Singularities in MB representations - examples

Ievgen Dubovyk

PARTICLEFACE: Case studies of analytical and numerical methods of multiloop calculations for future e+e- colliders

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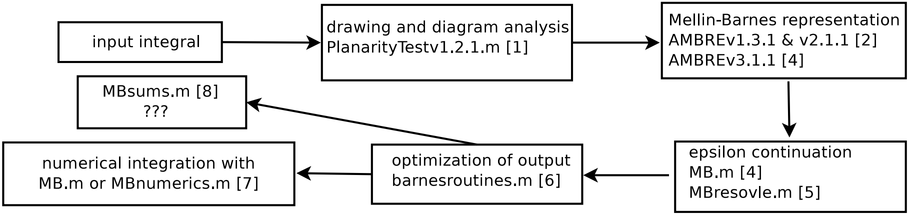
Outline

1. Introduction
2. Optimal MB representations
3. From `MB` integrals to convergent series

Calculation of Feynman integrals via MB method



[1] I. Dubovyk, K. Bielas: '13
http://us.edu.pl/~gluza/ambre/planarity/

[2] J Gluza, K. Kajda, I. Dubovyk: '11
http://prac.us.edu.pl/~gluza/ambre/

[3] I. Dubovyk: '15

[4] M. Czakon: '06
https://www.hepforge.org/downloads/mbtools

[5] A. Smirnov: '09

[6] D. Kosower: '09

[7] J. Usovitsch: '15

[8] J. Rieman. M. Ochman '15

Feynman parameters representation

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j - 1}\delta(1 - \sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}}$$

The functions U and F are called graph or Symanzik polynomials.

Some remarks

Change of variables in Symanzik polynomials U and F is effective as:

- They are homogeneous in the Feynman parameters, U is of degree L, F is of degree $L+1$
- U is linear in each Feynman parameter. If all internal masses are zero, then also F is linear in each Feynman parameter
- In expanded form each monomial of U has coefficient $+1$

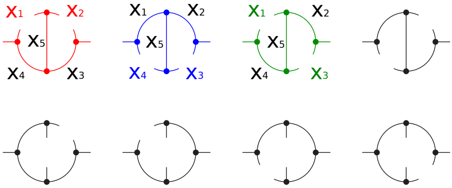
Construction of graph polynomials

- algebraic

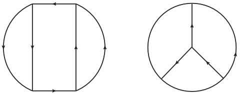
$$F = F_0 + U \sum_{i=1}^{N} x_i m_i^2, \quad U = det[M], \quad F_0 = Q M^T Q$$

with $M(x_i)$ – $(L \times L)$ matrix and $Q(x_i, p_k)$ – L-vector, L – number of loops.

- graphical:
spanning trees and the spanning 2-forests, C.Bogner, S.Weinzierl: '10



- graphical:
chain diagrams, Kinoshita: '74



- ....

Construction of Mellin-Barnes representation

"Om definita integraler", R. H. Mellin, Acta Soc. Sci. Fenn. 20(7), 1 (1895),
"The theory of the gamma function", E. W. Barnes Messenger Math. 29(2), 64 (1900).

General Mellin-Barnes relation:

$$\frac{1}{(A_1+\ldots+A_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i}\; A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\;\Gamma(\lambda+z_1+\ldots+z_{n-1})$$

Integration over Feynman parameters:

$$\int_0^1 \prod_{i=1}^{N} dx_i\; x_i^{n_i-1}\;\delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1+\ldots+n_N\right)}$$

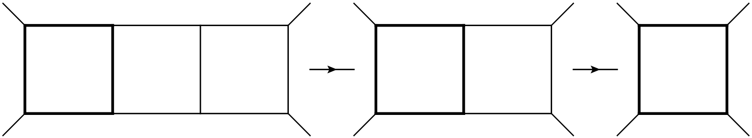
Optimal MB representations

- Feynman parameters + MB
  - iteratively to each subloop – loop-by-loop (LA) approach (`AMBREv1.3.1` & `AMBREv2.1.1`)
  - in one step to the complete U and F polynomials – global (GA) approach (`AMBREv3.1.1`)
  - combination of the above methods – Hybrid approach (under development - AMBREv4)

