IBP Bottlenecks

Mikołaj Misiak

University of Warsaw

PARTICLEFACE workshop, Katowice, October 2-5th, 2018

1. Introduction: Feynman integrals and integration by parts (IBP)
2. Master Integrals (MIs) and differential equations
3. The Laporta algorithm and technical challenges
4. Unitarity cuts (blackboard)
5. Algebraic structure of the IBP relations
6. Summary

Each Feynman diagram with loops is specified in terms of a Feynman integrand which, after some (computer) algebra, can be written as a linear combination of expressions of the form:

$$J_{n_1 n_2 \ldots n_k} = \frac{1}{A_1^{n_1} A_2^{n_2} \ldots A_k^{n_k}},$$

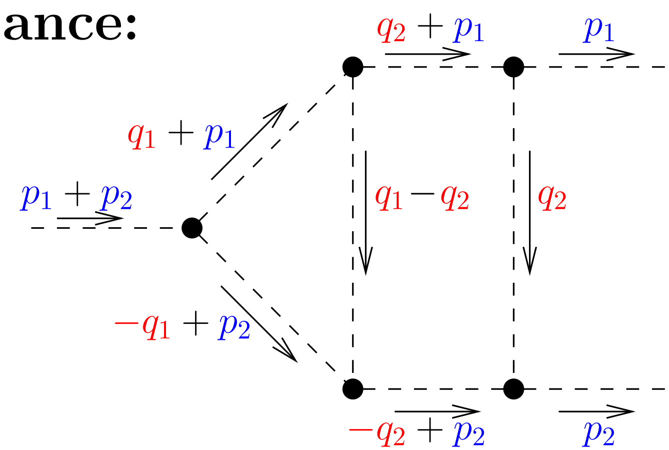
where $n_i \in \mathbb{Z}$, and A_i are linear functions of momentum products.

Each Feynman diagram with loops is specified in terms of a Feynman integrand which, after some (computer) algebra, can be written as a linear combination of expressions of the form:

$$J_{n_1 n_2 \ldots n_k} = \frac{1}{A_1^{n_1} A_2^{n_2} \ldots A_k^{n_k}},$$

where $n_i \in \mathbb{Z}$, and A_i are linear functions of momentum products.

For instance:



$$A_1 = M_1^2 - (q_1 + p_1)^2$$
$$A_2 = M_2^2 - (q_1 - q_2)^2$$
$$A_3 = M_3^2 - (-q_1 + p_2)^2 \qquad M_j^2 = m_j^2 - i\varepsilon$$
$$A_4 = M_4^2 - (q_2 + p_1)^2 \qquad \varepsilon \in \mathbb{R}_+, \ \underbrace{m_j \in \mathbb{R}_+ \cup \{0\}}_{\text{physical masses}}$$
$$A_5 = M_5^2 - q_2^2$$
$$A_6 = M_6^2 - (-q_2 + p_2)^2$$
$$A_7 = M_7^2 - q_1^2 \qquad m_7 = 0$$

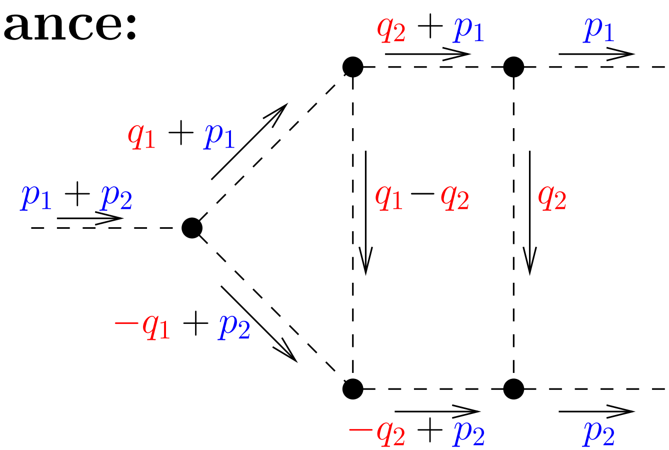
Bases: $\{A_1, \ldots, A_7\} \leftrightarrow \{q_1^2, q_2^2, q_1 q_2, q_1 p_1, q_1 p_2, q_2 p_1, q_2 p_2\}$

Each Feynman diagram with loops is specified in terms of a Feynman integrand which, after some (computer) algebra, can be written as a linear combination of expressions of the form:

$$J_{n_1 n_2 \ldots n_k} = \frac{1}{A_1^{n_1} A_2^{n_2} \ldots A_k^{n_k}},$$

where $n_i \in \mathbb{Z}$, and A_i are linear functions of momentum products.

For instance:



$$A_1 = M_1^2 - (q_1 + p_1)^2$$
$$A_2 = M_2^2 - (q_1 - q_2)^2$$
$$A_3 = M_3^2 - (-q_1 + p_2)^2$$
$$A_4 = M_4^2 - (q_2 + p_1)^2$$
$$A_5 = M_5^2 - q_2^2$$
$$A_6 = M_6^2 - (-q_2 + p_2)^2$$
$$A_7 = M_7^2 - q_1^2$$

$M_j^2 = m_j^2 - i\varepsilon$

$\varepsilon \in \mathbb{R}_+$, $\underbrace{m_j \in \mathbb{R}_+ \cup \{0\}}_{\text{physical masses}}$

$m_7 = 0$

Bases: $\{A_1, \ldots, A_7\} \leftrightarrow \{q_1^2, q_2^2, q_1 q_2, q_1 p_1, q_1 p_2, q_2 p_1, q_2 p_2\}$

Within dimensional regularization, we find contributions to scattering amplitudes by replacing

$$J_{n_1 n_2 \ldots n_k} \to I_{n_1 n_2 \ldots n_k} \equiv F[D, J_{n_1 n_2 \ldots n_k}],$$

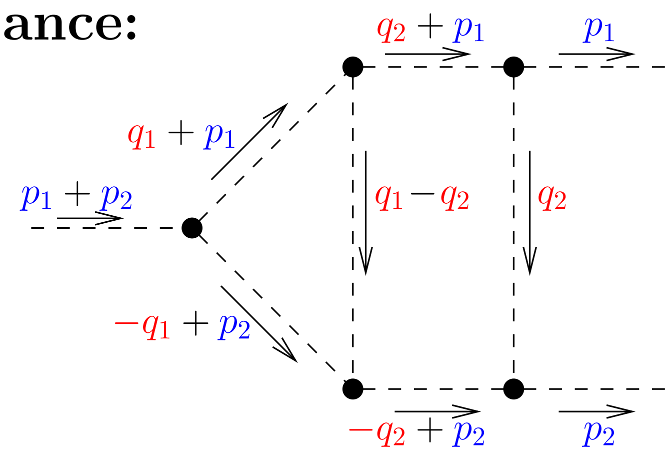
where the r.h.s is an analytic function of $D, M_j^2 \in \mathbb{C}$ and products of external momenta.

Each Feynman diagram with loops is specified in terms of a Feynman integrand which, after some (computer) algebra, can be written as a linear combination of expressions of the form:

$$J_{n_1 n_2 \ldots n_k} = \frac{1}{A_1^{n_1} A_2^{n_2} \ldots A_k^{n_k}},$$

where $n_i \in \mathbb{Z}$, and A_i are linear functions of momentum products.

For instance:



$$
\begin{aligned}
A_1 &= M_1^2 - (q_1 + p_1)^2 \\
A_2 &= M_2^2 - (q_1 - q_2)^2 \\
A_3 &= M_3^2 - (-q_1 + p_2)^2 \\
A_4 &= M_4^2 - (q_2 + p_1)^2 \\
A_5 &= M_5^2 - q_2^2 \\
A_6 &= M_6^2 - (-q_2 + p_2)^2 \\
A_7 &= M_7^2 - q_1^2
\end{aligned}
$$

$M_j^2 = m_j^2 - i\varepsilon$

$\varepsilon \in \mathbb{R}_+$, $\underbrace{m_j \in \mathbb{R}_+ \cup \{0\}}_{\text{physical masses}}$

$m_7 = 0$

Bases: $\{A_1, \ldots, A_7\} \leftrightarrow \{q_1^2, q_2^2, q_1 q_2, q_1 p_1, q_1 p_2, q_2 p_1, q_2 p_2\}$

Within dimensional regularization, we find contributions to scattering amplitudes by replacing

$$J_{n_1 n_2 \ldots n_k} \to I_{n_1 n_2 \ldots n_k} \equiv F[D, J_{n_1 n_2 \ldots n_k}],$$

where the r.h.s is an analytic function of $D, M_j^2 \in \mathbb{C}$ and products of external momenta.

The mapping F has the following properties:

(i) $F[D, X]$ is linear in X, while X is a rational function of momentum products and M_j^2.

(ii) $F[D, X] = 0$ when X depends neither on the external momenta nor on $m_j^2 \neq 0$.

(iii) For $D \in \mathbb{N}$, $F[D, X] = \int (d^D q_1) \ldots (d^D q_L) X$ when the integral is finite and (ii) does not apply.

