,x_d) \in \mathbb{K}[x_1,\ldots,x_d], \alpha_p \in \mathbb{K}$ for $1 \le p \le P$. With e.g., $\mathbb{K} = \mathbb{Q}(\varepsilon)$.

Then there exist

$L \in \mathbb{N}, e_0(n), e_1(n), \ldots, e_L(n) \in \mathbb{K}[n]$, *not all zero*, and $R_i(n;x_1,\ldots,x_d) \in \mathbb{K}(n,x_1,\ldots,x_d)$

such that

$$G_i(n;x_1,\ldots,x_d) := R_i(n;x_1,\ldots,x_d)F(n;x_1,\ldots,x_d)$$

satisfy

$$\sum_{i=0}^{L} e_i(n)F(n+i;x_1,\ldots,x_d) = \sum_{i=1}^{d} D_{x_i}G_i(n;x_1,\ldots,x_d).$$

In[1]:= **<< Sigma.m**

In[2]:= **<< HarmonicSums.m**

In[3]:= **<< MultiIntegrate.m**

Sigma - A summation package by Carsten Schneider -RISC Linz- V 1.0 (7/7/11)

HarmonicSums by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

MultiIntegrate by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

In[4]:= **mAZ[** $\frac{1}{(xy)^{n+1}(1-x-y+xy)^z}$ **, n, {x, y}, f]**

Out[4]=

$$\left\{\left\{1, \frac{(-1+x)^{-z}(-1+y)^{-z}\left(\frac{1}{xy}\right)^n}{xy}\right\}, \left\{\frac{(-1+x)^{-z}(-1+y)^{-z}(xy)^{-2-n}}{xy}, \left\{\{nx^2y - nxy + x^2yz + x^2y - xyz - xy, x\}, \{ny^2 - ny + 2y^2 - 2y, y\}\right\}, \left(-1-2n-n^2-2z-2nz-z^2\right) f(1+n) + \left(4+4n+n^2\right) f(2+n)\right\}\right\}$$

We now consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

Where for $F(n; x_1, \ldots, x_d)$ we have

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d).$$

We now consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

Where for $F(n; x_1, \ldots, x_d)$ we have

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d).$$

If

$$F(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) \quad = \quad F(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0,$$

we also have

$$G_i(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) \quad = \quad G_i(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0$$

We now consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

Where for $F(n; x_1, \ldots, x_d)$ we have

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d).$$

If

$$F(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) \quad = \quad F(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0,$$

we also have

$$G_i(n; \ldots, x_{i-1}, u_i, x_{i+1}, \ldots) \quad = \quad G_i(n; \ldots, x_{i-1}, o_i, x_{i+1}, \ldots) = 0$$

and hence $a(n)$ satisfies the homogenous linear recurrence

$$\sum_{i=0}^{L} e_i(n) a(n+i) = 0.$$

In[5]:= **mAZIntegrate[**$\frac{(a-x)(-1+x)(-2+y)(y-1)}{x^m y^n}$**, n, {{x, a, 1}, {y, 1, 2}}]**

Out[5]=
$$\frac{-5+15a+5m-5am+n-3an-mn+amn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+$$
$$\frac{2^{2-n}(1-3a-m+am+n-3an-mn+amn)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+$$
$$a^{-m}\left(\frac{-15a^2+5a^3+5a^2m-5a^3m+3a^2n-a^3n-a^2mn+a^3mn}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}+\right.$$
$$\left.\frac{2^{2-n}\left(3a^2-a^3-a^2m+a^3m+3a^2n-a^3n-a^2mn+a^3mn\right)}{(-3+m)(-2+m)(-1+m)(-3+n)(-2+n)(-1+n)}\right)$$

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \ldots, x_d)$ does not have to vanish at the bounds

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \ldots, x_d)$ does not have to vanish at the bounds
- then force the G_i to vanish at the integration bounds by modifying the ansatz, and look for G_i of the form

$$G_i(n; x_1, \ldots, x_d) = \overline{G}_i(n; x_1, \ldots, x_d)(x_i - u_i)(x_i - o_i),$$

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d,$$

- suppose $F(n; x_1, \ldots, x_d)$ does not vanish at the bounds
- $G_i(n; x_1, \ldots, x_d)$ does not have to vanish at the bounds
- then force the G_i to vanish at the integration bounds by modifying the ansatz, and look for G_i of the form

$$G_i(n; x_1, \ldots, x_d) = \overline{G}_i(n; x_1, \ldots, x_d)(x_i - u_i)(x_i - o_i),$$

- hence $a(n)$ satisfies again a homogenous linear recurrence of the form

$$\sum_{i=0}^{L} e_i(n) a(n+i) = 0.$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds.

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds. Using our ansatz we find

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)F(n) - (n+2)(5\varepsilon - 5n - 13)F(n+1) \\
&+(n+3)(4\varepsilon - 4n - 13)F(n+2) - (n+4)(\varepsilon - n - 4)F(n+3) \\
&= D_{x_1} F(n)(x_1 - 1)\, x_1\, (x_1 + 1)\, x_2\, ((n+3)x_1 x_2 + 2) \\
&+D_{x_2} F(n)(x_2 - 1)\, x_2 \left(x_2^2 x_1^3(-\varepsilon + n + 4) - (\varepsilon - 3)x_2 x_1^2 + (n+2)x_2 x_1 + 1\right).
\end{aligned}$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds. Using our ansatz we find

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)F(n) - (n+2)(5\varepsilon - 5n - 13)F(n+1) \\
&+(n+3)(4\varepsilon - 4n - 13)F(n+2) - (n+4)(\varepsilon - n - 4)F(n+3) \\
&= D_{x_1} F(n)(x_1 - 1)\, x_1\, (x_1 + 1)\, x_2\, ((n+3)x_1 x_2 + 2) \\
&+D_{x_2} F(n)(x_2 - 1)\, x_2 \left(x_2^2 x_1^3(-\varepsilon + n + 4) - (\varepsilon - 3)x_2 x_1^2 + (n+2)x_2 x_1 + 1\right).
\end{aligned}$$

