Symmetric ϵ- and $(\epsilon + 1/2)$-form and algebraic constraints in "elliptic" sectors

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Outline

Reminder

Criterion of (ir)reducibility

Two important observations.

Examples

Summary

Reminder

- IBP identities[Chetyrkin&Tkachov'81]
  $0 = \int \mathrm{d}^d l_1 \ldots \mathrm{d}^d l_L \, \partial_{l_i} \cdot q_j \prod_{\alpha=1}^{N} D_\alpha^{-n_\alpha}$
  - heuristic solutions, Laporta algorithm, finite fields, syzygies,...
  - Résumé: Variety of approaches to IBP reduction but we still want more.

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$$\boldsymbol{j} = (j_1(\boldsymbol{n}_1), \ldots, j_K(\boldsymbol{n}_1))^\intercal.$$

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  IBP reduction leads to a finite set of master integrals
  $$\boldsymbol{j} = (j_1(\boldsymbol{n}_1), \ldots, j_K(\boldsymbol{n}_1))^\intercal.$$
- Differential equations [Kotikov'91; Remiddi'97]
  $\frac{\partial}{\partial(p_1 \cdot p_2)} j(\boldsymbol{n}) = \sum \left[G^{-1}\right]_{i2} p_i \cdot \partial_{p_1} j(\boldsymbol{n})$. Using IBP reduction, one obtains the differential system for master integrals:
  $$\partial_x \boldsymbol{j}(x,\epsilon) = M(x,\epsilon)\boldsymbol{j}(x,\epsilon)$$
  - We need a few first coefficients $\boldsymbol{j}_n(x)$ in $\boldsymbol{j}(x,\epsilon) = \sum_{n=n_0}^{\infty} \epsilon^n \boldsymbol{j}_n(x)$.
  - The problem greatly simplifies [Henn'13] if masters $J(x,\epsilon)$ are chosen such that
  $$\partial_x \boldsymbol{J}(x,\epsilon) = \epsilon S(x) \boldsymbol{J}(x,\epsilon)$$

Reduction to ϵ-form

Given a differential system

$$\partial_x \boldsymbol{j}(x,\epsilon) = M(x,\epsilon)\boldsymbol{j}(x,\epsilon)$$

find the variable change

$$x = f(y)$$

and the function change

$$\boldsymbol{j}(x,\epsilon) = T(y,\epsilon)\boldsymbol{J}(y,\epsilon)\,,$$

where f and entries of T are rational functions of y, such that

$$\partial_y \boldsymbol{J}(y,\epsilon) = \epsilon S(y)\boldsymbol{J}(y,\epsilon)\,, \qquad (\epsilon\text{-form})$$

or prove that (f, T) does not exist.[1]

[1] We will also require that $S(y)$ has only simple poles and decays at ∞.

Algorithm

The algorithm was introduced in [RL'15] and a few important improvements [RL& Pomeransky'17]

- Reduce the system to Fuchsian form.
- Normalize residues
  - Find proper variable $x = f(y)$, $f(y)$ is a rational function.
  - (Ir)reducibility criterion: check if the system can be reduced.
- Factor out ϵ-dependence.

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Now the algorithm is implemented (except the latest improvements) in two public packages, `Fuchsia`[Gitiliar&Mageria'17] and `epsilon` [Prausa'17].

Irreducible cases

Had we been able to always find the ϵ-form, the problem of the evaluation of the multiloop integrals, essentially, would be solved (with some reservations of course).
However there are some nasty examples where the ϵ-form can not be achieved.

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Have we tried hard enough?

Strict criterion of irreducibility is very welcome. This criterion was derived in [RL& Pomeransky (arXiv:1707.07856)].

Criterion of (ir)reducibility[RL& Pomeransky'17]

The (ir)reducibility criterion is based on a simple but improtant

Proposition

Suppose the matrix $M(x, e)$ is normalized Fuchsian at $x = a$, i.e.,

$$M(x,\epsilon) = \frac{S(\epsilon)}{x-a} + O\left((x-a)^0\right), \quad \text{all } \underline{\text{evs}} \text{ of } S(\epsilon) \text{ are } \propto \epsilon.$$

Then, the transformation $T(x)$ preserves fuchsianity and normalization $\Leftrightarrow$ T is regular at $x = a$, i.e. $T(x,\epsilon) \overset{x\to a}{\longrightarrow} T(a,\epsilon) < \infty$ and $\det T(a,\epsilon) \neq 0$.