(iv) $F[D, X] = 0$ when X is a total derivative w.r.t. any of the loop momenta.
In our example $F\left[D, \frac{\partial}{\partial q_i^\alpha}(r^\alpha J)\right] = 0$, where $r \in \{q_1, q_2, p_1, p_2\}$.

Vanishing of F for total derivatives provides useful identities. Let us consider, for instance,

$$F\left[D, \frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right)\right] = 0.$$

A straightforward calculation gives

$$\frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right) = m^2\left(J_{2111110} - J_{1211110} + J_{1121110}\right) + \\ +(D-3)J_{1111110} + J_{1211010} - J_{21111(-1)} - J_{12111(-1)} - J_{11211(-1)},$$

where, for simplicity, $p_1^2 = p_2^2 = m_2^2 = 0$, while all the other masses m_i have been set to m. Moreover, all the $i\varepsilon$ terms in the numerators have been tacitly set to zero.

Vanishing of F for total derivatives provides useful identities. Let us consider, for instance,

$$F\left[D, \frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right)\right] = 0.$$

A straightforward calculation gives

$$\frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right) = m^2\left(J_{2111110} - J_{1211110} + J_{1121110}\right) + \\ +(D-3)J_{1111110} + J_{1211010} - J_{21111(-1)} - J_{12111(-1)} - J_{11211(-1)},$$

where, for simplicity, $p_1^2 = p_2^2 = m_2^2 = 0$, while all the other masses m_i have been set to m. Moreover, all the $i\varepsilon$ terms in the numerators have been tacitly set to zero.

Consequently, we get the following identity for the integrals $I_{n_1 n_2 \ldots n_7}$:

$$0 = m^2\left(I_{2111110} - I_{1211110} + I_{1121110}\right) + \\ +(D-3)I_{1111110} + I_{1211010} - I_{21111(-1)} - I_{12111(-1)} - I_{11211(-1)}.$$

Such relations are called the Integration By Parts (IBP) identities.

Vanishing of F for total derivatives provides useful identities. Let us consider, for instance,

$$F\left[D, \frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right)\right] = 0.$$

A straightforward calculation gives

$$\frac{\partial}{\partial q_1^\alpha}\left(q_1^\alpha J_{1111110}\right) = m^2\left(J_{2111110} - J_{1211110} + J_{1121110}\right) +$$
$$+(D-3)J_{1111110} + J_{1211010} - J_{21111(-1)} - J_{12111(-1)} - J_{11211(-1)},$$

where, for simplicity, $p_1^2 = p_2^2 = m_2^2 = 0$, while all the other masses m_i have been set to m. Moreover, all the $i\varepsilon$ terms in the numerators have been tacitly set to zero.

Consequently, we get the following identity for the integrals $I_{n_1 n_2 \ldots n_7}$:

$$0 = m^2\left(I_{2111110} - I_{1211110} + I_{1121110}\right) +$$
$$+(D-3)I_{1111110} + I_{1211010} - I_{21111(-1)} - I_{12111(-1)} - I_{11211(-1)}.$$

Such relations are called the Integration By Parts (IBP) identities.

We can view $I_{n_1 n_2 \ldots n_k}$ as a mapping

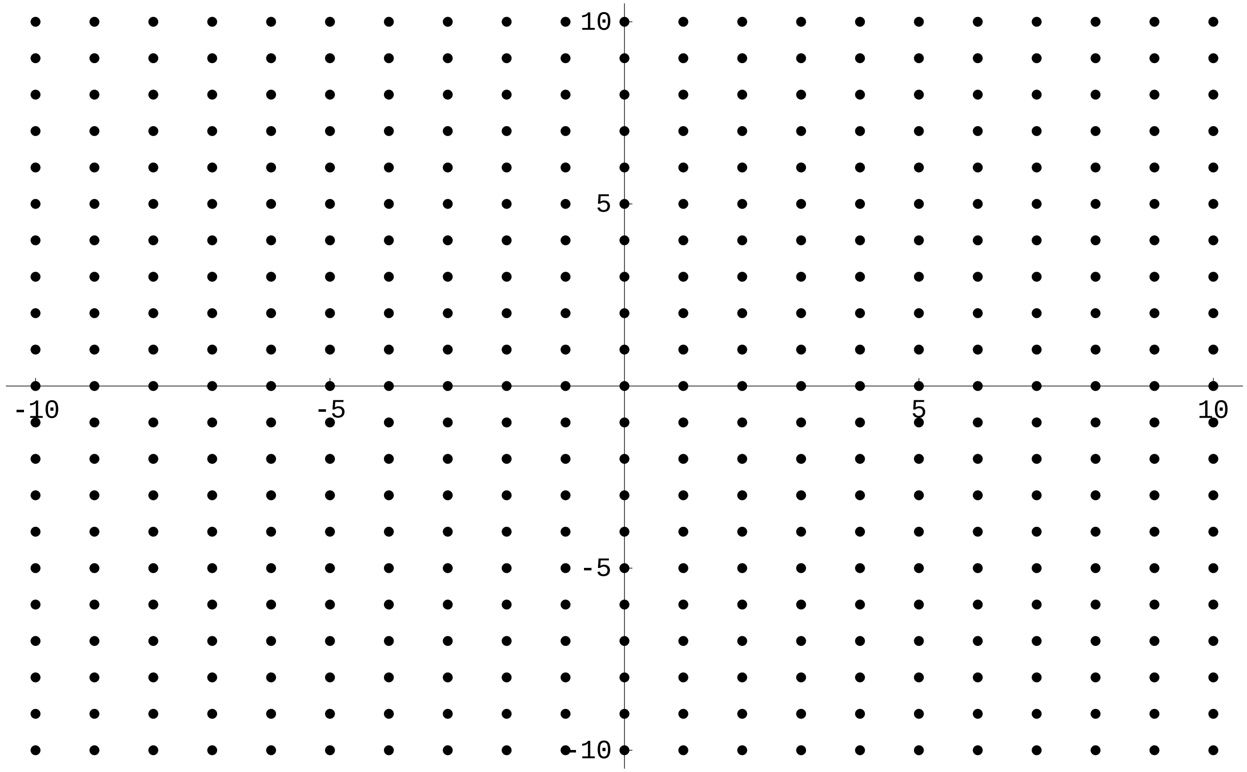
$$I : \mathbb{Z}^k \to \mathcal{C}\left(\mathbb{C}^N\right)$$

[Complex-valued functions of $D, M_j^2 \in \mathbb{C}$ and products of external momenta (treated as complex)]

The IBP identities give us linear relations between values of I at several nearest-neighbour points. Naively, we get "more relations than integrals".

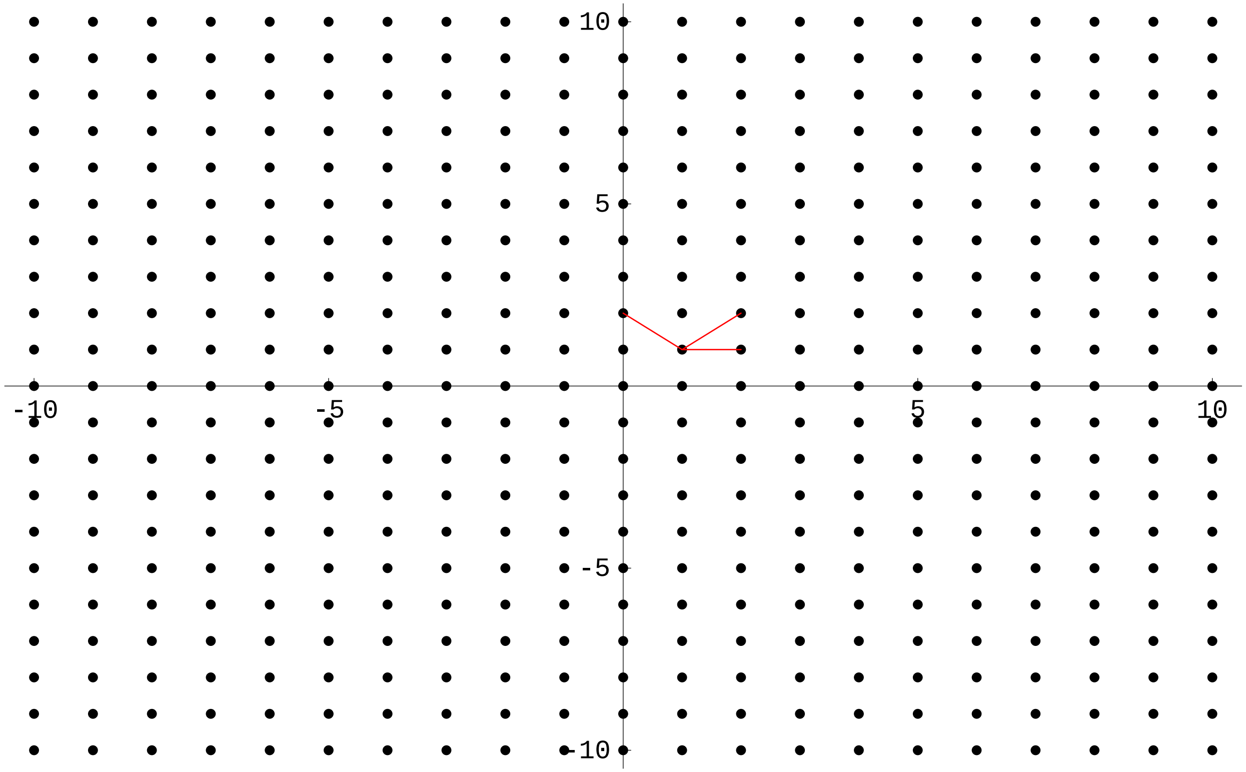
$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}}\left(\mathbb{C}^{N}\right)$$

The IBP identities give us linear relations between values of $\boldsymbol{I}$ at several nearest-neighbour points.



