Multi-Loop Amplitudes in the High-Energy Limit in $\mathcal{N} = 4$ SYM

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CP3 - UCLouvain

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Introduction

Goal: Analyze mathematical structure of scattering amplitudes.

Ideal playground for this is $\mathcal{N}=4$ SYM.

- Conformal symmetry
- Maximally transcendental

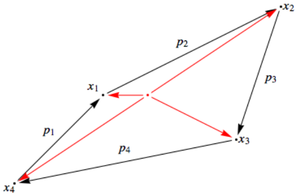
Polylogarithmic part of ℓ-loop amplitude has weight 2ℓ

$$\mathcal{A}_4^{(1)} \sim \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3}$$

The Planar Limit ('t Hooft limit)

Corresponds to $N_c \to \infty \, , a = \frac{N_c \, g^2}{8\pi^2}$ finite.

Define *dual coordinates* x_i, s.t. $p_i = x_i - x_{i-1}$



Dual conformal symmetry

[Drummond, Henn, Korchemsky, Sokatchev; Drummond, Henn, Smirnov, Sokatchev]

- Fixes all 4&5 point results
- $A_N = A_N^{\mathrm{BDS}} e^{R_N}$

[Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov]

[Drummond, Henn, Korchemsky, Sokatchev]

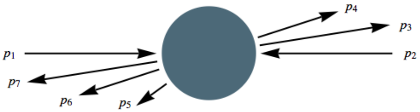
Multi Regge Kinematics (MRK)

Forward scattering:

- $s \gg |t|$
- $\mathcal{A}_4^{(1)} \sim \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3}$

Multi Regge kinematics:

- Hierarchy in rapidity
- No hierarchy in transverse plane

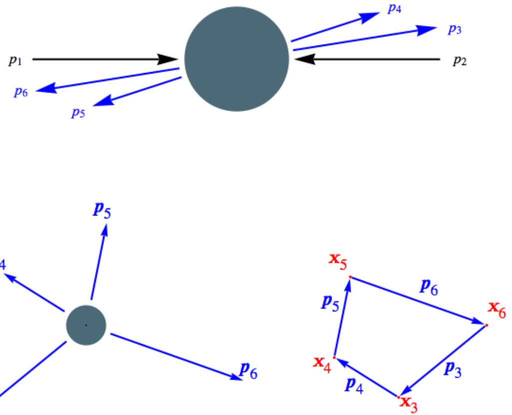


- $$R_N \sim a^2\left(\log\tau\, g^{(2)}_{N,\mathrm{LLA}} + g^{(2)}_{N,\mathrm{NLLA}}\right) + a^3\left(\log^2\tau\, g^{(3)}_{N,\mathrm{LLA}} + \log\tau\, g^{(3)}_{N,\mathrm{NLLA}} + g^{(3)}_{N,\mathrm{NNLLA}}\right) + \dots\ ,$$

  with $\log\tau \gg 1$

Multi Regge Kinematics

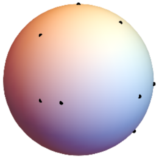
Kinematical dependence encoded in $N-2$ transverse momenta.



$$R_N \sim a^2 \left(\log \tau \, g^{(2)}_{N,\mathrm{LLA}}(\{\mathbf{x}_i\}) + g^{(2)}_{N,\mathrm{NLLA}}(\{\mathbf{x}_i\}) \right) + \ldots$$

The Riemann Sphere With m Marked Points

Riemann sphere $\cong \mathbb{C}$ (stereographic projection)



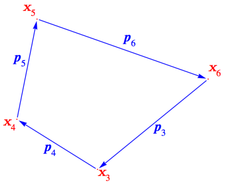
Riemann spheres with m marked points

$\Updownarrow$

'Phase space' of $N = m+2$-particle MRK amplitude.