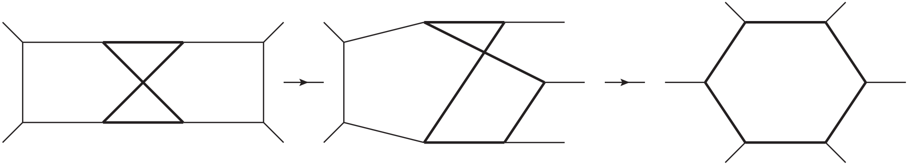
Examples, description, links to basic tools and literature:
http://prac.us.edu.pl/~gluza/ambre/

LA

Planar case:



Non-planar case:



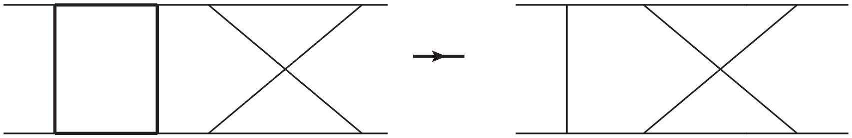
GA

U polynomial for non-planar 3-loop box (64 terms)

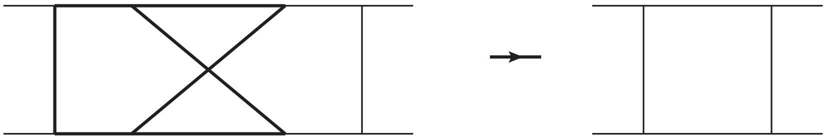
```
x[1] x[2] x[4] + x[1] x[3] x[4] + x[2] x[3] x[4] + x[1] x[2] x[5] +
x[1] x[3] x[5] + x[2] x[3] x[5] + x[1] x[4] x[5] + x[2] x[4] x[5] +
x[2] x[4] x[6] + x[3] x[4] x[6] + x[2] x[5] x[6] + x[3] x[5] x[6] +
x[4] x[5] x[6] + x[2] x[4] x[7] + x[3] x[4] x[7] + x[2] x[5] x[7] +
x[3] x[5] x[7] + x[4] x[5] x[7] + x[1] x[2] x[8] + x[1] x[3] x[8] +
x[2] x[3] x[8] + x[1] x[4] x[8] + x[2] x[4] x[8] + x[2] x[6] x[8] +
x[3] x[6] x[8] + x[4] x[6] x[8] + x[2] x[7] x[8] + x[3] x[7] x[8] +
x[4] x[7] x[8] + x[1] x[2] x[9] + x[1] x[3] x[9] + x[2] x[3] x[9] +
x[2] x[4] x[9] + x[3] x[4] x[9] + x[1] x[5] x[9] + x[3] x[5] x[9] +
x[4] x[5] x[9] + x[2] x[6] x[9] + x[3] x[6] x[9] + x[5] x[6] x[9] +
x[2] x[7] x[9] + x[3] x[7] x[9] + x[5] x[7] x[9] + x[1] x[8] x[9] +
x[3] x[8] x[9] + x[4] x[8] x[9] + x[6] x[8] x[9] + x[7] x[8] x[9] +
x[1] x[2] x[10] + x[1] x[3] x[10] + x[2] x[3] x[10] +
x[1] x[4] x[10] + x[2] x[4] x[10] + x[2] x[6] x[10] +
x[3] x[6] x[10] + x[4] x[6] x[10] + x[2] x[7] x[10] +
x[3] x[7] x[10] + x[4] x[7] x[10] + x[1] x[9] x[10] +
x[3] x[9] x[10] + x[4] x[9] x[10] + x[6] x[9] x[10] + x[7] x[9] x[10]
```

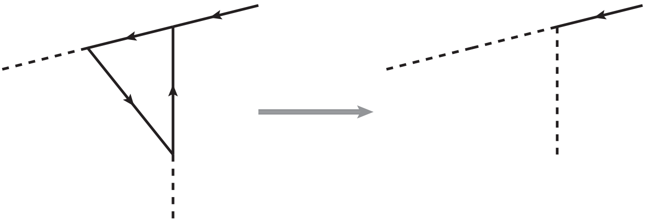
Hybrid approach

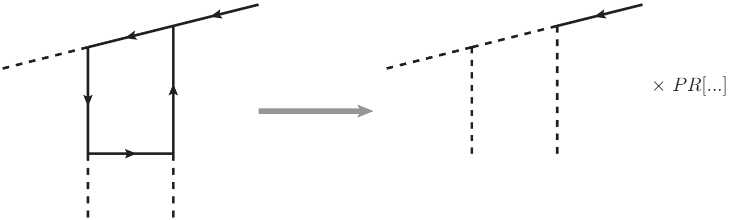
Mixed approach starting with planar subloop:



Mixed approach starting with non-planar subloop:

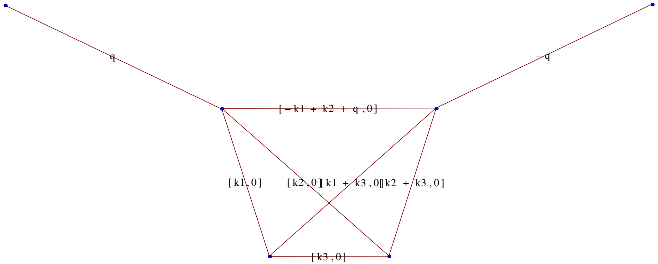





$\times\ PR[...]$

AMBRE I

- LA is basicaly for planar diagrams and GA – for non-planar.

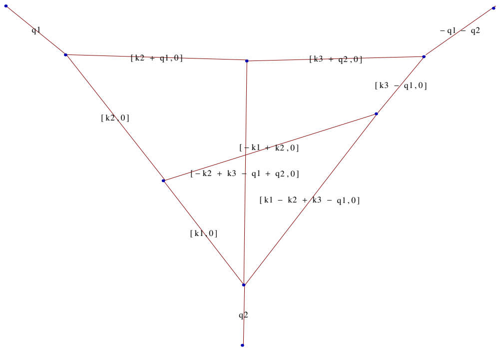


- For LA momentum flow really matters.
- 1st and 2nd Barnes lemmas are key ingredients

$$-sx_1x_2 - sx_1x_4 + \ldots = -sx_1(x_2 + x_4) + \ldots \leftrightarrow \text{1st Barnes lemma}$$

AMBRE II

- LA works also for non-planar diagrams:



  - dimensionality must be checked after expansion in ϵ
  - pros: AMBREv2.1.1 - 6-dim vs AMBREv3.1.1 - 13-dim
  - cons: always "minkowskian" - representation contains $(-s)^z s^z$
- GA works for both planar and non-planar diagrams

General structure of the MB integrals after expansion in ϵ

$$\frac{1}{(2\pi i)^r} \int\limits_{c_1-i\infty}^{c_1+i\infty} \cdots \int\limits_{c_r-i\infty}^{c_r+i\infty} \prod_i^r dz_i \; \mathbf{F}(Z,S) \frac{\prod\limits_j \mathbf{G_j}(N_j)}{\prod\limits_k \mathbf{G_k}(N_k)}$$

$\mathbf{F}$ depends on: Z – linear combinations of r complex variables z_i,
S – kinematic parameters and masses;

$\mathbf{G_i}$: Gamma and PolyGamma functions
N_i : linear combinations of z_i, e.g. $N_i = \sum_l \alpha_{il} z_l + \gamma_i$

In practice F is a product of powers of S:

$$\mathbf{F} \sim \prod_k X_k^{\sum\limits_i (\alpha_{ki} z_i + \gamma_k)}$$

$\alpha_{ij}, \gamma_i \in$ Integer, $\quad X = \left\{ -\frac{s}{m_1^2}, \frac{m_1^2}{m_2^2}, \frac{s}{t}, \ldots \right\}$.

MBsums – a Mathematica package for the representation of MB integrals by multiple sums

http://prac.us.edu.pl/~gluza/ambre/
T.Rieman,M.Ochman,arXiv:1511.01323

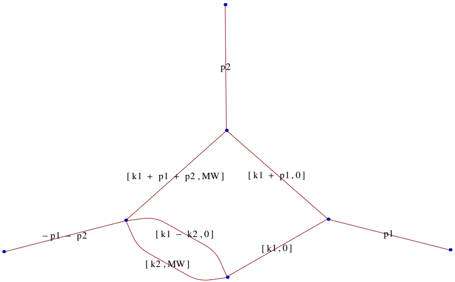
Input:

```
MBIntToSum[int,kinematics,contours]

int = MBInt[f,{{eps -> 0},{z1 -> c1, z2 -> c2, ... , zD -> cD}]

contours = {z1 -> L/R, z2 -> L/R, ... , zD -> L/R}

kinematics = {x_1 -> v_1, x_2 -> v_2, ... , x_K -> v_K} or {}
```