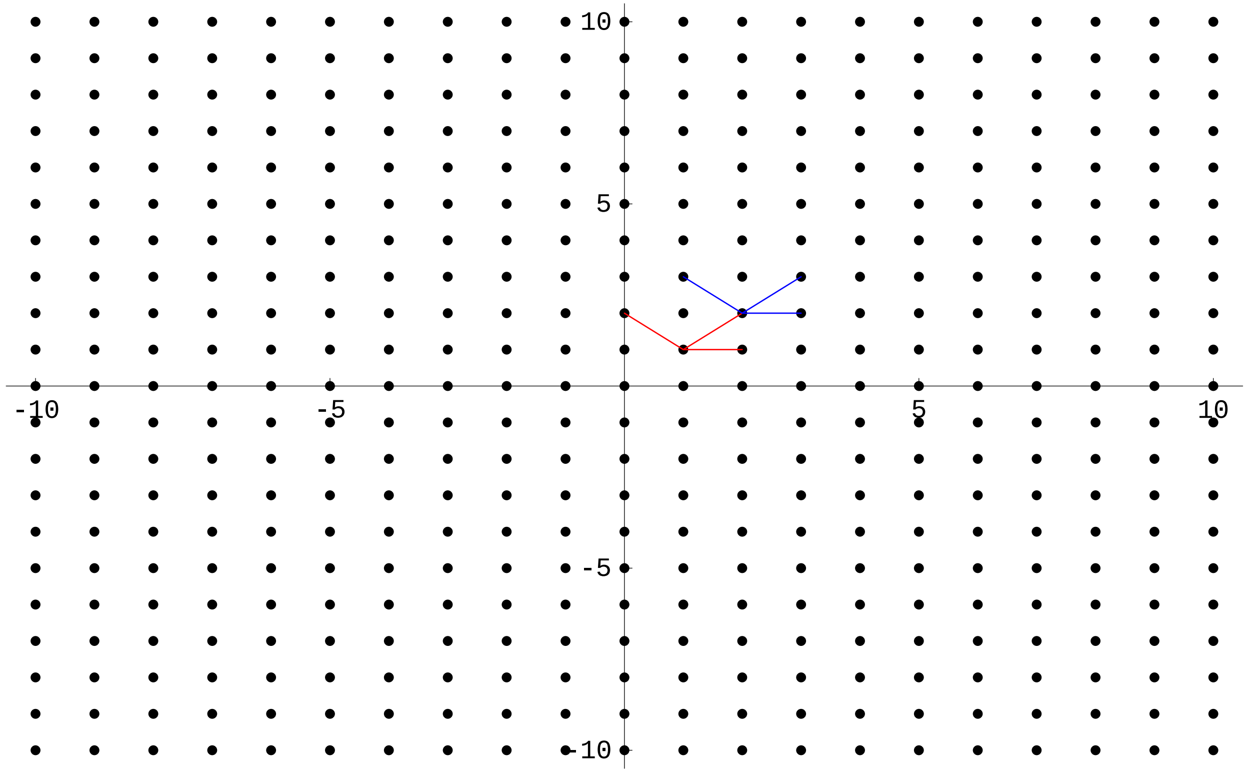
$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}}\left(\mathbb{C}^{\boldsymbol{N}}\right)$$

The IBP identities give us linear relations between values of $\boldsymbol{I}$ at several nearest-neighbour points.



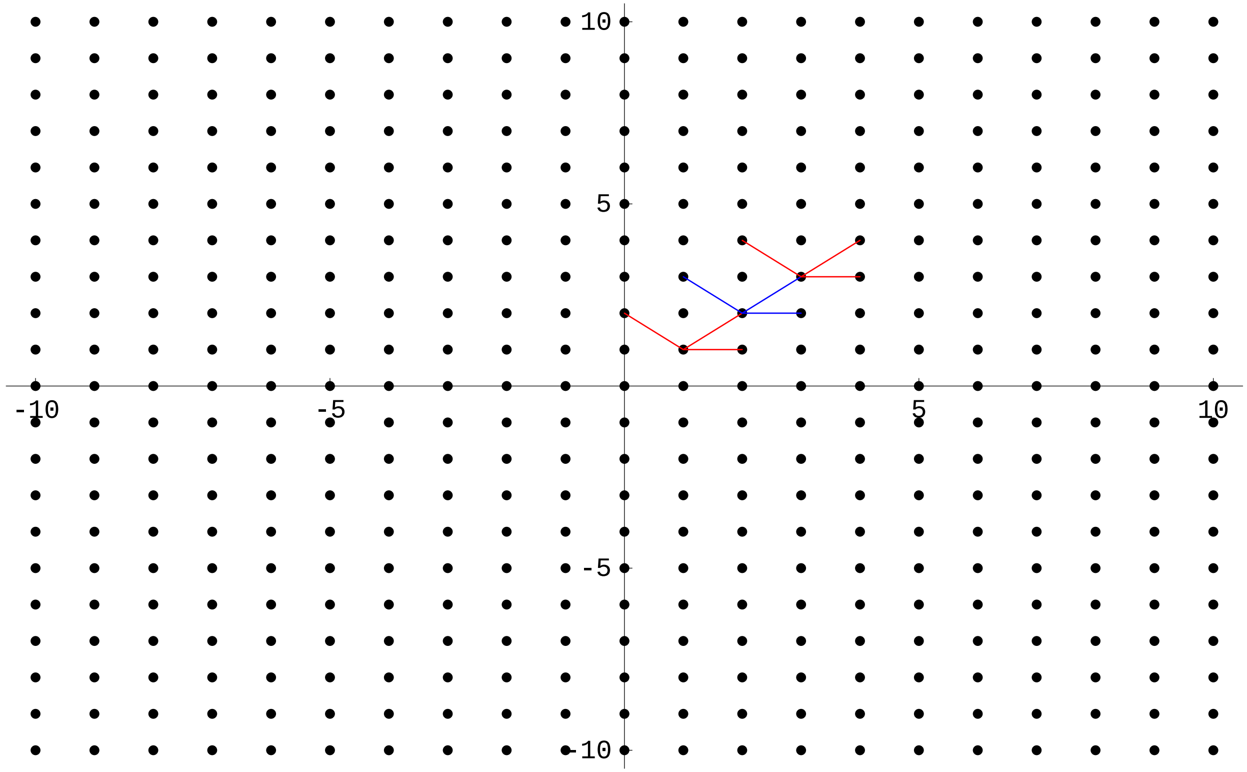
$$I : \mathbb{Z}^k \to \mathcal{C}\left(\mathbb{C}^N\right)$$

The IBP identities give us linear relations between values of I at several nearest-neighbour points.