Integration of this reccurence yields

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)I(\varepsilon, n) - (n+2)(5\varepsilon - 5n - 13)I(\varepsilon, n+1) \\
&+(n+3)(4\varepsilon - 4n - 13)I(\varepsilon, n+2) - (n+4)(\varepsilon - n - 4)I(\varepsilon, n+3) = 0.
\end{aligned}$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds. Using our ansatz we find

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)F(n) - (n+2)(5\varepsilon - 5n - 13)F(n+1) \\
&+(n+3)(4\varepsilon - 4n - 13)F(n+2) - (n+4)(\varepsilon - n - 4)F(n+3) \\
&= D_{x_1} F(n)(x_1 - 1)\, x_1\, (x_1 + 1)\, x_2\, ((n+3)x_1 x_2 + 2) \\
&+D_{x_2} F(n)(x_2 - 1)\, x_2 \left(x_2^2 x_1^3(-\varepsilon + n + 4) - (\varepsilon - 3)x_2 x_1^2 + (n+2)x_2 x_1 + 1\right).
\end{aligned}$$

Integration of this reccurence yields

$$\begin{aligned}
&2(n+1)(\varepsilon - n - 2)I(\varepsilon, n) - (n+2)(5\varepsilon - 5n - 13)I(\varepsilon, n+1) \\
&+(n+3)(4\varepsilon - 4n - 13)I(\varepsilon, n+2) - (n+4)(\varepsilon - n - 4)I(\varepsilon, n+3) = 0.
\end{aligned}$$

Solving the recurrence leads to

$$I(\varepsilon, n) = \frac{1}{n+1}\left(\sum_{i=1}^{n} \frac{1}{-i + \varepsilon - 1} - 2^{1-\varepsilon} \sum_{i=1}^{n} \frac{2^i}{-i + \varepsilon - 1} + \frac{2^{-\varepsilon}\,(2^\varepsilon - 2)}{\varepsilon - 1}\right)$$

In[6]:= **mAZDirectIntegrate[$\frac{(1+x1*x2)^n}{(1+x1)^\varepsilon}$, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[6]= $\frac{\sum_{\iota_1=1}^{n} \frac{1}{\iota_1-\varepsilon+1}}{-n-1} - \frac{2\sum_{\iota_1=1}^{n} \frac{2^{\iota_1}}{-\iota_1+\varepsilon-1}}{(n+1)2^\varepsilon} + \frac{2^\varepsilon-2}{(n+1)(\varepsilon-1)2^\varepsilon}$

In[7]:= **mAZDirectIntegrate[$\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[7]= $\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$

In[8]:= **mAZDirectIntegrate[$\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$, n, {{x1, 0, $\frac{1}{2}$}, {x2, 0, 2}}]**

Out[8]= $\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$

We look again at the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

We look again at the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

By integration with respect to $x_1, \ldots, x_d$ we get

$$\begin{aligned} \sum_{i=0}^{L} e_i(n) a(n+i) &= \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \\ &\quad - \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \end{aligned}$$

with

$$\begin{aligned} O_i(n) &:= G_i(n; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d) \\ U_i(n) &:= G_i(n; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d). \end{aligned}$$

We look again at the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

Suppose that we found

$$\sum_{i=0}^{L} e_i(n) F(n+i; x_1, \ldots, x_d) = \sum_{i=1}^{d} D_{x_i} G_i(n; x_1, \ldots, x_d)$$

By integration with respect to $x_1, \ldots, x_d$ we get

$$\begin{aligned} \sum_{i=0}^{L} e_i(n) a(n+i) &= \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \\ &\quad - \sum_{i=1}^{d} \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \end{aligned}$$

with

$$\begin{aligned} O_i(n) &:= G_i(n; x_1, \ldots, x_{i-1}, o_i, x_{i+1} \ldots, x_d) \\ U_i(n) &:= G_i(n; x_1, \ldots, x_{i-1}, u_i, x_{i+1} \ldots, x_d). \end{aligned}$$

Note that there are $2 \cdot d$ integrals of dimension $d-1$, to compute.

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

In the next step apply the algorithm to $I_1(n)$; we find

$$I_1(\varepsilon, n) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

In the next step apply the algorithm to $I_1(n)$; we find

$$I_1(\varepsilon, n) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

Plugging in yields

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

In the next step apply the algorithm to $I_1(n)$; we find

$$I_1(\varepsilon, n) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

Plugging in yields

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon (n + 2 - \varepsilon)}.$$

Solving this recurrence yields

$$I(\varepsilon, n) = \frac{1}{n+1} \left(\sum_{i=1}^{n} \frac{1}{-i + \varepsilon - 1} - 2^{1-\varepsilon} \sum_{i=1}^{n} \frac{2^i}{-i + \varepsilon - 1} + \frac{2^{-\varepsilon} (2^\varepsilon - 2)}{\varepsilon - 1} \right).$$

In[9]:= **mAZIntegrate[**$\frac{(1+x1*x2)^n}{(1+x1)^\varepsilon}$**, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[9]= $\frac{\sum_{\iota_1=1}^{n} \frac{1}{\iota_1-\varepsilon+1}}{-n-1} - \frac{2\sum_{\iota_1=1}^{n} \frac{2^{\iota_1}}{-\iota_1+\varepsilon-1}}{(n+1)2^\varepsilon} + \frac{2^\varepsilon-2}{(n+1)(\varepsilon-1)2^\varepsilon}$

In[10]:= **mAZIntegrate[**$\frac{(1+\mathbf{x1}+\mathbf{x2}+\mathbf{x1}*\mathbf{x2})^n}{(1+\mathbf{x1})^\varepsilon}$**, n, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[10]= $\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$

In[11]:= **mAZIntegrate[**$\frac{(1+\mathbf{x1}+\mathbf{x2}+\mathbf{x1}*\mathbf{x2})^n}{(1+\mathbf{x1})^\varepsilon}$**, n, {{x1, 0,** $\frac{\mathbf{1}}{\mathbf{2}}$**}, {x2, 0, 2}}]**

Out[11]= $\frac{2^{-2n-\varepsilon}\left(208-83\ 2^{2+n}+63\ 2^{1+2n}-2^{4+\varepsilon}+7\ 2^{2+n+\varepsilon}-13\ 2^{2n+\varepsilon}\right)}{(1+n)(1+n-\varepsilon)}$

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{l=-L}^{\infty} \varepsilon^l I_l(n).$$

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{l=-L}^{\infty} \varepsilon^l I_l(n).$$

- find $I_{-L}(n), I_{-L+1}(n), \ldots, I_u(n)$ in terms of indefinite nested product-sum expressions.