Use $SL(2,\mathbb{C})$ symmetry to fix three points to get $N-5$ cross ratios

$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}$$



Single-Valuedness

Optical theorem tells us that branch cuts start at

$$x_{ij}^2 = (x_i - x_j)^2 = 0$$

In MRK, this corresponds to branch cuts at

$$\mathbf{x}_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2 = 0$$

In the transverse part, however

$$\mathbf{x}_{ij}^2 \geq 0$$

$\Rightarrow$ Scattering amplitudes in planar $\mathcal{N} = 4$ SYM in MRK are single-valued

Single-valued Polylogarithms

Scattering amplitudes in planar $\mathcal{N} = 4$ SYM in MRK are single-valued iterated integrals

$\rightarrow$ Use *single-valued polylogarithms* $\mathcal{G}$ [Brown]

Fulfil same differential equation as regular Polylogs

$$\partial_z G_{a_1,\dots,a_n}(z) = \frac{1}{z - a_1} G_{a_2,\dots,a_n}(z)$$

$$\downarrow$$

$$\partial_z \mathcal{G}_{a_1,\dots,a_n}(z) = \frac{1}{z - a_1} \mathcal{G}_{a_2,\dots,a_n}(z)$$

Single-valued Polylogarithms

Single-valued polylogs are combinations of regular polylogs in z and $\overline{z}$ such that all branch cuts cancel

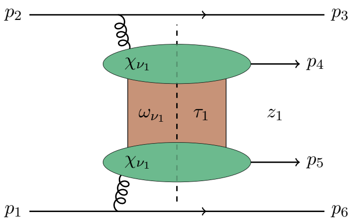
$$\begin{aligned}\mathcal{G}_0(z) &= G_0(z) + G_0(\overline{z}) \\ &= \log z + \log \overline{z} = \log |z|^2\end{aligned}$$

Bloch-Wigner dilog:

$$\begin{aligned}D_2(z) &= \operatorname{Im}\big(\operatorname{Li}_2(z) + \log|z| \log(1-z)\big) \\ &= \frac{1}{2\mathrm{i}}\left(G_{0,1}(\overline{z}) - G_{0,1}(z) + \frac{1}{2}(G_0(z) + G_0(\overline{z}))(G_1(z) - G_1(\overline{z}))\right) \\ &= \frac{1}{4\mathrm{i}}\left(\mathcal{G}_{1,0}(z) - \mathcal{G}_{0,1}(z)\right)\end{aligned}$$

The Remainder Function

The Remainder Function

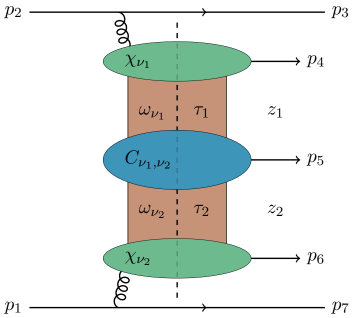


[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygarin]

$$\sim \sum_{n_1} \int \frac{\mathrm{d}\nu_1}{2\pi} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} |z_1|^{2\mathrm{i}\nu_1} \chi_1^+ \tau_1^{-\omega_1} \chi_1^-$$

$$\equiv \mathcal{F}_1 \left[\chi_1^+ \tau_1^{-\omega_1} \chi_1^- \right]$$

The Remainder Function

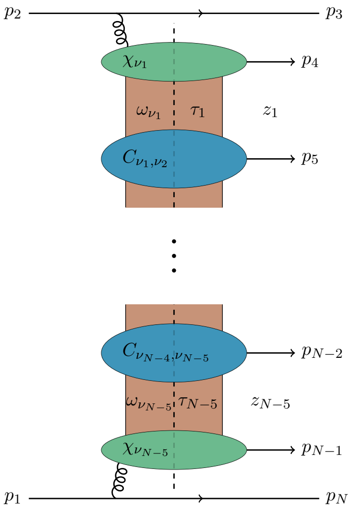


[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

$$\sim \mathcal{F}_2 \left[\mathcal{F}_1 \left[\chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^- \right] \right]$$

$$\equiv \mathcal{F}_{12} \left[\chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^- \right]$$

The Remainder Function



[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

The Remainder Function

At LLA, replace all building blocks by their LO approximation

$$\chi^{+}_{0,1}\tau_1^{aE_1}C^{+}_{0,12}\tau_2^{aE_2}\chi^{-}_{0,2}$$

$$\tau_k^{aE_k} = a\log\tau_k\,E_k + \frac{a^2}{2}\log^2\tau_k\,E_k^2 + \dots$$

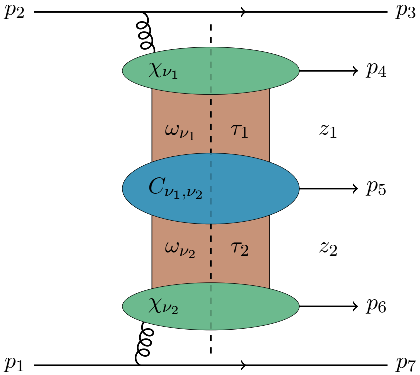
At LLA: $\ell - 1 = \#E_k$

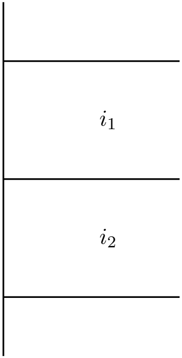
Define the *vacuum ladder*

$$\varpi_N = \chi^{+}_{0,1}C^{+}_{0,12}\cdots C^{+}_{0(N-6)(N-5)}\chi^{-}_{0,N-5}$$