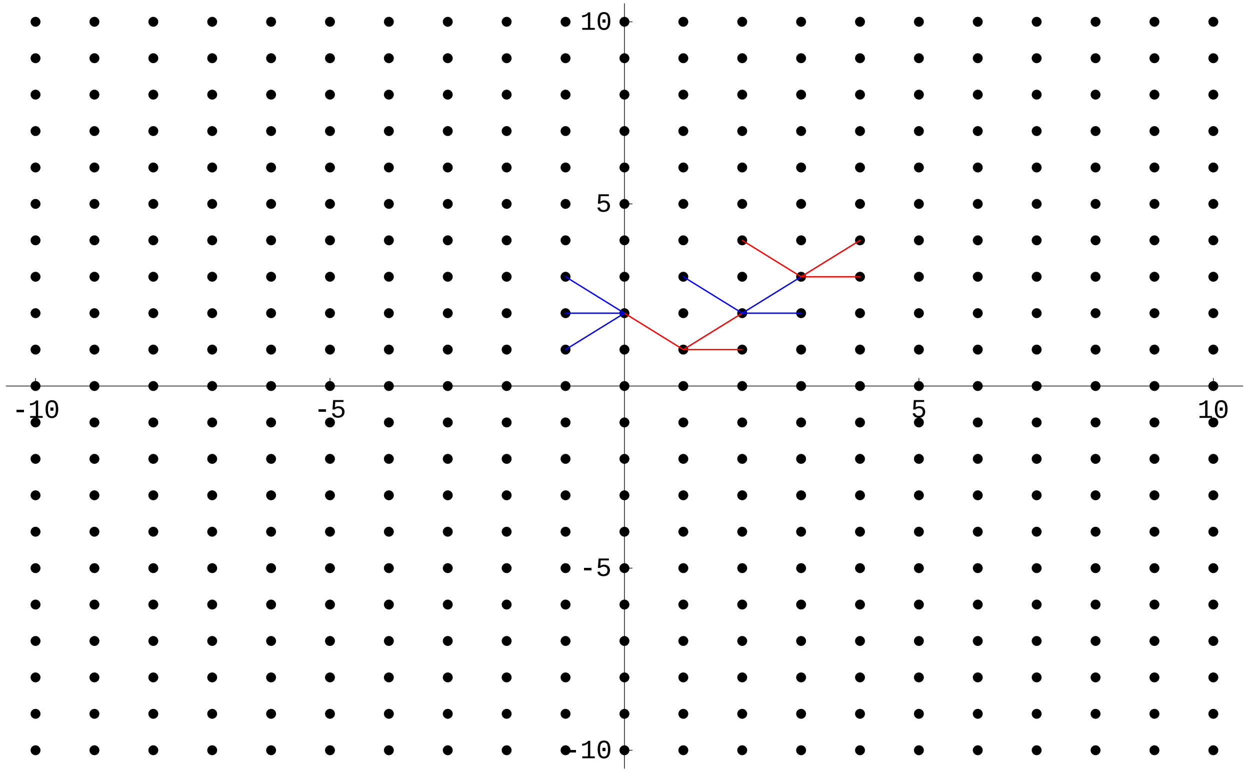
$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}}\left(\mathbb{C}^{\boldsymbol{N}}\right)$$

The IBP identities give us linear relations between values of $\boldsymbol{I}$ at several nearest-neighbour points.



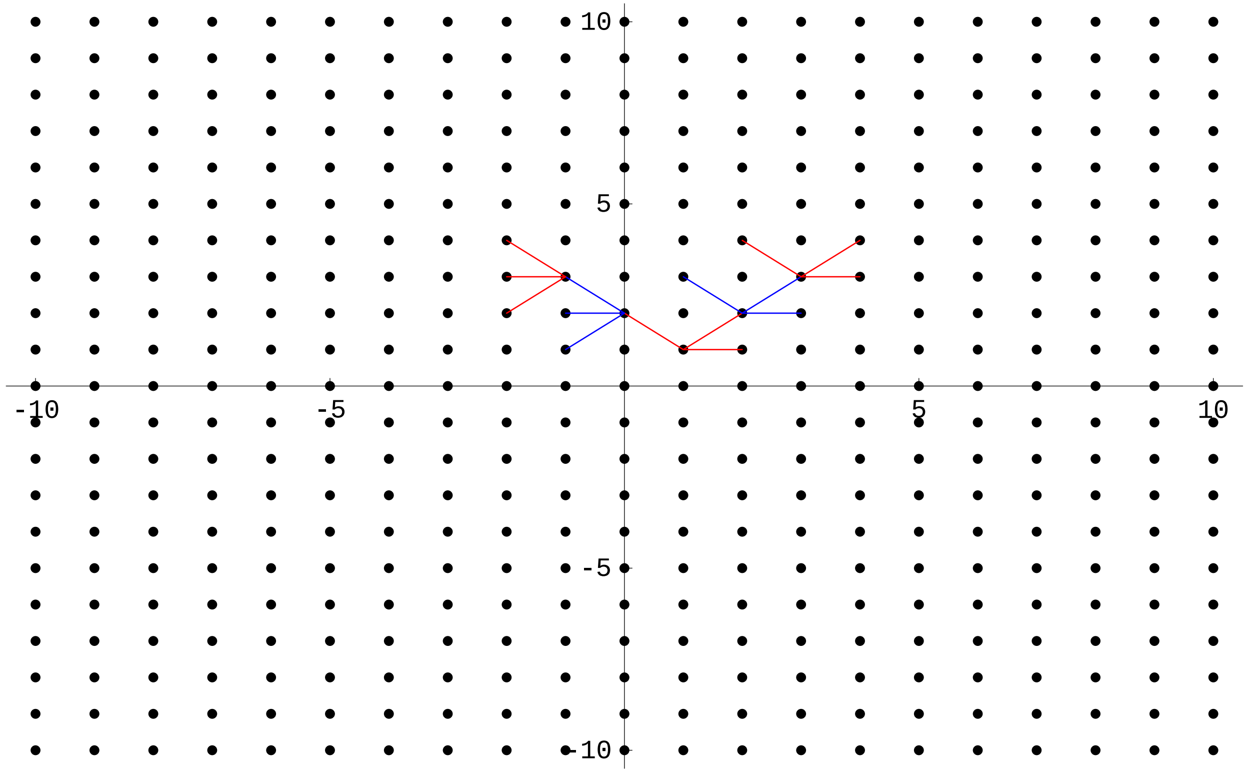
$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}}\left(\mathbb{C}^{\boldsymbol{N}}\right)$$

The IBP identities give us linear relations between values of $\boldsymbol{I}$ at several nearest-neighbour points.



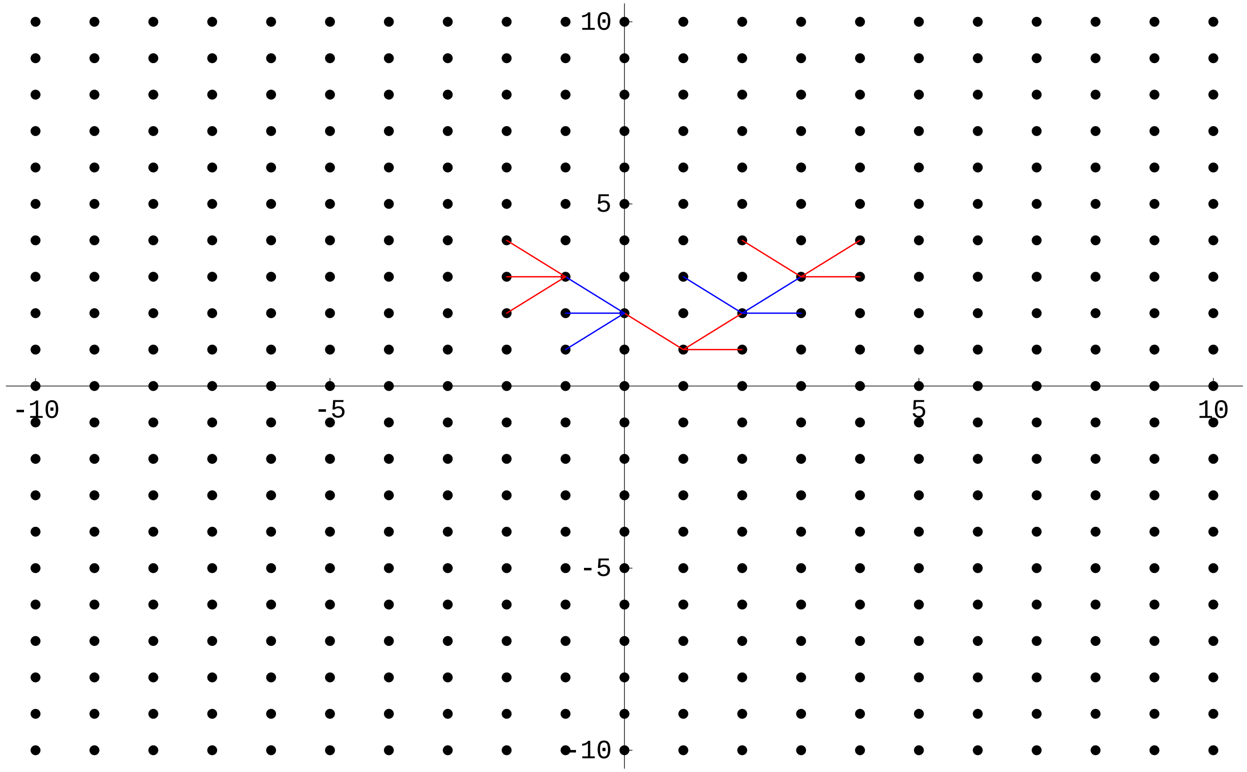
$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}}\left(\mathbb{C}^{\boldsymbol{N}}\right)$$

The IBP identities give us linear relations between values of $\boldsymbol{I}$ at several nearest-neighbour points.



$$I : \mathbb{Z}^k \to \mathcal{C}\ (\mathbb{C}^N)$$

The IBP identities give us linear relations between values of I at several nearest-neighbour points.