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{l=-L}^{\infty} \varepsilon^l I_l(n).$$

- find $I_{-L}(n), I_{-L+1}(n), \ldots, I_u(n)$ in terms of indefinite nested product-sum expressions.
- compute a recurrence for $\mathcal{I}(\varepsilon, n)$ in the form

$$a_0(\varepsilon, n)T(\varepsilon, n) + \cdots + a_d(\varepsilon, n)T(\varepsilon, n+d) = h_0(n) + \cdots + h_u(n)\varepsilon^u + O(\varepsilon^{u+1});$$

use one of the methods presented above

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{l=-L}^{\infty} \varepsilon^l I_l(n).$$

- find $I_{-L}(n), I_{-L+1}(n), \ldots, I_u(n)$ in terms of indefinite nested product-sum expressions.
- compute a recurrence for $\mathcal{I}(\varepsilon, n)$ in the form

$$a_0(\varepsilon, n)T(\varepsilon, n) + \cdots + a_d(\varepsilon, n)T(\varepsilon, n+d) = h_0(n) + \cdots + h_u(n)\varepsilon^u + O(\varepsilon^{u+1});$$

  use one of the methods presented above
- use algorithm FLSR implemented in `Sigma`.

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n;x_1,x_2):=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2 (x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

In the next step apply the method to $I_1(n)$; we find

$$\begin{aligned} I_1(\varepsilon, n) \quad = \quad & \frac{2^{n+2} - 1}{n+2} + \frac{\varepsilon \left(-2^{n+2} \left(\mathsf{H}_{-1}(1)(n+2) - 1\right) - 1\right)}{(n+2)^2} \\ & + \frac{\varepsilon^2 \left(2^{n+1} \left(\mathsf{H}_{-1}(1)^2 (n+2)^2 - 2\mathsf{H}_{-1}(1)(n+2) + 2\right) - 1\right)}{(n+2)^3} + O\left(\varepsilon^3\right). \end{aligned}$$

Plugging in and solving the resulting recurrence by means of FLSR and combining the solution with the initial values of the integral yields the result:

$$
\begin{aligned}
I(\varepsilon,n) \quad = \quad & \frac{\mathsf{S}_1(2;n)}{n+1} - \frac{\mathsf{S}_1(n)}{n+1} + \frac{2(n+1)^3 + 4\left(-n+2^n-1\right)(n+1)^2 + 2n(n+1)^2}{2(n+1)^4} \\
& + \varepsilon \left(\mathsf{H}_{-1}(1) \left(\frac{-2^{n+2}(n+1) - 2^{n+2}n(n+1)}{2(n+1)^4} - \frac{\mathsf{S}_1(2;n)}{n+1} \right) \right. \\
& + \frac{\left(2(n+1)n^2 + 4(n+1)n + 2(n+1)\right)\mathsf{S}_2(2;n)}{2(n+1)^4} - \frac{\mathsf{S}_2(n)}{n+1} \\
& \left. + \frac{2^{n+2}(n+1) - 2(n+1)}{2(n+1)^4} \right) + \varepsilon^2 \left(\mathsf{H}_{-1}(1)^2 \left(\frac{\mathsf{S}_1(2;n)}{2(n+1)} + \frac{2^{n+1}(n^2+2n+1)}{2(n+1)^4} \right) \right. \\
& + \mathsf{H}_{-1}(1) \left(\frac{-2^{n+2}n - 2^{n+2}}{2(n+1)^4} - \frac{\mathsf{S}_2(2;n)}{n+1} \right) + \frac{\mathsf{S}_3(2;n)}{n+1} - \frac{\mathsf{S}_3(n)}{n+1} + \frac{2^{n+2}-2}{2(n+1)^4} \Bigg) \\
& + O\left(\varepsilon^3\right).
\end{aligned}
$$

In[12]:= **mAZExpandedIntegrate[**$\frac{(1+x1*x2)^n}{(1+x1)^\varepsilon}$**, n, { ε, 0, 2}, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[12]=

$$\left\{\left\{\frac{S[1,\{2\},n]}{n+1}-\frac{S[1,n]}{n+1}+\frac{2^{n+1}-1}{(n+1)^2},-\frac{\log(2)S[1,\{2\},n]}{n+1}+\frac{S[2,\{2\},n]}{n+1}-\frac{S[2,n]}{n+1}+\frac{\log(2)\left(-2^{n+1}\right)(n+1)+2^{n+1}-1}{(n+1)^3},\frac{\log(2)^2S[1,\{2\},n]}{2n+2}-\frac{\log(2)S[2,\{2\},n]}{n+1}+\frac{S[3,\{2\},n]}{n+1}-\frac{S[3,n]}{n+1}+\frac{\log(2)^2 2^n(n+1)^2-\log(2)2^{n+1}(n+1)+2^{n+1}-1}{(n+1)^4}\right\},0,2\right\}$$

In[13]:= **mAZExpandedIntegrate[**$\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$**, n, { ε, 0, 1}, {{x1, 0, 1}, {x2, 0, 1}}]**

Out[13]=

$$\left\{\left\{\frac{\left(2^{n+1}-1\right)^2}{(n+1)^2},-\frac{\left(2^{n+1}-1\right)\left(-2^{n+1}+2^{n+1}n\log(2)+2^{n+1}\log(2)+1\right)}{(n+1)^3}\right\},0,1\right\}$$

In[14]:= **mAZExpandedIntegrate[**$\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$**, n, { ε, 0, 1}, {{x1, 0, $\frac{1}{2}$}, {x2, 0, 2}}]**

Out[14]=

$$\left\{\left\{-\frac{2^{-n-1}\left(2^{n+1}-3^{n+1}\right)\left(3^{n+1}-1\right)}{(n+1)^2},-\frac{2^{-n-1}\left(3^{n+1}-1\right)\left(2^{n+1}-3^{n+1}+3^{n+1}n\log\left(\frac{3}{2}\right)+3^{n+1}\log\left(\frac{3}{2}\right)\right)}{(n+1)^3}\right\},0,1\right\}$$