Output:

```
{MBsum_1, MBsum_2, ... , MBsum_Q}

MBsum_i = MBsum[Sum_Coefficient_i,Conditions_i,List_i]
```

Example 1



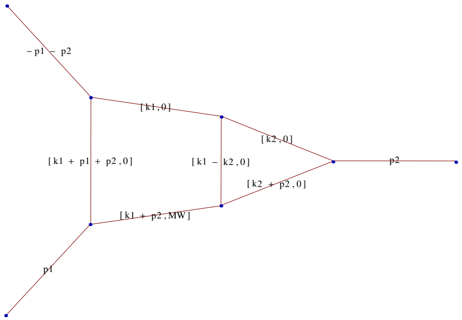

```
f = MBint[-(((-(MW^2/s))^z2 Gamma[-z2]^2 Gamma[1 + z2])/(
  eps^2 s Gamma[1 - z2])), {{eps -> 0}, {z2 -> -0.89507482768}}]
```

```
MBIntToSum[f, {}, {z2 -> L}]

z2->L ( Re z2 < -0.8950748276800662 )

{MBsum[((-1)^-n1 (-(MW^2/s))^-n1 n1!)/(eps^2 MW^2 (1 + n1)!),
  n1 >= 0, {n1}]}
```

```
 Sum[((-1)^-n1 (-(MW^2/s))^-n1 n1!)/(
 eps^2 MW^2 (1 + n1)!), {n1, 0, Infinity}]

 -(Log[(MW^2 - s)/MW^2]/(eps^2 s))
```

Example 2

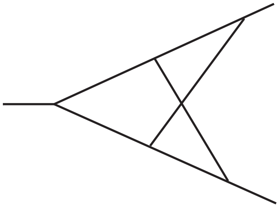
```
{k2 p1,k2 p1}
```

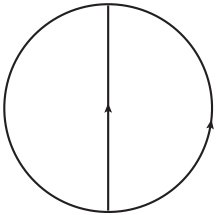



```
f = MBint[-((s Gamma[2 - z1] Gamma[-z1]^2 Gamma[1 + z1]^2)/(
  16 eps^2 MW^2 Gamma[1 - z1])), {{eps -> 0}, {z1 -> -0.895075}}]
```

```
MBIntToSum[f, {}, {z1 -> L}]

z1->L ( Re z1 < -35803/40000 )

{MBsum[-(((-1)^(-2 n1)
     s (2 + n1)! (HarmonicNumber[1 + n1] - HarmonicNumber[2 + n1]))/(
   16 eps^2 MW^2 (1 + n1)!)), n1 >= 0, {n1}]}



 In[1]:= Sum[-(((-1)^(-2 n1)
   s (2 + n1)! (HarmonicNumber[1 + n1] - HarmonicNumber[2 + n1]))/(
 16 eps^2 MW^2 (1 + n1)!)), {n1, 0, Infinity}]

During evaluation of In[1]:= Sum::div: Sum does not converge. >>

During evaluation of In[1]:= Sum::div: Sum does not converge. >>
```

GA 2-loop example (non-planar vertex)





$$\iint d^dk_1 d^dk_2 \frac{1}{[k_1^2]^{n_1}[(p_1-k_1)^2]^{n_2}[(p_1-k_1-k_2)^2]^{n_3}}$$
$$\frac{1}{[(p_2+k_1+k_2)^2]^{n_4}[(p_2+k_2)^2]^{n_5}[k_2^2]^{n_6}}$$

$$U = x_1x_2+x_1x_3+x_2x_3+x_1x_4+x_3x_4+x_1x_5+x_2x_5+x_4x_5+x_2x_6+x_3x_6+x_4x_6+x_5x_6$$

$$F = U\sum_{i=1}^{6} m_i^2 x_i - sx_1x_4x_5 - sx_1x_2x_6 - sx_1x_3x_6 - sx_2x_3x_6 - sx_1x_4x_6 - sx_1x_5x_6$$