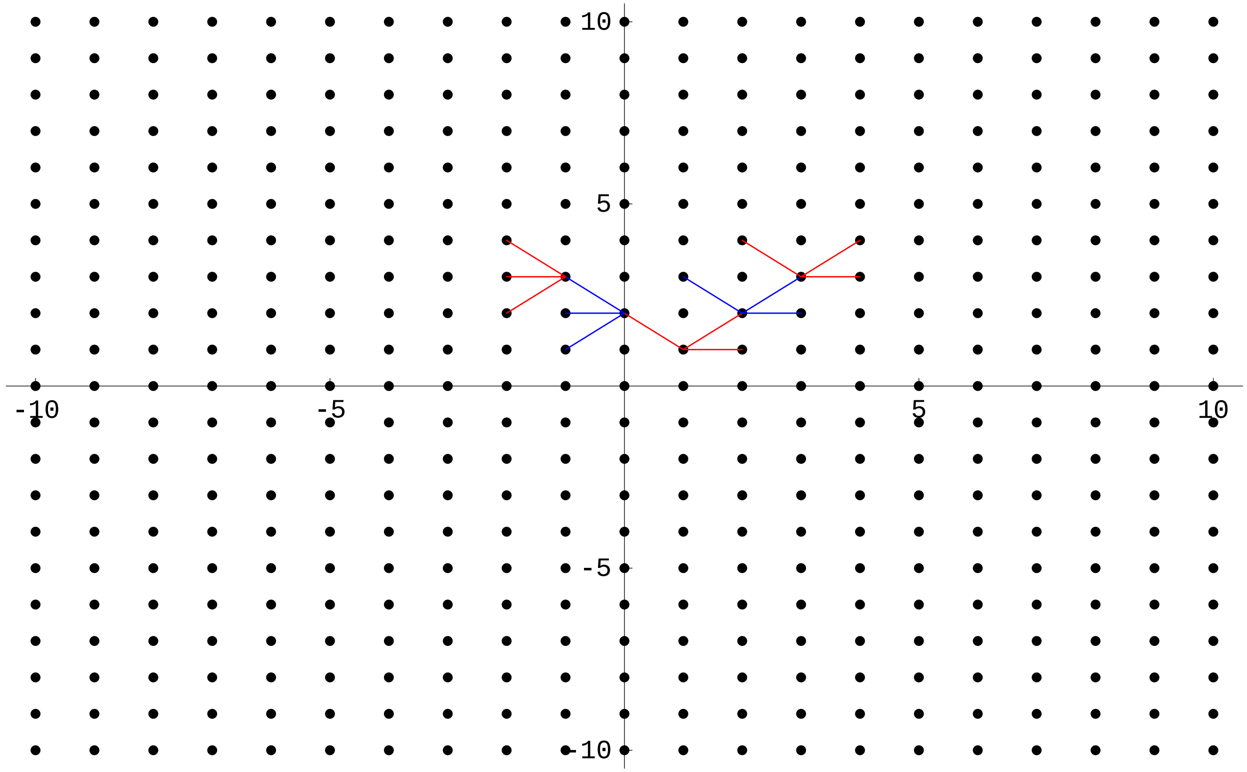
We get infinitely many linear relations among infinitely many functions.

This is similar to a linear recurrence relation, e.g., $H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)$.

[Here we get any H_n in terms of H_0 and H_1].

$$\boldsymbol{I} : \mathbb{Z}^{\boldsymbol{k}} \to \boldsymbol{\mathcal{C}} \, (\mathbb{C}^{\boldsymbol{N}})$$

The IBP identities give us linear relations between values of $\boldsymbol{I}$ at several nearest-neighbour points.



We get infinitely many linear relations among infinitely many functions.

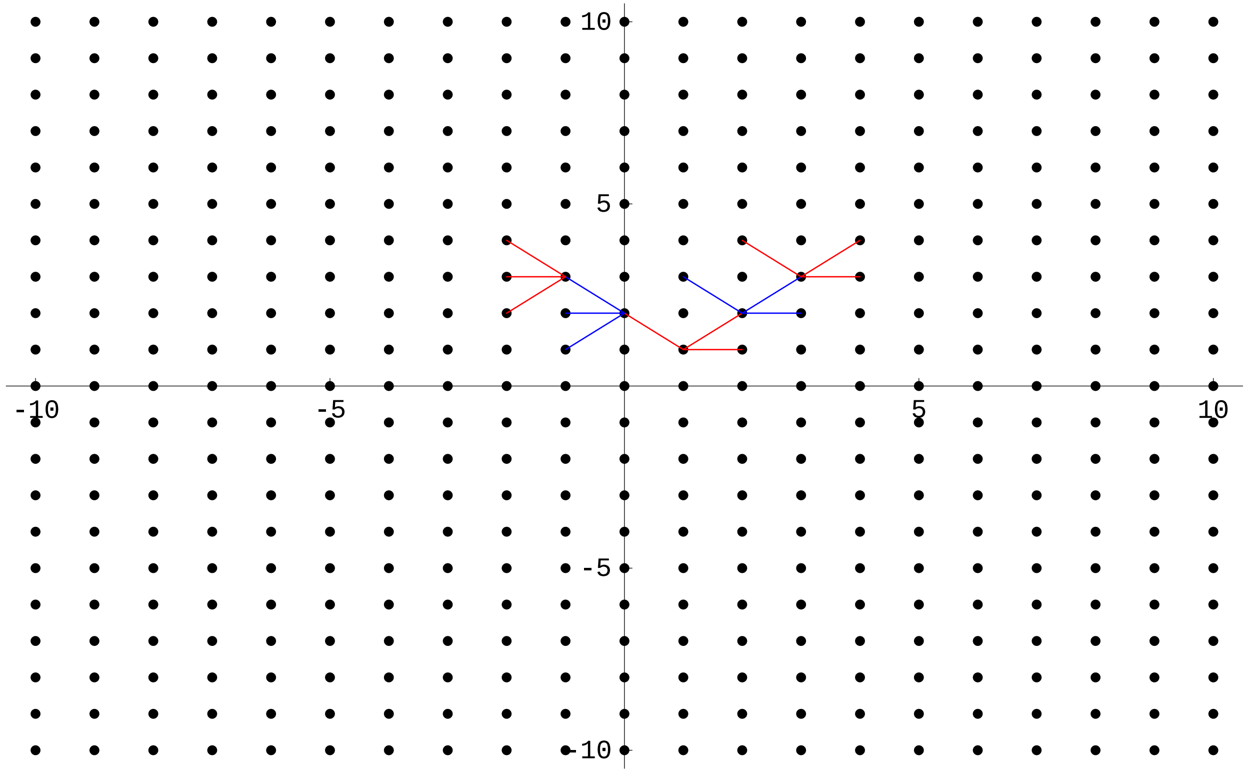
This is similar to a linear recurrence relation, e.g., $\boldsymbol{H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)}$.

[Here we get any $\boldsymbol{H_n}$ in terms of $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$].

Do the IBP give us a recurrence relation? Does a finite set of $\boldsymbol{I_{n_1 n_2 \ldots n_k}}$ determine all of these functions?

$$I : \mathbb{Z}^k \to \mathcal{C}\ (\mathbb{C}^N)$$

The IBP identities give us linear relations between values of I at several nearest-neighbour points.



We get infinitely many linear relations among infinitely many functions.

This is similar to a linear recurrence relation, e.g., $H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z)$.

[Here we get any H_n in terms of H_0 and H_1].

Do the IBP give us a recurrence relation? Does a finite set of $I_{n_1 n_2 \ldots n_k}$ determine all of these functions?

Answer: **Yes!**

Proof: A. V. Smirnov and A. V. Petukhov, "The Number of Master Integrals is Finite," Lett. Math. Phys. 97 (2011) 37 [arXiv:1004.4199].

Thanks to the fact that the number of Master Integrals (MIs) is finite, we can derive closed sets of differential equations (DEs) for them.

Thanks to the fact that the number of Master Integrals (MIs) is finite, we can derive closed sets of differential equations (DEs) for them.

In our example, when the integrand $J_{1111110}$ is differentiated w.r.t. m^2, we get:

$$\frac{\partial}{\partial(m^2)} J_{1111110} = -J_{2111110} - J_{1121110} - J_{1112110} - J_{1111210} - J_{1111120}.$$

Thus:

$$\frac{\partial}{\partial(m^2)} I_{1111110} = \frac{\partial}{\partial(m^2)} F[D, J_{1111110}] = F\left[D, \frac{\partial}{\partial(m^2)} J_{1111110}\right] =$$

$$= -I_{2111110} - I_{1121110} - I_{1112110} - I_{1111210} - I_{1111120}.$$

Writing such derivatives for all the MIs and expressing the r.h.s. in terms of the MIs (using the IBP identities), we obtain a closed set of linear DEs for the MIs.

Thanks to the fact that the number of Master Integrals (MIs) is finite, we can derive closed sets of differential equations (DEs) for them.

In our example, when the integrand $J_{1111110}$ is differentiated w.r.t. m^2, we get:

$$\frac{\partial}{\partial(m^2)} J_{1111110} = -J_{2111110} - J_{1121110} - J_{1112110} - J_{1111210} - J_{1111120}.$$

Thus:

$$\frac{\partial}{\partial(m^2)} I_{1111110} = \frac{\partial}{\partial(m^2)} F[D, J_{1111110}] = F\left[D, \frac{\partial}{\partial(m^2)} J_{1111110}\right] =$$

$$= -I_{2111110} - I_{1121110} - I_{1112110} - I_{1111210} - I_{1111120}.$$

Writing such derivatives for all the MIs and expressing the r.h.s. in terms of the MIs (using the IBP identities), we obtain a closed set of linear DEs for the MIs.