The following integral occurs in the direct computation of a 3-loop diagram of the ladder-type:

$$\int_0^1 \int_0^1 \left(\frac{(s(x-1)+t(u-1)+1)^n}{(w-1)(z-1)(sx-s+tu-t-u+1)(sx-s+tu-t-x+1)} \right.$$
$$+ \frac{1}{(z-1)(sx-s+tu-t-x+1)} \frac{(z(-s+tu-t+1)+x((s-1)z+1))^n}{-swx+sw+sxz-sz-tuw+tuz+tw-tz+uw-u-w-xz+x+z}$$
$$\left. + \frac{1}{(w-1)(sx-s+tu-t-u+1)} \frac{(u((t-1)w+1)-w(s(-x)+s+t-1))^n}{swx-sw-sxz+sz+tuw-tuz-tw+tz-uw+u+w+xz-x-z} \right) dudx$$

The following integral occurs in the direct computation of a 3-loop diagram of the ladder-type:

$$\int_0^1\int_0^1\left(\frac{(s(x-1)+t(u-1)+1)^n}{(w-1)(z-1)(sx-s+tu-t-u+1)(sx-s+tu-t-x+1)}\right.$$
$$+\frac{1}{(z-1)(sx-s+tu-t-x+1)}\frac{(z(-s+tu-t+1)+x((s-1)z+1))^n}{-swx+sw+sxz-sz-tuw+tuz+tw-tz+uw-u-w-xz+x+z}$$
$$\left.+\frac{1}{(w-1)(sx-s+tu-t-u+1)}\frac{(u((t-1)w+1)-w(s(-x)+s+t-1))^n}{swx-sw-sxz+sz+tuw-tuz-tw+tz-uw+u+w+xz-x-z}\right)dudx$$

$$=\frac{1}{(w-1)(z-1)(s+t-1)}\left(S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{sw-sz+z-1}{z-1};n\right)-S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{z(sw-sz+z-1)}{w(z-1)};n\right)\right.$$
$$-S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{(t-1)(sw-sz+z-1)}{(s+t-1)(z-1)};n\right)+S_{1,1}\left(\frac{s+t-1}{t-1},(1-t)w;n\right)+S_{1,1}\left(\frac{s+t-1}{t-1},-\frac{(s-1)(t-1)}{s+t-1};n\right)$$
$$+S_{1,1}\left(-\frac{(s+t-1)w(z-1)}{sw-sz+z-1},\frac{(sw-sz+z-1)(tz-1)}{(s+t-1)w(z-1)};n\right)-S_{1,1}\left(\frac{s+t-1}{t-1},1-t;n\right)+S_{1,1}\left(\frac{s+t-1}{s-1},-\frac{(s-1)(t-1)}{s+t-1};n\right)$$
$$+S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},\frac{-tw+w+tz-1}{w-1};n\right)-S_{1,1}\left(\frac{s+t-1}{t-1},-\frac{(t-1)(sw-1)}{s+t-1};n\right)+S_{1,1}\left(\frac{s+t-1}{s-1},(1-s)z;n\right)$$
$$-S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},\frac{(s-1)(-tw+w+tz-1)}{(s+t-1)(w-1)};n\right)-S_{1,1}\left(\frac{s+t-1}{s-1},1-s;n\right)-S_{1,1}\left(\frac{s+t-1}{s-1},-\frac{(s-1)(tz-1)}{s+t-1};n\right)$$
$$-S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},-\frac{w((t-1)w-tz+1)}{(w-1)z};n\right)+S_{1,1}\left(\frac{(s+t-1)(w-1)z}{(t-1)w-tz+1},-\frac{(sw-1)((t-1)w-tz+1)}{(s+t-1)(w-1)z};n\right)$$
$$-S_{1,1}(1,(1-s)z;n)+S_{1,1}(1,z-sz;n)+S_2((1-s)z;n)-S_{1,1}(1,(1-t)w;n)+S_{1,1}(1,w-tw;n)-S_2((-s-t+1)w;n)$$
$$\left.-S_2((-s-t+1)z;n)+2S_2(-s-t+1;n)-S_2(1-s;n)+S_2((1-t)w;n)-S_2(1-t;n)\right)$$

Holonomic Functions and Sequences

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).
- A sequence $(f_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* if there exist $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all $p_i = 0$) such that

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0.$$

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).
- A sequence $(f_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* if there exist $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all $p_i = 0$) such that

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0.$$

- A function $f = f(x)$ is called *holonomic* if there exist $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all $p_i = 0$) such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).
- A sequence $(f_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* if there exist $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all $p_i = 0$) such that

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0.$$

- A function $f = f(x)$ is called *holonomic* if there exist $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all $p_i = 0$) such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- The generating function of a holonomic sequence $(f_n)_{n\geq 0}$ is holonomic.

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).
- A sequence $(f_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* if there exist $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all $p_i = 0$) such that

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0.$$

- A function $f = f(x)$ is called *holonomic* if there exist $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all $p_i = 0$) such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- The generating function of a holonomic sequence $(f_n)_{n\geq 0}$ is holonomic.
- If $f(x)$ is holonomic, then the coefficients f_n of the formal power series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

form a holonomic sequence.

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).
- A sequence $(f_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* if there exist $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all $p_i = 0$) such that

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0.$$

- A function $f = f(x)$ is called *holonomic* if there exist $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all $p_i = 0$) such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- The generating function of a holonomic sequence $(f_n)_{n\geq 0}$ is holonomic.
- If $f(x)$ is holonomic, then the coefficients f_n of the formal power series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

form a holonomic sequence.

- The nested sums we are considering are holonomic.

- Let $\mathbb{K}$ be a field of characteristic 0 (i.e., $\mathbb{K}$ contains $\mathbb{Q}$).
- A sequence $(f_n)_{n\geq 0} \in \mathbb{K}^{\mathbb{N}}$ is called *holonomic* if there exist $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all $p_i = 0$) such that

$$p_d(n) f_{n+d} + \cdots + p_1(n) f_{n+1} + p_0(n) f_n = 0.$$

- A function $f = f(x)$ is called *holonomic* if there exist $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all $p_i = 0$) such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- The generating function of a holonomic sequence $(f_n)_{n\geq 0}$ is holonomic.
- If $f(x)$ is holonomic, then the coefficients f_n of the formal power series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

form a holonomic sequence.