In particular

- If M is normalized in all points, T is independent of x.
- If M is normalized in all points but $x = 0$, $T(x)$ and $T^{-1}(x)$ are both polynomial in x^{-1}.
- If M is normalized in all points but $x = 0$ and $x = \infty$, $T(x)$ and $T^{-1}(x)$ are both Laurent polynomial in x.

Criterion of (ir)reducibility[RL& Pomeransky'17]

- Pick two arbitrary singular points x_1 and x_2 and map them onto 0 and ∞ with Moebius transformation of variable: $x = \frac{x_1 + x_2 y}{1+y}$.

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- Let $U = T_\infty^{-1} T_0$. Then, necessarily, either the system is not reducible, or U decomposes as

$$U = Q_\infty(y) Q_0^{-1}\left(y^{-1}\right),$$

where Q_∞ and Q_0 are polynomial in their arguments, together with their inverse matrices.

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  where Q_∞ and Q_0 are polynomial in their arguments, together with their inverse matrices.
- Finding such a decomposition is a variant of the Riemann-Hilbert problem and can be done via simple algorithm (see [arXiv:1707.07856] for details).

Criterion of (ir)reducibility[RL& Pomeransky'17]

- If the decomposition exists, the normalized form is achieved by the transformation

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- If the decomposition does not exist, ϵ-form can not be found not only by the transformation rational in y, but by any transformation rational in z, related to y as $y = g(z)$ (g is an arbitrary rational function).
- If the third step, *factorization*, fails, the ϵ-form does not exists. By "fails" we mean that there is no invertible matrix T among the solutions of (overdetermined) linear system

$$T(\epsilon,\mu)(S_i(\mu)/\mu) = (S_i(\epsilon)/\epsilon)\,T(\epsilon,\mu)\,.$$

Multiscale integrals

Let us remark about the application of the reduction algorithm in multiscale setup. We have now several differential systems

$$\partial_i \boldsymbol{J} = M_i(\boldsymbol{x}, \epsilon) J$$

suppose we have managed to reduce the first system to ϵ-form:

$$\partial_1 \boldsymbol{J} = \epsilon S_1(\boldsymbol{x}) J$$

What transformations T can we use for the remaining systems?

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What transformations T can we use for the remaining systems?

- thanks to the formulated proposition, T can not depend on x_1.
- ϵ-dependence is likely to be factorized into a common factor.

So, $T = f(x_2, \ldots, x_n, \epsilon)\,\tilde{T}(x_2, \ldots x_n)$.

This allows one to use one-by-one approach: when passing to the next differential system consider only the transformations independent of all previous variables and depending on ϵ only via common factor.

Irreducible cases

If the differential system is reduced to ϵ-form, the general solution is readily expressed via multiple polylogarithmic function which have convenient representations (iterated integrals, iterated sums), well-understood analytical properties and can be effectively evaluated with arbitrary precision.

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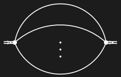
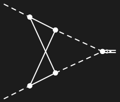
However there are some bad guys:

- $(L > 1)$-loop massive sunrises: 
- Two-loop nonplanar vertex with massive loop 
- Many other topologies (in particular, those presented by Hjalte Frellesvig yesterday).

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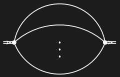
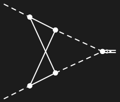
- $(L > 1)$-loop massive sunrises: 
- Two-loop nonplanar vertex with massive loop 
- Many other topologies (in particular, those presented by Hjalte Frellesvig yesterday).

What is the appropriate class of functions for irreducible DEs?

What counts: known analytic properties, minimal basis of functions, possibility to calculate efficiently with arbitrary precision.

Irreducible cases

Iterated elliptic and modular integrals

For some cases the question raised has an answer (or variants of answers): the ϵ-expansion can be expressed via iterated integrals over modular forms or via elliptic polylogs (Weinzierl, Bogner, Adams, Tancredi, Primo, Broedel, Duhr,...). Mostly these findings are applied to 2-loop massive sunrise and its supertopologies. However, it looks fair to say that **no general prescription of how to deal with irreducible cases exists**.

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My goal here

"Minimality" of the set of the functions, in particular, means the absence of the undiscovered algebraic relations. The result of the present work is the discovery of a large set of the quadratic identities for the terms of the ϵ expansion of the homogeneous solutions.

Symmetric ϵ- and $(\epsilon + 1/2)$- forms

Remark

Note that if ϵ-form of the differential system exists near $d = 4$, it necesarily exists near any even d, and vice versa.
In contrast, near odd d the ϵ-form can or can not exist independently.