Variables transformation

$$\{\vec{x}\}_i : x_k \to v_i \xi_{ik}$$

i denotes a subset of feynman parameters associated to propagators with different combinations of loop momenta

$$\begin{aligned} m^2 = \sum x_i D_i &= x_1 (p_1 - k_1 - k_2)^2 & x_1 &\to v_1 \xi_{11} \\ &+ x_2 (p_2 + k_1 + k_2)^2 & x_2 &\to v_1 \xi_{12} \\ &+ x_3 (k_1)^2 & x_3 &\to v_2 \xi_{21} \\ &+ x_4 (p_1 - k_1)^2 & x_4 &\to v_2 \xi_{22} \\ &+ x_5 (p_2 + k_2)^2 & x_5 &\to v_3 \xi_{31} \\ &+ x_6 (k_2)^2 & x_6 &\to v_3 \xi_{32} \end{aligned}$$

$$\delta\left(1 - \sum_{i=1}^{6} x_i\right) \Rightarrow \delta(1 - v_1 - v_2 - v_3)\delta(1 - \xi_{11} - \xi_{12})\delta(1 - \xi_{21} - \xi_{22})\delta(1 - \xi_{31} - \xi_{32})$$

Jacobian:

$$J = v_1^{N_{\xi_1} - 1} v_2^{N_{\xi_2} - 1} v_3^{N_{\xi_3} - 1} = v_1 v_2 v_3$$

- Using $\prod_i \delta\left(1 - \sum_k \xi_{ik}\right)$ we can simplify U and F

$$U = v_1 v_2 + v_1 v_3 + v_2 v_3 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1v_2v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1v_2v_3 - s\xi_{31}\xi_{32}v_1v_3^2 - s\xi_{31}\xi_{32}v_2v_3^2$$

Chang–Wu theorem:
delta function in the feyman parameters representation can be replaced by

$$\delta\left(\sum_{i\in\Omega} x_i - 1\right)$$

where Ω is an arbitrary subset of the lines $1, \ldots, L$, when the integration over the rest of the variables, i.e. for $i \notin \Omega$, is extended to the integration from zero to infinity.

- Choose now v_3 as Chang-Wu variable $\int_0^\infty dv_3 \int_0^1 dv_1 dv_2 \delta(1 - v_1 - v_2)$

$$U = v_3 + v_1 v_2 \quad F = -s\xi_{11}\xi_{22}\xi_{31}v_1v_2v_3 - s\xi_{12}\xi_{21}\xi_{32}v_1v_2v_3 - s\xi_{31}\xi_{32}v_1v_3^2$$

- Apply MB relation for F

$$\frac{1}{(A_1+\ldots+A_n)^{\lambda}} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi i)^{n-1}}\int_{-i\infty}^{i\infty} dz_1\ldots dz_{n-1}$$
$$\times \prod_{i=1}^{n-1} A_i^{z_i}\; A_n^{-\lambda-z_1-\ldots-z_{n-1}} \prod_{i=1}^{n-1}\Gamma(-z_i)\;\Gamma(\lambda+z_1+\ldots+z_{n-1})$$

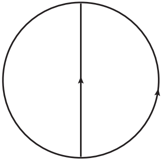
- Iintegrate over $v3$ using

$$\int\limits_0^{\infty} dx\; x^{z_1}(x+y)^{z_2} = \frac{y^{1+z_1+z_2}\Gamma(1+z_1)\Gamma(-1-z_1-z_2)}{\Gamma(-z_2)}$$

- Iintegrate over each subset of variables $\{v,\xi_i\}$ separately using

$$\int_0^1 \prod_{i=1}^{N} dx_i\; x_i^{n_i-1}\;\delta(1-x_1-\ldots-x_N) = \frac{\Gamma(n_1)\ldots\Gamma(n_N)}{\Gamma\left(n_1+\ldots+n_N\right)}$$

U polynomial gives no additional MB integration and final dimentionality depends only on length of F $\longrightarrow$ similar to one loop integrals and/or LA approach



$$U = v_1 v_2 + v_1 v_3 + v_2 v_3$$
$$F = -p^2 v_1 v_2 v_3 + U \sum_i v_i m_i^2$$

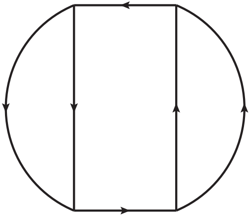
$$G(X) \sim \int \prod d\xi_{ik} \delta \left(1 - \sum_k \xi_{ik} \right)$$
$$\int d^d k_1 d^d k_2 \frac{1}{[k_1^2 - m_1^2(S,\xi_{ik})]^{n_1} [k_2^2 - m_2^2(S,\xi_{ik})]^{n_2} [(p+k_1+k_2)^2 - m_3^2(S,\xi_{ik})]^{n_3}}$$
$$p^2 = Q(S,\xi_{ik})$$