- The nested sums we are considering are holonomic.
- The iterated integrals we are considering are holonomic.

- Assume that $f(x) = \sum_{n \geq 0} f_n x^n$ is holonomic such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- Assume that $f(x) = \sum_{n \geq 0} f_n x^n$ is holonomic such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- It is easy to check that

$$x^k f^{(j)}(x) = \sum_{n \geq 0} \prod_{i=1}^{j} (n + i - k) f_{n+j-k} x^n \tag{1}$$

- Assume that $f(x) = \sum_{n \geq 0} f_n x^n$ is holonomic such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- It is easy to check that

$$x^k f^{(j)}(x) = \sum_{n \geq 0} \prod_{i=1}^{j} (n + i - k) f_{n+j-k} x^n \tag{1}$$

- Transform the differential equation according to this relation.

- Assume that $f(x) = \sum_{n \geq 0} f_n x^n$ is holonomic such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- It is easy to check that

$$x^k f^{(j)}(x) = \sum_{n \geq 0} \prod_{i=1}^{j} (n + i - k) f_{n+j-k} x^n \tag{1}$$

- Transform the differential equation according to this relation.
- Equate coefficients of same powers of x on both sides.

- Assume that $f(x) = \sum_{n \geq 0} f_n x^n$ is holonomic such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

- It is easy to check that

$$x^k f^{(j)}(x) = \sum_{n \geq 0} \prod_{i=1}^{j} (n + i - k) f_{n+j-k} x^n \tag{1}$$

- Transform the differential equation according to this relation.
- Equate coefficients of same powers of x on both sides.
- We get a linear recurrence equation with polynomial coefficients, satisfied by $(f_n)_{n \geq 0}$.

Note: The reverse direction can be performed as well.

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic differential equation for $f(x)$.

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the idea above to compute a holonomic recurrence for $(f_n)_{n\geq 0}$.

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the idea above to compute a holonomic recurrence for $(f_n)_{n\geq 0}$.
3. Compute initial values for the recurrence.

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the idea above to compute a holonomic recurrence for $(f_n)_{n\geq 0}$.
3. Compute initial values for the recurrence.
4. Solve the recurrence to get a closed form representation for $(f_n)_{n\geq 0}$.

Given a holonomic function $f(x)$.
Find a sequence f_n given as a linear combination of indefinite nested sums such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the idea above to compute a holonomic recurrence for $(f_n)_{n\geq 0}$.
3. Compute initial values for the recurrence.
4. Solve the recurrence to get a closed form representation for $(f_n)_{n\geq 0}$.

Note: This is implemented in `HarmonicSums` using the recurrence solver of `Sigma`.

Consider

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} f_n x^n.$$

Consider

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} f_n x^n.$$

We can derive the differential equation:

$$(x+1)(x-1)f'''(x) + (3x-1)f''(x) + f'(x) = 0.$$

Consider

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} f_n x^n.$$

We can derive the differential equation:

$$(x+1)(x-1)f'''(x) + (3x-1)f''(x) + f'(x) = 0.$$

Expanding leads to

$$x^2 f'''(x) - f'''(x) + 3xf''(x) - f''(x) + f'(x) = 0.$$

Consider

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} f_n x^n.$$

We can derive the differential equation:

$$(x+1)(x-1)f'''(x) + (3x-1)f''(x) + f'(x) = 0.$$

Expanding leads to

$$x^2 f'''(x) - f'''(x) + 3xf''(x) - f''(x) + f'(x) = 0.$$

Using (1) results in:

$$\sum_{n=0}^{\infty}(n+1)f_{n+1}x^n + 3\sum_{n=0}^{\infty} n(n+1)f_{n+1}x^n + \sum_{n=0}^{\infty}(n-1)n(n+1)f_{n+1}x^n$$
$$-\sum_{n=0}^{\infty}(n+1)(n+2)f_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)f_{n+3}x^n = 0.$$

Consider

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} f_n x^n.$$

We can derive the differential equation:

$$(x+1)(x-1)f'''(x) + (3x-1)f''(x) + f'(x) = 0.$$

Expanding leads to

$$x^2 f'''(x) - f'''(x) + 3xf''(x) - f''(x) + f'(x) = 0.$$

Using (1) results in:

$$\sum_{n=0}^{\infty}(n+1)f_{n+1}x^n + 3\sum_{n=0}^{\infty} n(n+1)f_{n+1}x^n + \sum_{n=0}^{\infty}(n-1)n(n+1)f_{n+1}x^n$$
$$-\sum_{n=0}^{\infty}(n+1)(n+2)f_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)(n+2)(n+3)f_{n+3}x^n = 0.$$

Hence

$$(n+1)^3 f_{n+1} - (n+2)(n+1)f_{n+2} - (n+2)(n+3)(n+1)f_{n+3} = 0$$

holds for $(f_n)_{n>0}$.

Initial values can be computed easily:

$$f_1 = 0,\ f_2 = \frac{1}{2},\ f_3 = -\frac{1}{6}.$$

Initial values can be computed easily:

$$f_1 = 0,\ f_2 = \frac{1}{2},\ f_3 = -\frac{1}{6}.$$

Solving (using the recurrence solver of `Sigma`)

$$(n+1)^3 f_{n+1} - (n+2)(n+1) f_{n+2} - (n+2)(n+3)(n+1) f_{n+3} = 0$$

leads to

$$f_n = \frac{(-1)^n \sum_{i=2}^{n} \frac{(-1)^i}{i-1}}{n}.$$

Initial values can be computed easily:

$$f_1 = 0,\ f_2 = \frac{1}{2},\ f_3 = -\frac{1}{6}.$$

Solving (using the recurrence solver of `Sigma`)

$$(n+1)^3 f_{n+1} - (n+2)(n+1) f_{n+2} - (n+2)(n+3)(n+1) f_{n+3} = 0$$

leads to

$$f_n = \frac{(-1)^n \sum_{i=2}^{n} \frac{(-1)^i}{i-1}}{n}.$$

Hence

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} \frac{(-1)^n \sum_{i=2}^{n} \frac{(-1)^i}{i-1}}{n} x^n.$$