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Observation 1

For many known irreducible cases (i.e., when ϵ-form does not exist near even d), there exists ϵ-form near odd d. To preserve the meaning of ϵ as twice a deviation from $d = 4$, we will call the latter the $(\epsilon + 1/2)$-form.

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Observation 2

The matrix S in the ϵ- or $(\epsilon + 1/2)$-forms can be chosen symmetric.

Symmetric ϵ- and $(\epsilon + 1/2)$- forms

Both properties can be checked for each specific topology by already available methods.

- Existence of $(\epsilon + 1/2)$-form can be established by the algorithm of Refs. [RL'15, RL& Pomeransky'17]

Symmetric ϵ- and $(\epsilon + 1/2)$- forms

Both properties can be checked for each specific topology by already available methods.

- Existence of $(\epsilon + 1/2)$-form can be established by the algorithm of Refs. [RL'15, RL& Pomeransky'17]
- When ϵ- or $(\epsilon + 1/2)$-form is achieved,

$$\partial \boldsymbol{J} = \mu S(x) \boldsymbol{J}, \quad (\mu = \epsilon \text{ or } \epsilon + 1/2),$$

one can search for constant transformation L which results in symmetric $\tilde{S} = L^{-1} S L$. Since we want $\tilde{S}^\intercal = \tilde{S}$, we have

$$L^{-1} S L = L^\intercal S^\intercal L^{\intercal -1}$$

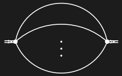
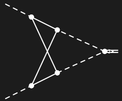
Multiplying by $L \times \bullet \times L^\intercal$, we obtain a system of linear equations

$$S\mathcal{L} = \mathcal{L} S^\intercal$$

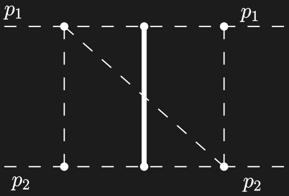
for the elements of the symmetric matrix $\mathcal{L} = L L^\intercal$. When $\mathcal{L}$ is found, L can be obtain using Cholesky-type decomposition.

Symmetric ϵ- and $(\epsilon + 1/2)$- forms

Both properties appear to hold in many available examples. In particular, for all examples reducible to ϵ-form that I have checked so far, the symmetric ϵ-form exists. For irreducible cases I have checked the existence of the symmetric $(\epsilon + 1/2)$ for

- $(L = 2, 3, 4, \ldots)$-loop equal-mass sunrises 
- Two-loop nonplanar vertex 
- Found only one exception: 3-loop forward box from

[Mistlberger (arXiv:1802.00833)]



Quadratic constraints for ϵ-form I

Suppose first that we have achieved symmetric ϵ-form:

$$\partial \boldsymbol{J} = \epsilon S \boldsymbol{J}\,, \quad S^{\intercal} = S\,.$$

As we are interested in the constraints for general solution of this equation, it is convenient to pass to fundamental matrix F, satisfying the same equation

$$\partial F = \epsilon S F\,.$$

What algebraic constraints can we obtain for the ϵ-expansion of F? Let us write the general solution as path-ordered exponent:

$$F(x, x_0, \epsilon) = \mathrm{Pexp}[\epsilon \int\limits_{x_0}^{x} dx S(x)]$$

Quadratic constraints for ϵ-form II

Now note that

$$[F(x, x_0, \epsilon)]^{-1} = \left[\text{Pexp}[-\epsilon \int_{x_0}^{x} dx S^{\intercal}(x)]\right]^{\intercal} = F^{\intercal}(x, x_0, -\epsilon) \qquad (1)$$

Therefore, we have a constraint

$$F^{\intercal}(x, x_0, -\epsilon) F(x, x_0, \epsilon) = I\,.$$

Note that the ϵ-expansion of $F^{\intercal}(x, x_0, -\epsilon)$ is the same, up to an alternating sign, as that of $F^{\intercal}(x, x_0, \epsilon)$, so we have constraints for each order in ϵ. Expressing $F(x, x_0, \epsilon)$ via generalized polylogs, we obtain constraints for the latter.

It is quite expected that these constraints should not give any unknown relations between the polylogarithmic functions. And indeed, for several examples e have checked these constraints follow from known shuffling algebra.

Quadratic constraints for $(\epsilon+\frac{1}{2})$-form I

But for irreducible cases similar constraints would be very meaningful!