An Example at 7 Points

A Graphical Representation - LLA

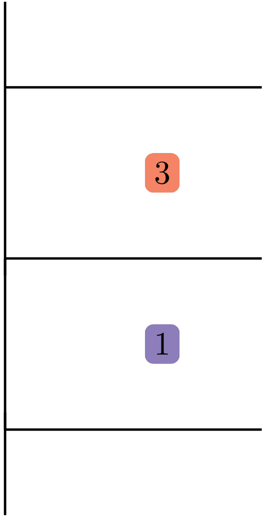




$$\mathcal{R}_{7,\mathrm{LLA}}^{(\ell)} \sim \sum_{i_1+i_2=\ell-1} \frac{a^\ell}{i_1! i_2!} \log^{i_1} \tau_1 \log^{i_2} \tau_2 \; g^{(i_1,i_2)}$$

A Graphical Representation - LLA

$$g^{(3,1)} = \mathcal{F}_{12}\left[\varpi_7 E_1^3 E_2\right] \equiv$$



Adding Loops

The number of E_k in the integrand is related to the loop order of LLA amplitudes

$\rightarrow$ Increase the loop order at a fixed logarithmic accuracy by inserting E_k into the integrand.

...but how?

Fourier-Mellin Convolutions

The Fourier-Mellin transform

$$\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}\nu}{2\pi} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} |z|^{2\mathrm{i}\nu} F(\nu,n)$$

Products are mapped to convolutions

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = \frac{1}{\pi} \int \frac{\mathrm{d}^2 w}{|w|^2} \mathcal{F}[F](w) \mathcal{F}[G]\left(\frac{z}{w}\right)$$

Use this to raise loop order

$$g^{(i_1,i_2)} = g^{(i_1-1,i_2)} * \mathcal{F}_1[E_1] = g^{(0,0)} * \mathcal{F}[E_1]^{*i_1} * \mathcal{F}_2[E_2]^{*i_2}$$

Fourier-Mellin Convolutions

$$\mathcal{F}_k[E_k] = -\frac{z_k + \overline{z}_k}{2|1 - z_k|^2}$$

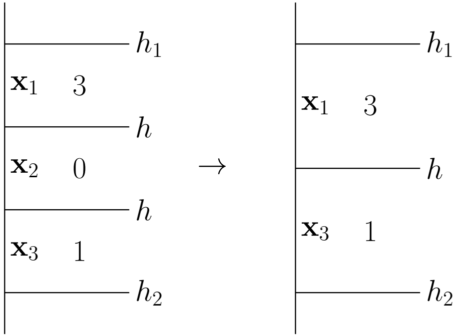
Single-valuedness allows us to solve the convolution integral by computing residues

[Schnetz]

$$\int \frac{\mathrm{d}^2 z}{\pi} f(z) = \mathrm{Res}_{z=\infty} F(z) - \sum_i \mathrm{Res}_{z=a_i} F(z) \qquad \partial_{\overline{z}} F(z) = f(z)$$

→ Increasing loop order as simple as computing residues!

Faces with no Insertions



$$f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f(\mathbf{x}_1, \mathbf{x}_3)$$

[Del Duca, Druc, Drummond, Duhr ,Dulat, RM, Papathanasiou, Verbeek]