Initial values can be computed easily:

$$f_1 = 0,\ f_2 = \frac{1}{2},\ f_3 = -\frac{1}{6}.$$

Solving (using the recurrence solver of `Sigma`)

$$(n+1)^3 f_{n+1} - (n+2)(n+1) f_{n+2} - (n+2)(n+3)(n+1) f_{n+3} = 0$$

leads to

$$f_n = \frac{(-1)^n \sum_{i=2}^{n} \frac{(-1)^i}{i-1}}{n}.$$

Hence

$$f(x) = \mathsf{G}\left(\frac{1}{1+\tau}, \frac{1}{1-\tau}, x\right) = \sum_{n>0} \frac{(-1)^n \sum_{i=2}^{n} \frac{(-1)^i}{i-1}}{n} x^n.$$

Note: This is implemented in `HarmonicSums`.

Given a holonomic sequence $(f_n)_{n\geq 0}$.
Find a function $f(x)$ given as a linear combination of iterated integrals such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Given a holonomic sequence $(f_n)_{n\geq 0}$.
Find a function $f(x)$ given as a linear combination of iterated integrals such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic recurrence equation for $(f_n)_{n\geq 0}$.

Given a holonomic sequence $(f_n)_{n\geq 0}$.
Find a function $f(x)$ given as a linear combination of iterated integrals such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic recurrence equation for $(f_n)_{n\geq 0}$.
2. Compute a holonomic differential for $f(x)$.

Given a holonomic sequence $(f_n)_{n\geq 0}$.
Find a function $f(x)$ given as a linear combination of iterated integrals such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic recurrence equation for $(f_n)_{n\geq 0}$.
2. Compute a holonomic differential for $f(x)$.
3. Compute initial values for the differential equation.

Given a holonomic sequence $(f_n)_{n\geq 0}$.
Find a function $f(x)$ given as a linear combination of iterated integrals such that we have

$$f(x) = \sum_{n=0}^{\infty} x^n f_n.$$

Method:

1. Compute a holonomic recurrence equation for $(f_n)_{n\geq 0}$.
2. Compute a holonomic differential for $f(x)$.
3. Compute initial values for the differential equation.
4. Solve the differential equ. to get a closed form representation for $f(x)$.

For example given

$$f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i^2 \binom{2i}{i}}$$

we find:

$$\begin{aligned} f(x) &= \frac{1}{32}\left(16x - 20x^2 + 8x^3 - x^4\right) + \frac{x-2}{8}\sqrt{(4-x)x}\,\mathsf{G}\left(\sqrt{4-\tau}\sqrt{\tau}, x\right) \\ &\quad + \frac{1}{8}\mathsf{G}\left(\sqrt{4-\tau}\sqrt{\tau}, x\right)^2 \end{aligned}$$

Mellin transform of holonomic functions

Let $f(x)$ be a holonomic function such that

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx$$

exists and let $p_i(x) \in \mathbb{K}[x]$ such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

Let $f(x)$ be a holonomic function such that

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx$$

exists and let $p_i(x) \in \mathbb{K}[x]$ such that

$$p_d(x) f^{(d)}(x) + \cdots + p_1(x) f'(x) + p_0(x) f(x) = 0.$$

Since we have

$$\begin{aligned} \mathbf{M}[x^m f^{(p)}(x)](n) &= \frac{(-1)^p (n+m)!}{(n+m-p)!} \mathbf{M}[f(x)](n+m-p) \\ &\quad + \sum_{i=0}^{p-1} \frac{(-1)^i (n+m)!}{(n+m-i)!} f^{(p-1-i)}(1), \end{aligned} \tag{2}$$

Let $f(x)$ be a holonomic function such that

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x)dx$$

exists and let $p_i(x) \in \mathbb{K}[x]$ such that

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0.$$

Since we have

$$\begin{aligned}\mathbf{M}[x^m f^{(p)}(x)](n) &= \frac{(-1)^p(n+m)!}{(n+m-p)!}\mathbf{M}[f(x)](n+m-p) \\ &\quad + \sum_{i=0}^{p-1} \frac{(-1)^i(n+m)!}{(n+m-i)!} f^{(p-1-i)}(1), \qquad (2)\end{aligned}$$

we can conclude:

If the Mellin transform of a holonomic function is defined i.e., the integral $\int_0^1 x^n f(x)dx$ *exists, then it is holonomic.*

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

Method:

1. Compute a holonomic differential equation for $f(x)$.

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the proposition above to compute a holonomic recurrence for $\mathbf{M}[f(x)](n)$.

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the proposition above to compute a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
3. Compute initial values for the recurrence.

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the proposition above to compute a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
3. Compute initial values for the recurrence.
4. Solve the recurrence to get a closed form representation for $\mathbf{M}[f(x)](n)$.

Given a holonomic function $f(x)$.
Find an expression $F(n)$ given as a linear combination of indefinite nested sums such that for all $n \in \mathbb{N}$ (from a certain point on) we have

$$\mathbf{M}[f(x)](n) := \int_0^1 x^n f(x) dx = F(n).$$

Method:

1. Compute a holonomic differential equation for $f(x)$.
2. Use the proposition above to compute a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
3. Compute initial values for the recurrence.
4. Solve the recurrence to get a closed form representation for $\mathbf{M}[f(x)](n)$.

Note: This is implemented in `HarmonicSums` using the recurrence solver of `Sigma`.