Suppose that we have managed to reduce ϵ-irreducible case to symmetric $(\epsilon+\frac{1}{2})$-form:

$$\partial F = (\epsilon+\tfrac{1}{2})SF\,.$$

Let us try to follow the same path as before. Writing the general solution as path-ordered exponent:

$$F(x,x_0,\epsilon) = \mathrm{Pexp}[(\epsilon+\tfrac{1}{2})\int_{x_0}^{x} dx S(x)]\,,\quad S^{\intercal} = S\,,$$

we obtain for the inverse matrix

$$[F(x,x_0,\epsilon)]^{-1} = \left[\mathrm{Pexp}[-(\epsilon+\tfrac{1}{2})\int_{x_0}^{x} dx S^{\intercal}(x)]\right]^{\intercal} = F^{\intercal}(x,x_0,-\epsilon-1)$$

Quadratic constraints for $(\epsilon + \frac{1}{2})$-form II

So we have

$$F^{\intercal}(x, x_0, -\epsilon-1)F(x, x_0, \epsilon) = I \quad \text{or}$$
$$F^{\intercal}(x, x_0, -\epsilon)F(x, x_0, \epsilon-1) = I$$

Problem

In contrast to the ϵ-reducible case the ϵ-expansion of $F^{\intercal}(x, x_0, -\epsilon-1)$ is not directly expressed via that of $F(x, x_0, \epsilon)$ due to -1 shift in the argument.

Quadratic constraints for $(\epsilon + \frac{1}{2})$-form III

Solution

Fortunately, we have dimension shifting relations [Tarasov'96]:

$$J(x, \epsilon - 1) = R(x, \epsilon) J(\epsilon)$$

which, for F, translates to

$$F(x, x_0, \epsilon - 1) = R(x, \epsilon) F(x, x_0, \epsilon) R^{-1}(x_0, \epsilon) \qquad (*)$$

The matrix $R(x, \epsilon)$ is rational in x and ϵ and can be routinely found via IBP reduction.

Using (*), we obtain

$$F^{\mathsf{T}}(x, x_0, -\epsilon) R^{\mathsf{T}}(x, -\epsilon) F(x, x_0, \epsilon) = R^{\mathsf{T}}(x_0, -\epsilon)$$
$$F^{\mathsf{T}}(x, x_0, -\epsilon) R(x, \epsilon) F(x, x_0, \epsilon) = R(x_0, \epsilon)$$

The two above constraints seem to be the same since for all cases we have checked we observed that $R^{\mathsf{T}}(x, -\epsilon) \propto R(x, \epsilon)$.

Quadratic constraints for $(\epsilon + \frac{1}{2})$-form, summary I

Let us summarize what we have obtained.

Complex object

For each case reducible to symmetric $(\epsilon + \frac{1}{2})$-form (including, in particular, cases irreducible to ϵ-form) we have the formal solution

$$F(x, x_0, \epsilon) = \mathrm{Pexp}[(\epsilon + \tfrac{1}{2}) \int_{x_0}^{x} dx S(x)] ,$$

whose ϵ-expansion is no more simple and includes non-polylogarithmic functions. For a few known cases those appear to be iterated integrals over modular forms or elliptic functions.

Quadratic constraints for $(\epsilon + \frac{1}{2})$-form, summary II

Simple constraints

Nevertheless, we have a simple way to obtain quadratic constraints

$$F^{\intercal}(x, x_0, -\epsilon) R(x, \epsilon) F(x, x_0, \epsilon) = R(x_0, \epsilon)$$

for any order in ϵ with coefficients being rational functions which can be determined using standard procedures.

Remark I

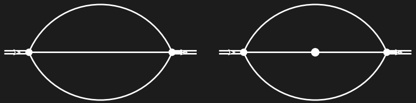
The constraints can be written directly for any two solutions $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ (including the case $\boldsymbol{J}_1 = \boldsymbol{J}_2$) as

$$\boldsymbol{J}_1^{\intercal}(x, -\epsilon) R(x, \epsilon) \boldsymbol{J}_2(x, \epsilon) = \mathrm{const}(\epsilon)$$

Remark II

The constraints for multivariate setup have literally the same form.