Factorization of Remainder Functions - $N = 7$

$$R^{(3)}_{7,\mathrm{LLA}} \sim \log^2\tau_1 \left|\begin{array}{ll} \hline \mathbf{x}_1 & 2 \\ \hline \mathbf{x}_2 & 0 \\ \hline \end{array}\right. + \log^2\tau_2 \left|\begin{array}{ll} \hline \mathbf{x}_1 & 0 \\ \hline \mathbf{x}_2 & 2 \\ \hline \end{array}\right. + \log\tau_1 \log\tau_2 \left|\begin{array}{ll} \hline \mathbf{x}_1 & 1 \\ \hline \mathbf{x}_2 & 1 \\ \hline \end{array}\right.$$

$$= \log^2\tau_1 \left|\begin{array}{ll} \hline \mathbf{x}_1 & 2 \\ \hline \end{array}\right. + \log^2\tau_2 \left|\begin{array}{ll} \hline \mathbf{x}_2 & 2 \\ \hline \end{array}\right. + \log\tau_1 \log\tau_2 \left|\begin{array}{ll} \hline \mathbf{x}_1 & 1 \\ \hline \mathbf{x}_2 & 1 \\ \hline \end{array}\right.$$

Factorization of Remainder Functions - $N = 8$

$$R^{(3)}_{8,\mathrm{LLA}} \sim \log^2\tau_1 \left|\begin{array}{l}\hline \mathbf{x}_1\ \ 2 \\ \hline \end{array}\right. + \log^2\tau_2 \left|\begin{array}{l}\hline \mathbf{x}_2\ \ 2 \\ \hline \end{array}\right. + \log^2\tau_3 \left|\begin{array}{l}\hline \mathbf{x}_3\ \ 2 \\ \hline \end{array}\right.$$

$$+ \log\tau_1\log\tau_2 \left|\begin{array}{l}\hline \mathbf{x}_1\ \ 1 \\ \hline \mathbf{x}_2\ \ 1 \\ \hline \end{array}\right. + \log\tau_1\log\tau_3 \left|\begin{array}{l}\hline \mathbf{x}_1\ \ 1 \\ \hline \mathbf{x}_3\ \ 1 \\ \hline \end{array}\right. + \log\tau_2\log\tau_3 \left|\begin{array}{l}\hline \mathbf{x}_2\ \ 1 \\ \hline \mathbf{x}_3\ \ 1 \\ \hline \end{array}\right.$$

Factorization of Remainder Functions

Reminder:

$$\text{LLA} \Leftrightarrow \sum i_k = \ell - 1$$

$\rightarrow$ can only have limited number of 'insertions' at a given order

At MHV, a finite number of building blocks are enough to describe the ℓ-loop amplitude for all numbers of particles

Beyond MHV and LLA

Going beyond MHV - The Ratio Function

Can flip helicities with convolutions, too!

$$\mathcal{F}_{12}\left[\chi_1^- C_{12}^+ \chi_2^-\right] = \mathcal{F}_{12}\left[\chi_1^+ C_{12}^+ \chi_2^-\right] * \mathcal{F}_1\left[\frac{\chi_1^-}{\chi_1^+}\right]$$

$$\mathcal{F}_1\left[\frac{\chi_1^-}{\chi_1^+}\right] = -\frac{z_1}{(1-z_1)^2}$$

Beyond MHV, replace $e^{R_N} \to \mathcal{R}_N \equiv \frac{A_N}{A_N^{\mathrm{BDS}}}$

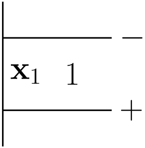
Going beyond MHV - The Ratio Function

Certain amplitudes will never factorize . . .

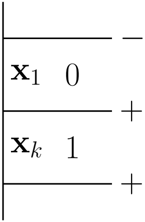
- $\mathcal{R}_{N,+-+-\dots}$ won't simplify

. . . but others still do

$$\mathcal{R}^{(2)}_{N,-+\dots+} \sim \log \tau_1$$



$$+ \sum_{k=2}^{N-5} \log \tau_k$$



Extension to N^{k}LLA

Formalism should still be applicable

- Has been applied at NLLA arXiv:1801.10605
- No conceptually new hurdles beyond NLLA

Corrections to BFKL building blocks prohibit factorization in certain faces, but their number is also limited at a given logarithmic order

$\rightarrow$ MHV amplitudes for all N still have finite number of building blocks

Conclusions

- Developed a framework that allows us to compute virtually any amplitude in MRK in $\mathcal{N} = 4$ SYM at LLA and NLLA
- A finite set of building blocks encodes amplitudes for any number of particles.
- Computed the building blocks to yield:
  - All MHV 5-loop amplitudes at LLA
  - 8-point LLA amplitudes for any helicity configuration up to 4 loops
  - All MHV 3-loop amplitudes at NLLA (Not yet published)
  - 7-point NLLA NMHV up to 3 loops
- Determined the function space of MRK amplitudes at LLA and NLLA to all orders