Analogously, for an integral $I(p_1 \cdot p_2, m^2) = F[D, J]$:

$$\frac{\partial}{\partial(p_1 \cdot p_2)} I = \frac{p_1^\alpha}{p_1 \cdot p_2} \frac{\partial}{\partial p_1^\alpha} I = F\left[D, \frac{p_1^\alpha}{p_1 \cdot p_2} \frac{\partial}{\partial p_1^\alpha} J\right] = \text{(linear combination of MIs)}.$$

Thanks to the fact that the number of Master Integrals (MIs) is finite, we can derive closed sets of differential equations (DEs) for them.

In our example, when the integrand $J_{1111110}$ is differentiated w.r.t. m^2, we get:

$$\frac{\partial}{\partial(m^2)} J_{1111110} = -J_{2111110} - J_{1121110} - J_{1112110} - J_{1111210} - J_{1111120}.$$

Thus:

$$\frac{\partial}{\partial(m^2)} I_{1111110} = \frac{\partial}{\partial(m^2)} F[D, J_{1111110}] = F\left[D, \frac{\partial}{\partial(m^2)} J_{1111110}\right] =$$

$$= -I_{2111110} - I_{1121110} - I_{1112110} - I_{1111210} - I_{1111120}.$$

Writing such derivatives for all the MIs and expressing the r.h.s. in terms of the MIs (using the IBP identities), we obtain a closed set of linear DEs for the MIs.

Analogously, for an integral $I(p_1 \cdot p_2, m^2) = F[D, J]$:

$$\frac{\partial}{\partial(p_1 \cdot p_2)} I = \frac{p_1^\alpha}{p_1 \cdot p_2} \frac{\partial}{\partial p_1^\alpha} I = F\left[D, \frac{p_1^\alpha}{p_1 \cdot p_2} \frac{\partial}{\partial p_1^\alpha} J\right] = \text{(linear combination of MIs)}.$$

Initial conditions for the DEs are set in regions where the evaluation of MIs is easier, e.g., for (masses) $\gg$ (products of external momenta).

The recurrence relations following from the IBP have been solved analytically in several cases [e.g., K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159; F.V. Tkachov, Phys. Lett. B100 (1981) 65].

However, in general, brute-force computer algebra methods are applied.

The most popular approach is the Laporta algorithm [S. Laporta, hep-ph/0102033].

The recurrence relations following from the IBP have been solved analytically in several cases [e.g., K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159;
F.V. Tkachov, Phys. Lett. B100 (1981) 65].

However, in general, brute-force computer algebra methods are applied.

The most popular approach is the **Laporta algorithm** [S. Laporta, hep-ph/0102033].

Consider a family of integrals $I_{n_1 \ldots n_k}$ defined by a set of denominators $\{A_1, \ldots, A_k\}$.
In any particular physical calculation only a finite set of them is relevant.

(i) We extend this set by including all the integrals with
(sum of positive indices) $\leq N_1$ and |sum of negative indices| $\leq N_2$,
with N_j fixed in a quasi-intuitive manner.

(ii) Derive all the IBP relations involving *only* the selected integrals.

(iii) Establish an absolute "simplicity" ordering in the selected set. Roughly:
First criterion: number of positive indices,
Second criterion: sum of positive indices,
Third criterion: |sum of negative indices|,
Next criteria: values of indices at particular positions.

(iv) Solve the system of linear equations, expressing the "least simple" integral in terms of "simpler" ones.

The recurrence relations following from the IBP have been solved analytically in several cases [e.g., K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159; F.V. Tkachov, Phys. Lett. B100 (1981) 65].

However, in general, brute-force computer algebra methods are applied.

The most popular approach is the **Laporta algorithm** [S. Laporta, hep-ph/0102033].

Consider a family of integrals $I_{n_1 \ldots n_k}$ defined by a set of denominators $\{A_1, \ldots, A_k\}$.
In any particular physical calculation only a finite set of them is relevant.

(i) We extend this set by including all the integrals with
(sum of positive indices) $\leq N_1$ and |sum of negative indices| $\leq N_2$,
with N_j fixed in a quasi-intuitive manner.

(ii) Derive all the IBP relations involving *only* the selected integrals.

(iii) Establish an absolute "simplicity" ordering in the selected set. Roughly:
First criterion: number of positive indices,
Second criterion: sum of positive indices,
Third criterion: |sum of negative indices|,
Next criteria: values of indices at particular positions.

(iv) Solve the system of linear equations, expressing the "least simple" integral in terms of "simpler" ones.

Publicly available IBP codes:
AIR, hep-ph/0404258
FIRE, arXiv:0807.3243, 1408.2372
Reduze, arXiv:0912.2546, 1201.4330
LiteRed, arXiv:1212.2685, 1310.1145
KIRA, arXiv:1705.05610

The recurrence relations following from the IBP have been solved analytically in several cases [e.g., K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159;
F.V. Tkachov, Phys. Lett. B100 (1981) 65].

However, in general, brute-force computer algebra methods are applied.

The most popular approach is the **Laporta algorithm** [S. Laporta, hep-ph/0102033].

Consider a family of integrals $I_{n_1 \ldots n_k}$ defined by a set of denominators $\{A_1, \ldots, A_k\}$.
In any particular physical calculation only a finite set of them is relevant.

(i) We extend this set by including all the integrals with
(sum of positive indices) $\leq N_1$ and |sum of negative indices| $\leq N_2$,
with N_j fixed in a quasi-intuitive manner.

(ii) Derive all the IBP relations involving *only* the selected integrals.

(iii) Establish an absolute "simplicity" ordering in the selected set. Roughly:
First criterion: number of positive indices,
Second criterion: sum of positive indices,
Third criterion: |sum of negative indices|,
Next criteria: values of indices at particular positions.

(iv) Solve the system of linear equations, expressing the "least simple" integral in terms of "simpler" ones.

Publicly available IBP codes:
AIR, hep-ph/0404258
FIRE, arXiv:0807.3243, 1408.2372
Reduze, arXiv:0912.2546, 1201.4330
LiteRed, arXiv:1212.2685, 1310.1145
KIRA, arXiv:1705.05610

Unfortunately, this method is computationally heavy.

My current project: 451 families with $\mathcal{O}(1000)$ integrals each, depending on two variables: D and m_1/m_2.
For some families, few weeks with 1TB RAM and 2TB disk space are insufficient for FIRE 5.7.

Structure of the IBP relations [see R.N. Lee, arXiv:0804.3008]

Let us consider the operators $O_{ik} = \frac{\partial}{\partial q_i} r_k$ acting on the integrands J, where $r_k \in \{q_1, \ldots, q_L, p_1, \ldots p_E\}$.
They form a closed Lie algebra with the commutation relations

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}.$$

Structure of the IBP relations [see R.N. Lee, arXiv:0804.3008]

Let us consider the operators $O_{ik} = \frac{\partial}{\partial q_i} r_k$ acting on the integrands J, where $r_k \in \{q_1, \ldots, q_L, p_1, \ldots p_E\}$.
They form a closed Lie algebra with the commutation relations

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}.$$

They are related to infinitesimal redefinitions of loop momenta $q_i \to q_i' = q_i + \eta_{ik} r_k$ with constant η_{ik}, under which the integrals are invariant:

$$F[D, J(q', p)] = F\left[D,\ J(q, p) + \underbrace{\eta_{ik} O_{ik} J(q, p)}_{\text{zero by IBP}}\ \right] + \mathcal{O}(\eta^2).$$

Structure of the IBP relations [see R.N. Lee, arXiv:0804.3008]

Let us consider the operators $O_{ik} = \frac{\partial}{\partial q_i} r_k$ acting on the integrands J, where $r_k \in \{q_1, \ldots, q_L, p_1, \ldots p_E\}$.

They form a closed Lie algebra with the commutation relations

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}.$$

They are related to infinitesimal redefinitions of loop momenta $q_i \to q_i' = q_i + \eta_{ik} r_k$ with constant η_{ik}, under which the integrals are invariant:

$$F[D, J(q', p)] = F\,[D,\ J(q,p) + \underbrace{\eta_{ik} O_{ik} J(q,p)}_{\text{zero by IBP}}\ ] + \mathcal{O}(\eta^2).$$

A one-dimensional analogy:

$$f_\eta(x) \ = \ (1+\eta)\, g[(1+\eta)x + 2\eta] \quad \Rightarrow \quad \int_{-\infty}^{+\infty} f_\eta(x) dx \ = \ \int_{-\infty}^{+\infty} g(x) dx.$$

$$f_\eta(x) \ = \ g(x) + \eta \tfrac{d}{dx} [(x+2) g(x)] + \mathcal{O}(\eta^2).$$

Thus, we may consider the IBP relations as following from the fact that the integrals are constant on the orbits corresponding to the generators Q_{ik}.