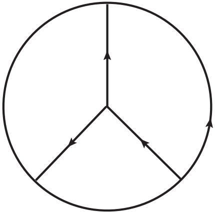
In case of massless non-planar vertex from above

$$p^2 = -s(\xi_{12}\xi_{22}\xi_{31} + \xi_{12}\xi_{21}\xi_{32} - \xi_{31}\xi_{32})$$

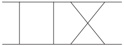
$$m_1^2 = -s\xi_{31}\xi_{32}, \quad m_2^2 = m_3^2 = 0.$$

3-loop GA



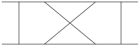


Case I:



$$U = v_1v_2v_3 + v_1v_2v_4 + v_2v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_2v_3v_5 + v_1v_4v_5 + v_3v_4v_5$$

Case II:



$$\begin{aligned} U = {} & v_1v_2v_3 + v_1v_2v_4 + v_1v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_2v_3v_5 + v_2v_4v_5 + v_3v_4v_5 \\ & + v_1v_2v_6 + v_2v_3v_6 + v_1v_4v_6 + v_2v_4v_6 + v_3v_4v_6 + v_1v_5v_6 + v_3v_5v_6 + v_4v_5v_6 \end{aligned}$$

Now in the Chang-Wu theorem we choose 3 variables

$$\int_0^\infty dv_2 dv_3 dv_4 \int_0^1 dv_1 dv_5 dv_6 \delta(1 - v_1 - v_5 - v_6)$$

$$U_{CW} = v_2v_3 + v_2v_4 + v_3v_4 + v_1v_2v_5 + v_1v_3v_5 + v_1v_2v_6 + v_1v_4v_6 + v_1v_5v_6 + v_3v_5v_6 + v_4v_5v_6$$

Factorization trick:

$$U_{CW} = v_2(v_3 + v_4 + v_1v_5) + v_3(v_4 + v_1v_5) + v_1v_6(v_2 + v_5) + v_4v_6(v_1 + v_5) + v_3v_5v_6$$

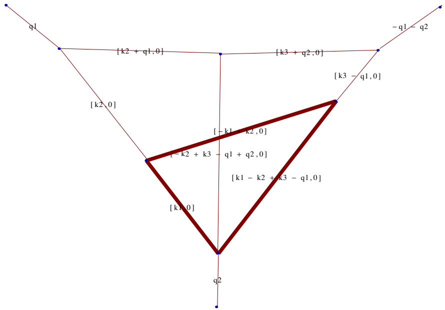
24 possibilities to choose of CW variables and factorize U

U polynomial gives 4 additional MB integration!

GA usually gives optimal representation if from the beginning Length(U) $\gtrsim$ Length(F)

3-loop mixed approach: example

```
PR[k1, 0, n1] PR[-k1 + k2, 0, n2] PR[k2, 0, n3] PR[k2 + q1, 0, n4] PR[k3 + q2, 0, n5]
PR[k3 - q1, 0, n6] PR[-k2 + k3 - q1 + q2, 0, n7] PR[k1 - k2 + k3 - q1, 0, n8]

--iteration nr: 1 with momentum: k1

   F polynomial during this iteration
-PR[k2,0] X[1] X[2]-PR[k2-k3+q1,0] X[1] X[3]-PR[k3-q1,0] X[2] X[3]
```




```
PR[k2, 0, nz3] PR[k3 - q1, 0, nz6] PR[k2 + q1, 0, n4] PR[k3 + q2, 0,n5] PR[-k2 + k3 - q1 + q2, 0, n7] PR[k2 - k3 + q1, 0, z]
```

Conclusions and Outlook

- Dimensionality of MB representations strongly depends on topology, number of legs and loops, internal and external masses. But can be optimized by application of Barnes lemmas after non-trivial variables transformation.
- `AMBRE` software is based on two different approaches:
  LA – general planar and some non-planar diagrams
  GA – 2-loop planar and non-planar, 3-loop non-planar diagrams with massless external legs
- new `AMBREv4` which combines all advantages of methods above is underway

```
MBrepr[{numerator}, {propagators}, {{k1, k3, k2}}];
                                    {k1, {k2, k3}}
                                 {{k1, k2}, {k3, k4}}
```

- MB-suite is ready to 3-loop challenge
- next step – 4-loop
- MB approach to Feynman integrals is suited to numerical integration (including Minkowskian region)
- much must be done on tools and methods to get it beyond the present status for analytical solution of Feynman integrtals