We want to compute the Mellin transform of

$$f(x) := \mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, x\right).$$

We want to compute the Mellin transform of

$$f(x) := \mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, x\right).$$

We find that

$$(-3+x)f(x)' + 2(-1+x)(1+x)f(x)'' = 0$$

We want to compute the Mellin transform of

$$f(x) := \mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, x\right).$$

We find that

$$(-3+x)f(x)' + 2(-1+x)(1+x)f(x)'' = 0$$

which leads to the recurrence

$$\begin{aligned} 6\int_0^1 \frac{\sqrt{1-\tau}}{1+\tau}\,d\tau \quad &= \quad -2(n-1)n\,\mathbf{M}[f(x)](n-2) + 3n\,\mathbf{M}[f(x)](n-1) \\ &\quad +(n+1)(2n+3)\,\mathbf{M}[f(x)](n). \end{aligned}$$

We want to compute the Mellin transform of

$$f(x) := \mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, x\right).$$

We find that

$$(-3+x)f(x)' + 2(-1+x)(1+x)f(x)'' = 0$$

which leads to the recurrence

$$\begin{aligned} 6\int_0^1 \frac{\sqrt{1-\tau}}{1+\tau}\,d\tau \quad &= \quad -2(n-1)n\,\mathbf{M}[f(x)](n-2) + 3n\,\mathbf{M}[f(x)](n-1) \\ &\quad +(n+1)(2n+3)\,\mathbf{M}[f(x)](n). \end{aligned}$$

Initial values can be computed easily

We want to compute the Mellin transform of

$$f(x) := \mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, x\right).$$

We find that

$$(-3+x)f(x)' + 2(-1+x)(1+x)f(x)'' = 0$$

which leads to the recurrence

$$\begin{aligned} 6\int_0^1 \frac{\sqrt{1-\tau}}{1+\tau}\,d\tau &= -2(n-1)n\,\mathbf{M}[f(x)](n-2) + 3n\,\mathbf{M}[f(x)](n-1) \\ &\quad +(n+1)(2n+3)\,\mathbf{M}[f(x)](n). \end{aligned}$$

Initial values can be computed easily and solving the recurrence leads to

$$\begin{aligned} \mathbf{M}[f(x)](n) &= (-1)^n\left(\frac{4^{n+1}}{(2n+1)(2n+3)\binom{2n}{n}} + \frac{\mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, 1\right) - 2}{n+1}\right) \\ &\quad - \frac{4(-1)^n\sum_{i=1}^{n}\frac{4^i}{(2i+1)\binom{2i}{i}}}{n+1} + \frac{\mathsf{G}\left(\frac{\sqrt{1-\tau}}{1+\tau}, 1\right)}{n+1}. \end{aligned}$$

Inverse Mellin transform

Given a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
Find a differential equation for $f(x)$.

Given a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
Find a differential equation for $f(x)$.

We observe that

$$\begin{aligned} n^p\,\mathbf{M}[f(x)](n+m) \quad &= \quad \mathbf{M}[(-1)^p x^{m+p} f^{(p)}(x)](n) - a(n)\,\mathbf{M}[f(x)](n+m) \\ &\quad - \sum_{i=0}^{p-1} \frac{(-1)^{i+p}(n+m+p)!}{(n+m+p-i)!} f^{(p-1-i)}(1), \end{aligned}$$

where $a(n) \in \mathbb{K}[n]$ with $\deg(a(n)) < p$.

Given a holonomic recurrence for $\mathbf{M}[f(x)](n)$.
Find a differential equation for $f(x)$.

We observe that

$$\begin{aligned} n^p\,\mathbf{M}[f(x)](n+m) \quad &= \quad \mathbf{M}[(-1)^p x^{m+p} f^{(p)}(x)](n) - a(n)\,\mathbf{M}[f(x)](n+m) \\ &\quad - \sum_{i=0}^{p-1} \frac{(-1)^{i+p}(n+m+p)!}{(n+m+p-i)!} f^{(p-1-i)}(1), \end{aligned}$$

where $a(n) \in \mathbb{K}[n]$ with $\deg(a(n)) < p$.

We can use this observation to compute the differential equation recursively.

Given a nested sum of the form

$$F(N) := F_0(N) \sum_{i_1=1}^{N} F_1(i_1) \sum_{i_2=1}^{i_1} F_2(i_2) \cdots \sum_{i_{k-1}=1}^{i_{k-2}} F_{k-1}(i_{k-1}) \sum_{i_k=1}^{i_{k-1}} F_k(i_k). \quad (3)$$

Given a nested sum of the form

$$F(N) := F_0(N) \sum_{i_1=1}^{N} F_1(i_1) \sum_{i_2=1}^{i_1} F_2(i_2) \cdots \sum_{i_{k-1}=1}^{i_{k-2}} F_{k-1}(i_{k-1}) \sum_{i_k=1}^{i_{k-1}} F_k(i_k). \quad (3)$$

Find a representation in the form

$$G(N) = \sum_{j=0}^{k} v_j^N \sum_{i=0}^{b_j} d_{i,j} \int_0^1 (x^N - a_j^N) f_{i,j}(x) dx \quad (4)$$

such that

$$F(N) = G(N)$$

for all $N \in \mathbb{N}$ with $N \geq N_0$.

- $N_0 \in \mathbb{N}$
- $v_j,\ a_j,\ d_{i,j} \in \mathbb{K}(\eta)$
- $f_{i,j}(x)$ are expressions of the form $p(\eta, x)\ g(x)$ with $p(\eta, x) \in \mathbb{K}(\eta)(x)$ and $g(x)$ is an iterated integral.

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions

Sketch of the Method

(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$

Sketch of the Method

(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$
- hence we can write

$$F(N) = \mathbf{M}[\sum_{i=1}^{k} c_i s_i(x)](N) + R(N),$$

where $R(N)$ is an expression in terms of nested sums (which are simpler then $F(N)$)

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$
- hence we can write

$$F(N) = \mathbf{M}[\sum_{i=1}^{k} c_i s_i(x)](N) + R(N),$$

  where $R(N)$ is an expression in terms of nested sums (which are simpler then $F(N)$)
- apply the same strategy to all nested sums in $R(N)$ recursively

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$
- hence we can write

$$F(N) = \mathbf{M}[\sum_{i=1}^{k} c_i s_i(x)](N) + R(N),$$

  where $R(N)$ is an expression in terms of nested sums (which are simpler then $F(N)$)
- apply the same strategy to all nested sums in $R(N)$ recursively
- eventually we will arrive at the desired solution

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$
- hence we can write

$$F(N) = \mathbf{M}[\sum_{i=1}^{k} c_i s_i(x)](N) + R(N),$$

  where $R(N)$ is an expression in terms of nested sums (which are simpler then $F(N)$)
- apply the same strategy to all nested sums in $R(N)$ recursively
- eventually we will arrive at the desired solution

Note:

- this method works

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$
- hence we can write

$$F(N) = \mathbf{M}[\sum_{i=1}^{k} c_i s_i(x)](N) + R(N),$$

  where $R(N)$ is an expression in terms of nested sums (which are simpler then $F(N)$)
- apply the same strategy to all nested sums in $R(N)$ recursively
- eventually we will arrive at the desired solution

Note:

- this method works
- was not efficient enough for our recent work

Sketch of the Method
(compare arXiv:1606.02845 [cs.SC] and arXiv:1801.01039 [cs.SC])

- compute a recurrence for the nested sum $F(N)$
- derive a differential equation (using the method mentioned above)
- solve the differential equation; let $\{s_1(x), \ldots, s_k(x)\}$ be a set of linear independent solutions
- we know that $F(N)$ will be in the Mellin transform of $\sum_{i=1}^{k} c_i s_i(x)$
- hence we can write

$$F(N) = \mathbf{M}[\sum_{i=1}^{k} c_i s_i(x)](N) + R(N),$$

  where $R(N)$ is an expression in terms of nested sums (which are simpler then $F(N)$)
- apply the same strategy to all nested sums in $R(N)$ recursively
- eventually we will arrive at the desired solution

Note:

- this method works
- was not efficient enough for our recent work
- used insight gained from this method to create a more efficient one

The new method was developed and heavily used in the frame of our recent work:

The two-mass contribution to the three-loop pure singlet operator matrix element.
(arXiv:1711.06717 [hep-ph])

Typical expressions that we considered are:

$$4^{-N}\binom{2N}{N}\sum_{\tau_1=1}^{N}\frac{4^{\tau_1}\left(\frac{\eta}{\eta-1}\right)^{\tau_1}}{\binom{2\tau_1}{\tau_1}\tau_1}$$

$$\sum_{\tau_1=1}^{N}\frac{4^{-\tau_1}\binom{2\tau_1}{\tau_1}\sum_{\tau_2=1}^{\tau_1}\frac{4^{\tau_2}\sum_{\tau_3=1}^{\tau_2}\frac{1}{\tau_3}}{\binom{2\tau_2}{\tau_2}\tau_2^2}}{\tau_1}$$

$$\sum_{\tau_1=1}^{N}\frac{4^{-\tau_1}\binom{2\tau_1}{\tau_1}\sum_{\tau_2=1}^{\tau_1}\frac{4^{\tau_2}\left(\frac{\eta}{\eta-1}\right)^{\tau_2}\sum_{\tau_3=1}^{\tau_2}\frac{\left(\frac{\eta-1}{\eta}\right)^{\tau_3}\sum_{\tau_4=1}^{\tau_3}\frac{1}{\tau_4}}{\tau_3}}{\binom{2\tau_2}{\tau_2}\tau_2}}{\tau_1}$$

$$
\begin{aligned}
&4^{-N}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\left(\frac{1}{1-\eta}\right)^i\sum_{j=1}^{i}\frac{(1-\eta)^j}{j}}{i^2\binom{2i}{i}} = \\
&\quad\int_0^1 x^N\frac{1}{12\pi\sqrt{1-x}\sqrt{x}(1-\eta)^3}\Bigg(3\pi^2(1-\eta)^3-4\left(1+11\eta+11\eta^2+\eta^3\right)\\
&\quad-192\sqrt{-\eta}(1+\eta)\mathsf{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},1\right)\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)\\
&\quad+384\mathsf{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},1\right)^2\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)-24\pi(1-\eta)^3(1+\eta)\mathsf{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},1\right)\\
&\quad+96\pi(1-\eta)^4\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},\sqrt{1-\tau}\sqrt{\tau},1\right)+192(1-\eta)^3(1+\eta)\mathsf{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},\sqrt{1-\tau}\sqrt{\tau},1\right)\\
&\quad-768(1-\eta)^4\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},\sqrt{1-\tau}\sqrt{\tau},\sqrt{1-\tau}\sqrt{\tau},1\right)\Bigg)dx\\
&\quad+\int_0^1 x^N\frac{2(1-\eta)\left(-\sqrt{1-x}\sqrt{x}+2\sqrt{1-x}x^{3/2}-4\mathsf{G}\left(\sqrt{1-\tau}\sqrt{\tau},x\right)\right)\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)}{\sqrt{1-x}\sqrt{x}}dx\\
&\quad+\int_0^1 x^N\Bigg(-\frac{4\mathsf{G}\left(\sqrt{1-\tau}\sqrt{\tau},x\right)}{\sqrt{1-x}\sqrt{x}}+\frac{2(1+\eta)\mathsf{G}\left(\frac{\sqrt{1-\tau}\sqrt{\tau}}{1-\tau+\eta\tau},x\right)}{\sqrt{1-x}\sqrt{x}}\\
&\quad+\frac{2\left(\sqrt{1-x}\sqrt{x}-2\sqrt{1-x}x^{3/2}-\sqrt{1-x}\sqrt{x}\eta+2\sqrt{1-x}x^{3/2}\eta\right)\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},x\right)}{\sqrt{1-x}\sqrt{x}}\\
&\quad+\frac{2(4-4\eta)\mathsf{G}\left(\sqrt{1-\tau}\sqrt{\tau},\frac{1}{1-\tau+\eta\tau},x\right)}{\sqrt{1-x}\sqrt{x}}\Bigg)dx\\
&\quad+\frac{1}{(1-\eta)^N}\Bigg(\int_0^1 x^N\frac{2\left(\eta+\eta^2+4\sqrt{-\eta}\mathsf{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},1\right)\right)\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)}{\sqrt{x}\sqrt{1-x-\eta}(-1+\eta)\sqrt{-\eta}}dx\\
&\quad+\int_0^1 x^N\frac{2\left(-\sqrt{x}\sqrt{1-x-\eta}+2x^{3/2}\sqrt{1-x-\eta}+\sqrt{x}\sqrt{1-x-\eta}\eta-4\mathsf{G}\left(\sqrt{1-\eta-\tau}\sqrt{\tau},x\right)\right)\mathsf{G}\left(\frac{1}{1-\tau+\eta\tau},1\right)}{\sqrt{x}\sqrt{1-x-\eta}(-1+\eta)}dx\Bigg)
\end{aligned}
$$