Structure of the IBP relations [see R.N. Lee, arXiv:0804.3008]

Let us consider the operators $O_{ik} = \frac{\partial}{\partial q_i} r_k$ acting on the integrands J, where $r_k \in \{q_1, \ldots, q_L, p_1, \ldots p_E\}$.

They form a closed Lie algebra with the commutation relations

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}.$$

They are related to infinitesimal redefinitions of loop momenta $q_i \to q_i' = q_i + \eta_{ik} r_k$ with constant η_{ik}, under which the integrals are invariant:

$$F[D, J(q', p)] = F\,[D,\ J(q,p) + \underbrace{\eta_{ik} O_{ik} J(q,p)}_{\text{zero by IBP}}\ ] + \mathcal{O}(\eta^2).$$

A one-dimensional analogy:

$$f_\eta(x) \ = \ (1+\eta)\, g[(1+\eta)x + 2\eta] \quad \Rightarrow \quad \int_{-\infty}^{+\infty} f_\eta(x) dx \ = \ \int_{-\infty}^{+\infty} g(x) dx.$$

$$f_\eta(x) \ = \ g(x) + \eta \tfrac{d}{dx} \left[(x+2) g(x)\right] + \mathcal{O}(\eta^2).$$

Thus, we may consider the IBP relations as following from the fact that the integrals are constant on the orbits corresponding to the generators Q_{ik}.

This is the starting point of the proof that the number of MIs is finite.

Details in arXiv:1004.4199: “The reader familiar with the theory of holonomic D-modules should consider Theorem 2 as an exercise.”

Structure of the IBP relations [see R.N. Lee, arXiv:0804.3008]

Let us consider the operators $O_{ik} = \frac{\partial}{\partial q_i} r_k$ acting on the integrands J, where $r_k \in \{q_1, \ldots, q_L, p_1, \ldots p_E\}$.
They form a closed Lie algebra with the commutation relations

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}.$$

They are related to infinitesimal redefinitions of loop momenta $q_i \to q_i' = q_i + \eta_{ik} r_k$ with constant η_{ik}, under which the integrals are invariant:

$$F[D, J(q', p)] = F\,[D,\ J(q,p) + \underbrace{\eta_{ik} O_{ik} J(q,p)}_{\text{zero by IBP}}\ ] + \mathcal{O}(\eta^2).$$

A one-dimensional analogy:

$$f_\eta(x) \ = \ (1+\eta)\, g[(1+\eta)x + 2\eta] \quad \Rightarrow \quad \int_{-\infty}^{+\infty} f_\eta(x) dx \ = \ \int_{-\infty}^{+\infty} g(x) dx.$$

$$f_\eta(x) \ = \ g(x) + \eta \tfrac{d}{dx} [(x+2) g(x)] + \mathcal{O}(\eta^2).$$

Thus, we may consider the IBP relations as following from the fact that the integrals are constant on the orbits corresponding to the generators Q_{ik}.

This is the starting point of the proof that the number of MIs is finite.

Details in arXiv:1004.4199: “The reader familiar with the theory of holonomic D-modules should consider Theorem 2 as an exercise.”

Algorithms and codes for solving the IBP relations based on their “group structure” have been developed. [R.N. Lee, arXiv:1212.2685, 1310.1145]. However, they are not general.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $I_{n_1,\ldots,n_k} = f(n_1, \ldots, n_k)$

Keep the external momenta and parameters fixed, so now $f : \mathbb{Z}^k \to \mathbb{C}$.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $I_{n_1,\ldots,n_k} = f(n_1,\ldots,n_k)$

Keep the external momenta and parameters fixed, so now $f : \mathbb{Z}^k \to \mathbb{C}$.

We introduce two types of operators acting on such functions:

$$(A_\alpha f)(n_1,\ldots,n_k) = n_\alpha\, f(n_1,\ldots,n_\alpha+1,\ldots,n_k)$$
$$(B_\alpha f)(n_1,\ldots,n_k) = f(n_1,\ldots,n_\alpha-1,\ldots,n_k)$$

It is straightforward to verify that $[A_\alpha, B_\beta] = \delta_{\alpha\beta}$.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $I_{n_1,\ldots,n_k} = f(n_1,\ldots,n_k)$

Keep the external momenta and parameters fixed, so now $f : \mathbb{Z}^k \to \mathbb{C}$.

We introduce two types of operators acting on such functions:

$$(A_\alpha f)(n_1,\ldots,n_k) = n_\alpha\, f(n_1,\ldots,n_\alpha+1,\ldots,n_k)$$
$$(B_\alpha f)(n_1,\ldots,n_k) = f(n_1,\ldots,n_\alpha-1,\ldots,n_k)$$

It is straightforward to verify that $[A_\alpha, B_\beta] = \delta_{\alpha\beta}$.

Let $\mathcal{W}$ be the (Weyl) algebra of all possible polynomials in such operators.

Let $\mathcal{L}$ be the left ideal in $\mathcal{W}$ generated by P_{ij} : $(P_{ij}f)(n_1,\ldots,n_k) = F[D, O_{ij}J_{n_1,\ldots,n_k}]$.

It consists of all operators of the form $\sum_{ij} C_{ij}P_{ij}$ with $C_{ij} \in \mathcal{W}$.

All the P_{ij} have the form $a_{ij}^{\alpha\beta}A_\alpha B_\beta + b_{ij}^{\alpha}A_\alpha + c_{ij}$, with $a_{ij}^{\alpha\beta}, b_{ij}^{\alpha}, c_j \in \mathbb{C}$.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $I_{n_1,\ldots,n_k} = f(n_1,\ldots,n_k)$

Keep the external momenta and parameters fixed, so now $f : \mathbb{Z}^k \to \mathbb{C}$.

We introduce two types of operators acting on such functions:

$$(A_\alpha f)(n_1,\ldots,n_k) = n_\alpha f(n_1,\ldots,n_\alpha+1,\ldots,n_k)$$
$$(B_\alpha f)(n_1,\ldots,n_k) = f(n_1,\ldots,n_\alpha-1,\ldots,n_k)$$

It is straightforward to verify that $[A_\alpha, B_\beta] = \delta_{\alpha\beta}$.

Let $\mathcal{W}$ be the (Weyl) algebra of all possible polynomials in such operators.

Let $\mathcal{L}$ be the left ideal in $\mathcal{W}$ generated by P_{ij} : $(P_{ij}f)(n_1,\ldots,n_k) = F[D, O_{ij}J_{n_1,\ldots,n_k}]$.

It consists of all operators of the form $\sum_{ij} C_{ij}P_{ij}$ with $C_{ij} \in \mathcal{W}$.

All the P_{ij} have the form $a_{ij}^{\alpha\beta}A_\alpha B_\beta + b_{ij}^\alpha A_\alpha + c_{ij}$, with $a_{ij}^{\alpha\beta}, b_{ij}^\alpha, c_j \in \mathbb{C}$.

Let $\mathcal{R}$ be the right ideal in $\mathcal{W}$ that consists of all operators $\sum_\alpha B_\alpha A_\alpha C_\alpha$ with $C_\alpha \in \mathcal{W}$.

For any $R \in \mathcal{R}$, we have $(Rf)(1,\ldots,1) = 0$.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $I_{n_1,\ldots,n_k} = f(n_1,\ldots,n_k)$

Keep the external momenta and parameters fixed, so now $f : \mathbb{Z}^k \to \mathbb{C}$.

We introduce two types of operators acting on such functions:

$$(A_\alpha f)(n_1,\ldots,n_k) = n_\alpha f(n_1,\ldots,n_\alpha+1,\ldots,n_k)$$
$$(B_\alpha f)(n_1,\ldots,n_k) = f(n_1,\ldots,n_\alpha-1,\ldots,n_k)$$

It is straightforward to verify that $[A_\alpha, B_\beta] = \delta_{\alpha\beta}$.

Let $\mathcal{W}$ be the (Weyl) algebra of all possible polynomials in such operators.

Let $\mathcal{L}$ be the left ideal in $\mathcal{W}$ generated by P_{ij} : $(P_{ij}f)(n_1,\ldots,n_k) = F[D, O_{ij}J_{n_1,\ldots,n_k}]$.
It consists of all operators of the form $\sum_{ij} C_{ij}P_{ij}$ with $C_{ij} \in \mathcal{W}$.

All the P_{ij} have the form $a_{ij}^{\alpha\beta}A_\alpha B_\beta + b_{ij}^\alpha A_\alpha + c_{ij}$, with $a_{ij}^{\alpha\beta}, b_{ij}^\alpha, c_j \in \mathbb{C}$.

Let $\mathcal{R}$ be the right ideal in $\mathcal{W}$ that consists of all operators $\sum_\alpha B_\alpha A_\alpha C_\alpha$ with $C_\alpha \in \mathcal{W}$.
For any $R \in \mathcal{R}$, we have $(Rf)(1,\ldots,1) = 0$.

Our goal is to find a decomposition $w = L + R + r$ for any $w \in \mathcal{W}$ such that $L \in \mathcal{L}$, $R \in \mathcal{R}$, and r is the simplest possible.
Next, we focus on such w that $(wf)(1,\ldots,1) = f(n_1,\ldots,n_k)$ for all the indices of interest.