Example I: 2-loop sunrise for $d = 2 - 2\epsilon$ I



The homogeneous differential system has the form

$$\partial_s \boldsymbol{j}(s,\epsilon) = \begin{bmatrix} -\frac{2\epsilon+1}{s} & -\frac{3}{s} \\ -\frac{(s-3)(2\epsilon+1)(3\epsilon+1)}{(s-9)(s-1)s} & -\frac{s^2\epsilon+s^2+10s\epsilon-27\epsilon-9}{(s-9)(s-1)s} \end{bmatrix} \boldsymbol{j}(s,\epsilon)\,,$$

This system can not be reduced to ϵ-form but can be reduced to $(\epsilon + 1/2)$-form. We pass to the variable $x = \sqrt{s}$ and apply the algorithm of [RL'14]. Then we search for the constant matrix L to

Example I: 2-loop sunrise for $d = 2 - 2\epsilon$ II

obtain the symmetric $(\epsilon + \frac{1}{2})$-form. This gives us the following transformation

$$\boldsymbol{j}(x^2, \epsilon) = T(x, \epsilon)\boldsymbol{J}(x, \epsilon),$$

$$T(x, \epsilon) = \frac{4^\epsilon \Gamma\left(\epsilon + \frac{1}{2}\right)\Gamma(3\epsilon + 1)}{\sqrt{\pi}\Gamma(\epsilon + 1)} \begin{pmatrix} 1 & \frac{\sqrt{3}}{x} \\ 0 & -\frac{3\epsilon+1}{\sqrt{3}x} \end{pmatrix},$$

where $\boldsymbol{J}(x, \epsilon)$ are the new functions. The overall factor $\frac{4^\epsilon \Gamma\left(\epsilon+\frac{1}{2}\right)\Gamma(3\epsilon+1)}{\sqrt{\pi}\Gamma(\epsilon+1)}$ in the definition of $T(x, \epsilon)$ is not important for the form of the resulting differential system, but simplifies the matrix $R(x, \epsilon)$ entering the dimensional recurrence system. The differential system and dimensional recurrence relations have the forms

$$\partial_x \boldsymbol{J}(x, \epsilon) = \left(\epsilon + \frac{1}{2}\right) S(x)\boldsymbol{J}(x, \epsilon),$$

$$\boldsymbol{J}(x, \epsilon - 1) = R(x, \epsilon)\boldsymbol{J}(x, \epsilon),$$

Example I: 2-loop sunrise for $d = 2 - 2\epsilon$ III

where

$$S(x) = \begin{bmatrix} -\frac{4x(x^2-7)}{(x^2-9)(x^2-1)} & \frac{4\sqrt{3}(x^2-3)}{(x^2-9)(x^2-1)} \\ \frac{4\sqrt{3}(x^2-3)}{(x^2-9)(x^2-1)} & -\frac{2(x^4+4x^2-9)}{x(x^2-9)(x^2-1)} \end{bmatrix}$$

$$R(x,\epsilon) = \begin{bmatrix} (x^4 - 30x^2 + 45)\,\epsilon & -\sqrt{3}\frac{(x^2-9)(x^2-1)+2(x^4-9)\epsilon}{x} \\ \sqrt{3}\frac{(x^2-9)(x^2-1)-2(x^4-9)\epsilon}{x} & -\frac{3(5x^4-30x^2+9)\epsilon}{x^2} \end{bmatrix},$$

Note that $R(x,\epsilon)$ is a linear function of ϵ with the property $R(x,\epsilon) = -R^{\mathsf{T}}(x,-\epsilon)$. The ϵ-expansion of the (cut) sunrise integral is known in terms of the iterated integrals over modular forms. The first two terms are expressed via complete elliptic integrals K and E.

We have checked that the quadratic constraints for two first orders in ϵ lead to the Legendre relation for the elliptic integrals:

$$\mathrm{K}\mathrm{E}' + \mathrm{E}\mathrm{K}' - \mathrm{K}\mathrm{K}' = \frac{\pi}{2}$$

Example I: 2-loop sunrise for $d = 2 - 2\epsilon$ IV

Remarkably, in Ref. [Tarasov'06] the sunrise has been calculated exactly in d in terms of the hypergeometric function. In particular, the two solutions of the homogeneous system have been found:

$$j_1^{(1)}(s,\epsilon) = \frac{\left(-\frac{s}{(s-1)^2}\right)^{\epsilon}}{s+3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1-\epsilon;y\right) \tag{2}$$

$$j_1^{(2)}(s,\epsilon) = \frac{(9-s)^{-2\epsilon}}{s+3}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\epsilon+1;y\right)\,, \tag{3}$$

where

$$y = \frac{27(s-1)^2}{(s+3)^3}\,.$$

So we can see how the exact constraints look like. The combinations $\boldsymbol{J}^{(a)\intercal}(x,-\epsilon)R(x,\epsilon)\boldsymbol{J}^{(b)}(x,\epsilon)$ for various a and b are

Example I: 2-loop sunrise for $d = 2 - 2\epsilon$ V

independent of x (here $\boldsymbol{J}^{(a)} = T^{-1}\boldsymbol{j}^{(a)}$, $a = 1, 2$). The constants can be easily fixed by taking the limit $x \to 0$. We have

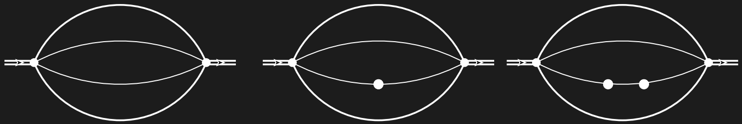
$$\boldsymbol{J}^{(1)\intercal}(-\epsilon)R(\epsilon)\boldsymbol{J}^{(1)}(\epsilon) = -\boldsymbol{J}^{(2)\intercal}(-\epsilon)R(\epsilon)\boldsymbol{J}^{(2)}(\epsilon) = \tfrac{1}{3}\epsilon\sin(3\pi\epsilon)\cot(\pi\epsilon),$$
$$\boldsymbol{J}^{(1)\intercal}(-\epsilon)R(\epsilon)\boldsymbol{J}^{(2)}(\epsilon) = \boldsymbol{J}^{(2)\intercal}(-\epsilon)R(\epsilon)\boldsymbol{J}^{(1)}(\epsilon) = 0\,.$$

The two first constraints result in the following curious identity

$$\begin{aligned}
&{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1-\epsilon; y\right)\,{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \epsilon+1; y\right) \\
&\quad + \tfrac{(y-1)}{3\epsilon}\,{}_2F_1\left(\tfrac{2}{3}, \tfrac{4}{3}; 1-\epsilon; y\right)\,{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; \epsilon+1; y\right) \\
&\quad + \tfrac{(1-y)}{3\epsilon}\,{}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1-\epsilon; y\right)\,{}_2F_1\left(\tfrac{2}{3}, \tfrac{4}{3}; \epsilon+1; y\right) = 1 \quad (4)
\end{aligned}$$

Indeed, this identity is valid, which can be checked independently by first differentiating it and then finding the constant, e.g., via substitution $y \to 0$.

Example II: 3-loop sunrise for $d = 2 - 2\epsilon$ I



The differential system and dimensional recurrence relations for the new functions $\boldsymbol{J}(x,\epsilon)$ have the forms

$$\partial_x \boldsymbol{J}(x,\epsilon) = \left(\epsilon + \tfrac{1}{2}\right) S(x)\boldsymbol{J}(x,\epsilon)\,,$$
$$\boldsymbol{J}(x,\epsilon-1) = R(x,\epsilon)\boldsymbol{J}(x,\epsilon)\,,$$

Example II: 3-loop sunrise for $d = 2 - 2\epsilon$ II

where

$$S(x) = S^{\mathsf{T}}(x) = \begin{bmatrix} -\frac{5x^2-8}{x(x^2-4)} & \frac{2\sqrt{6}}{x^2-4} & -\frac{\sqrt{3}x}{x^2-4} \\ \frac{2\sqrt{6}}{x^2-4} & -\frac{4x\left(x^2-10\right)}{(x^2-16)(x^2-4)} & \frac{2\sqrt{2}\left(5x^2-32\right)}{(x^2-16)(x^2-4)} \\ -\frac{\sqrt{3}x}{x^2-4} & \frac{2\sqrt{2}\left(5x^2-32\right)}{(x^2-16)(x^2-4)} & -\frac{\left(x^2+8\right)\left(3x^2-16\right)}{x(x^2-16)(x^2-4)} \end{bmatrix}$$

$$R(x,\epsilon) = R_0(x) + \epsilon R_1(x) + \epsilon^2 R_2(x)$$

Example II: 3-loop sunrise for $d = 2 - 2\epsilon$ III

$$R_0(x) = \begin{pmatrix} \frac{1}{4}x^2\left(x^2-8\right) & \frac{x\left(3x^2-32\right)}{2\sqrt{6}} & -\frac{x^4-28x^2+128}{4\sqrt{3}} \\ \frac{x\left(3x^2-32\right)}{2\sqrt{6}} & \frac{1}{3}\left(x^4-28x^2+64\right) & -\frac{x\left(5x^2-16\right)}{6\sqrt{2}} \\ -\frac{x^4-28x^2+128}{4\sqrt{3}} & -\frac{x\left(5x^2-16\right)}{6\sqrt{2}} & -\frac{1}{12}x^2\left(3x^2-32\right) \end{pmatrix}$$