Description in terms of functions on $\mathbb{Z}^k$

[see: R.N. Lee, arXiv:0804.3008]

[more: T. Bitoun, C. Bogner, R.P. Klausen, E. Panzer, arXiv:1712:09215]

Change of notation: $I_{n_1,\ldots,n_k} = f(n_1, \ldots, n_k)$

Keep the external momenta and parameters fixed, so now $f : \mathbb{Z}^k \to \mathbb{C}$.

We introduce two types of operators acting on such functions:

$$(A_\alpha f)(n_1, \ldots, n_k) = n_\alpha\, f(n_1, \ldots, n_\alpha + 1, \ldots, n_k)$$
$$(B_\alpha f)(n_1, \ldots, n_k) = f(n_1, \ldots, n_\alpha - 1, \ldots, n_k)$$

It is straightforward to verify that $[A_\alpha, B_\beta] = \delta_{\alpha\beta}$.

Let $\mathcal{W}$ be the (Weyl) algebra of all possible polynomials in such operators.

Let $\mathcal{L}$ be the left ideal in $\mathcal{W}$ generated by P_{ij} : $(P_{ij} f)(n_1, \ldots, n_k) = F[D, O_{ij} J_{n_1,\ldots,n_k}]$.

It consists of all operators of the form $\sum_{ij} C_{ij} P_{ij}$ with $C_{ij} \in \mathcal{W}$.

All the P_{ij} have the form $a_{ij}^{\alpha\beta} A_\alpha B_\beta + b_{ij}^\alpha A_\alpha + c_{ij}$, with $a_{ij}^{\alpha\beta}, b_{ij}^\alpha, c_j \in \mathbb{C}$.

Let $\mathcal{R}$ be the right ideal in $\mathcal{W}$ that consists of all operators $\sum_\alpha B_\alpha A_\alpha C_\alpha$ with $C_\alpha \in \mathcal{W}$.

For any $R \in \mathcal{R}$, we have $(Rf)(1, \ldots, 1) = 0$.

Our goal is to find a decomposition $w = L + R + r$ for any $w \in \mathcal{W}$ such that $L \in \mathcal{L}$, $R \in \mathcal{R}$, and r is the simplest possible.

Next, we focus on such w that $(wf)(1, \ldots, 1) = f(n_1, \ldots, n_k)$ for all the indices of interest.

To my knowledge, this problem still awaits a solution for generic $k \in \mathbb{N}$ and $a_{ij}^{\alpha\beta}, b_{ij}^\alpha, c_j \in \mathbb{C}$.

Summary

- **One of the most powerful methods for computing dimensionally regularized Feynman integrals relies on expressing them in terms of MIs, with the help of IBP relations.**

Summary

- **One of the most powerful methods for computing dimensionally regularized Feynman integrals relies on expressing them in terms of MIs, with the help of IBP relations.**

- **Thanks to the IBP reduction, we are able to derive DEs for the MIs that can be solved either analytically or numerically, with initial conditions in "easy" regions.**

Summary

- **One of the most powerful methods for computing dimensionally regularized Feynman integrals relies on expressing them in terms of MIs, with the help of IBP relations.**

- **Thanks to the IBP reduction, we are able to derive DEs for the MIs that can be solved either analytically or numerically, with initial conditions in "easy" regions.**

- **Unfortunately, the IBP reduction is often computationally heavy, mainly due to simplification of huge numbers of rational functions. Further progress in understanding the algebraic structure of the IBP relations is necessary.**

Summary

- **One of the most powerful methods for computing dimensionally regularized Feynman integrals relies on expressing them in terms of MIs, with the help of IBP relations.**

- **Thanks to the IBP reduction, we are able to derive DEs for the MIs that can be solved either analytically or numerically, with initial conditions in "easy" regions.**

- **Unfortunately, the IBP reduction is often computationally heavy, mainly due to simplification of huge numbers of rational functions. Further progress in understanding the algebraic structure of the IBP relations is necessary.**

- **High parallelization of the IBP reduction can be achieved with methods employing modular algebra $\rightarrow$ next talk.**

BACKUP SLIDES

Structure of the differential equations [see J.M. Henn, arXiv:1412.2296]

Suppose our MIs depend only on two parameters: $\epsilon = (4 - D)/2$ and a single dimensionless ratio t of two kinematical variables (masses or momentum products).

Structure of the differential equations [see J.M. Henn, arXiv:1412.2296]

Suppose our MIs depend only on two parameters: $\epsilon = (4 - D)/2$ and a single dimensionless ratio t of two kinematical variables (masses or momentum products).
Let's write the MIs as N components of a vector $\psi(t, \epsilon)$. Then the DEs for them take the form:

$$\frac{\partial}{\partial t} \psi(t, \epsilon) = H(t, \epsilon)\, \psi(t, \epsilon),$$

where the $N \times N$ matrix H is a rational function of t and ϵ.

Structure of the differential equations [see J.M. Henn, arXiv:1412.2296]

Suppose our MIs depend only on two parameters: $\epsilon = (4 - D)/2$ and a single dimensionless ratio t of two kinematical variables (masses or momentum products).
Let's write the MIs as N components of a vector $\psi(t, \epsilon)$. Then the DEs for them take the form:

$$\frac{\partial}{\partial t}\psi(t,\epsilon) \; = \; H(t,\epsilon)\,\psi(t,\epsilon),$$

where the $N \times N$ matrix H is a rational function of t and ϵ.

A simplification of the DEs can often be achieved via redefining the MI basis according to $\psi = X\widetilde{\psi}$, where $X(t,\epsilon)$ is an invertible matrix. Then:

$$\frac{\partial}{\partial t}\widetilde{\psi}(t,\epsilon) \; = \; \widetilde{H}(t,\epsilon)\,\widetilde{\psi}(t,\epsilon),$$

with $\widetilde{H} = X^{-1}HX - X^{-1}\frac{\partial}{\partial t}X.$

Structure of the differential equations [see J.M. Henn, arXiv:1412.2296]

Suppose our MIs depend only on two parameters: $\epsilon = (4 - D)/2$ and a single dimensionless ratio t of two kinematical variables (masses or momentum products).
Let's write the MIs as N components of a vector $\psi(t, \epsilon)$. Then the DEs for them take the form:

$$\frac{\partial}{\partial t} \psi(t, \epsilon) \;=\; H(t, \epsilon)\, \psi(t, \epsilon),$$

where the $N \times N$ matrix H is a rational function of t and ϵ.

A simplification of the DEs can often be achieved via redefining the MI basis according to $\psi = X \widetilde{\psi}$, where $X(t, \epsilon)$ is an invertible matrix. Then:

$$\frac{\partial}{\partial t} \widetilde{\psi}(t, \epsilon) \;=\; \widetilde{H}(t, \epsilon)\, \widetilde{\psi}(t, \epsilon),$$

with $\widetilde{H} = X^{-1} H X - X^{-1} \frac{\partial}{\partial t} X$.

Dedicated codes (see, e.g., arXiv:1705.06252) search (often successfully) for such matrices X that

$$\widetilde{H}(t, \epsilon) = \epsilon \frac{d}{dt} \sum_j K_j \; \ln h_k(t),$$

where K_j are constant matrices, while the scalar functions $h_k(t)$ are called "letters".

Structure of the differential equations [see J.M. Henn, arXiv:1412.2296]

Suppose our MIs depend only on two parameters: $\epsilon = (4-D)/2$ and a single dimensionless ratio t of two kinematical variables (masses or momentum products).
Let's write the MIs as N components of a vector $\psi(t,\epsilon)$. Then the DEs for them take the form:

$$\frac{\partial}{\partial t}\psi(t,\epsilon) \;=\; H(t,\epsilon)\,\psi(t,\epsilon),$$

where the $N \times N$ matrix H is a rational function of t and ϵ.

A simplification of the DEs can often be achieved via redefining the MI basis according to $\psi = X\widetilde{\psi}$, where $X(t,\epsilon)$ is an invertible matrix. Then:

$$\frac{\partial}{\partial t}\widetilde{\psi}(t,\epsilon) \;=\; \widetilde{H}(t,\epsilon)\,\widetilde{\psi}(t,\epsilon),$$

with $\widetilde{H} = X^{-1}HX - X^{-1}\frac{\partial}{\partial t}X$.

Dedicated codes (see, e.g., arXiv:1705.06252) search (often successfully) for such matrices X that

$$\widetilde{H}(t,\epsilon) = \epsilon\,\frac{d}{dt}\,\sum_j K_j\;\ln h_k(t),$$

where K_j are constant matrices, while the scalar functions $h_k(t)$ are called "letters".

This is advantageous because in practice we are interested in Laurent expansions

$$\widetilde{\psi}(t,\epsilon) = \sum_{n=n_{\rm min}}^{\infty} \epsilon^n\,\widetilde{\psi}_n(t).$$

Next: Getting $\widetilde{\psi}_n(t)$ via iterative integration. Harmonic Polylogarithms (HPLs).