$$R_1(x) = \begin{pmatrix} 0 & \frac{x\left(x^2-64\right)\left(x^2-6\right)}{2\sqrt{6}} & -\frac{5\left(x^2-8\right)\left(x^2+8\right)}{2\sqrt{3}} \\ -\frac{x\left(x^2-64\right)\left(x^2-6\right)}{2\sqrt{6}} & 0 & -\frac{\left(x^2-16\right)\left(x^4+42x^2-64\right)}{6\sqrt{2}x} \\ \frac{5\left(x^2-8\right)\left(x^2+8\right)}{2\sqrt{3}} & \frac{\left(x^2-16\right)\left(x^4+42x^2-64\right)}{6\sqrt{2}x} & 0 \end{pmatrix}$$

$$R_2(x) = \begin{pmatrix} -\frac{1}{16}x^2\left(x^4-104x^2+832\right) & \frac{x\left(x^2-16\right)\left(x^2+20\right)}{\sqrt{6}} & -\frac{\left(x^2-16\right)\left(x^4-40x^2-192\right)}{16\sqrt{3}} \\ \frac{x\left(x^2-16\right)\left(x^2+20\right)}{\sqrt{6}} & \frac{8}{3}\left(x^4-56x^2+64\right) & \frac{x^6-76x^4-256x^2+1024}{3\sqrt{2}x} \\ -\frac{\left(x^2-16\right)\left(x^4-40x^2-192\right)}{16\sqrt{3}} & \frac{x^6-76x^4-256x^2+1024}{3\sqrt{2}x} & -\frac{x^8-8x^6+3392x^4-20480x^2+16384}{48x^2} \end{pmatrix}.$$

Example II: 3-loop sunrise for $d = 2 - 2\epsilon$ IV

Only the ϵ^0 term is known [Primo&Tancredi17]:

$$j_1^{(1)}(s) = \mathrm{K}_1\mathrm{K}_2\,,\ j_1^{(2)}(s) = \mathrm{K}_1\mathrm{K}_3\,,\ j_1^{(3)}(s) = \mathrm{K}_4\mathrm{K}_3\,,$$

$$\omega_\pm = \frac{1}{2} + \frac{s-8}{32}\sqrt{4-s} \pm \frac{s}{32}\sqrt{16-s}\,,$$

$$\mathrm{K}_{1,2} = \mathrm{K}(\omega_\pm)\,,\ \mathrm{K}_{3,4} = \mathrm{K}(1-\omega_\pm)\,,\ \mathrm{E}_{1,2} = \mathrm{E}(\omega_\pm)\,,\ \mathrm{E}_{3,4} = \mathrm{E}(1-\omega_\pm)\,.$$

Already for this term the constraints are highly nontrivial:

$$\begin{aligned}
3\mathrm{K}_2\mathrm{K}_3 - \mathrm{K}_1\mathrm{K}_4 &= 0,\\
-6\mathrm{E}_2\mathrm{K}_1 + 2\mathrm{E}_1\mathrm{K}_2 y^2 - \mathrm{K}_1\mathrm{K}_2(y-3)(y+1) &= 0,\\
18\mathrm{E}_2\mathrm{K}_3 - 2\mathrm{E}_1\mathrm{K}_4 y^2 + \mathrm{K}_1\mathrm{K}_4(y-3)(y+1) &= 0,\\
-6\mathrm{E}_4\mathrm{K}_1 + 6\mathrm{E}_3\mathrm{K}_2 y^2 - \mathrm{K}_1\mathrm{K}_4(y-1)(y+3) &= 0,\\
6\mathrm{E}_4\mathrm{K}_3 - 2\mathrm{E}_3\mathrm{K}_4 y^2 + \mathrm{K}_3\mathrm{K}_4(y-1)(y+3) &= 0,\\
4y^2\left(3\mathrm{E}_3\mathrm{K}_2 + \mathrm{E}_1\mathrm{K}_4 - \mathrm{K}_1\mathrm{K}_4\right)^2 - 9\pi^2 &= 0\,.
\end{aligned}$$

Example III: Broadhurst' quadratic relations for IKM

Let us introduce notation

$$\mathrm{IKM}(N_1, N_2, M) = \int [I(x)]^{N_1} [K(x)]^{N_2} x^M \, dx \,,$$

where $I(x)$ and $K(x)$ are modified Bessel functions, $0 < N_1 < N_2$. These integrals are related to $N_1 + N_2 - 1$-loop fully massive tadpole integrals.

Example III: Broadhurst' quadratic relations for IKM

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where $I(x)$ and $K(x)$ are modified Bessel functions, $0 < N_1 < N_2$. These integrals are related to $N_1 + N_2 - 1$-loop fully massive tadpole integrals. More precisely, the column

$$P = \begin{pmatrix} \mathrm{IKM}(N_1, N_2, 1) \\ \mathrm{IKM}(N_1, N_2, 3) \\ \vdots \\ \mathrm{IKM}(N_1, N_2, 2\lfloor \frac{L}{2} \rfloor - 1) \end{pmatrix}$$

is the solution of the homogeneous part of DE for $L = N_1 + N_2 - 1$-loop tadpole (for suitable choice of masters).

Example III: Broadhurst' quadratic relations for IKM

David Broadhurst has conjectured the following quadratic relations

$$P^{\intercal}(N_1, N - N_1) D P(M_1, N - M_1) = \pi^{N - N_1 - M_1 + 1} B$$

where D and B are $N \times N$ matrices with elements being rational numbers (constructed in a specified way).

Example III: Broadhurst' quadratic relations for IKM

David Broadhurst has conjectured the following quadratic relations

$$P^{\mathsf{T}}(N_1, N-N_1)DP(M_1, N-M_1) = \pi^{N-N_1-M_1+1}B$$

where D and B are $N \times N$ matrices with elements being rational numbers (constructed in a specified way).
Some examples of Broadhurst relations:

2 loops: $\mathrm{IKM}(1,2,1)^2 = \frac{\pi^2}{27}$

4 loops:

$$\frac{13}{8}\mathrm{IKM}(1,4,1)^2 - \frac{225}{32}\mathrm{IKM}(1,4,1)\mathrm{IKM}(1,4,3) = \frac{\pi^4}{80}$$

$$\frac{13}{8}\mathrm{IKM}(2,3,1)^2 - \frac{225}{32}\mathrm{IKM}(2,3,1)\mathrm{IKM}(2,3,3) = -\frac{3\pi^2}{64}$$

$$\frac{13}{8}\mathrm{IKM}(1,4,1)\mathrm{IKM}(2,3,1) - \frac{225}{64}\mathrm{IKM}(1,4,3)\mathrm{IKM}(2,3,1) - \frac{225}{64}\mathrm{IKM}(1,4,1)\mathrm{IKM}(2,3,3) = 0$$

10 loops:

$$\frac{125}{16}\mathrm{IKM}(3,8,1)^2 - \frac{65679}{4}\mathrm{IKM}(3,8,3)\mathrm{IKM}(3,8,1) + \frac{25484133}{64}\mathrm{IKM}(3,8,5)\mathrm{IKM}(3,8,1)$$
$$- \frac{322307685}{512}\mathrm{IKM}(3,8,7)\mathrm{IKM}(3,8,1) + \frac{108056025\mathrm{IKM}(3,8,9)\mathrm{IKM}(3,8,1)}{2048}$$
$$+ \frac{2475315}{256}\mathrm{IKM}(3,8,3)^2 + \frac{108056025\mathrm{IKM}(3,8,5)^2}{4096} - \frac{64674153}{512}\mathrm{IKM}(3,8,3)\mathrm{IKM}(3,8,5)$$
$$+ \frac{108056025\mathrm{IKM}(3,8,3)\mathrm{IKM}(3,8,7)}{2048} = \frac{105\pi^6}{8}$$

Example III: Broadhurst' quadratic relations for IKM

Work in progress with Andrei Pomeransky.

Using the differential equations in symmetric $(\epsilon + 1/2)$-form for tadpoles with arbitrary masses we have the Broadhurst relations for each specific $L = 2, 3, \ldots,$ at least to 10.

We have also obtained various generalizations of these relations. In particular, the relations connecting the higher orders in ϵ expansion of different/equal mass tadpoles and sunrises.

Summary

- An infinite set of quadratic constraints for the ϵ-expansion of the solutions of the differential equations reducible to ϵ- or $(\epsilon + \frac{1}{2})$-form is discovered.

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- While for the DEs reducible to ϵ-form these constraints seem to be always the consequences of the known relations for polylogs (namely, the shuffling relations), for irreducible cases their origin is yet to be understood.
- We have explicitly checked using Frobenius approach that these identities also hold for the cases for which no closed form expressions exist.

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- An infinite set of quadratic constraints for the ϵ-expansion of the solutions of the differential equations reducible to ϵ- or $(\epsilon+\frac{1}{2})$-form is discovered.
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- We have explicitly checked using Frobenius approach that these identities also hold for the cases for which no closed form expressions exist.
- Stay tuned: we will probably have more to say about "elliptic" cases (work in progress with A. Pomeransky).