Progress on elliptic Feynman integrals and linear relations

Christian Bogner

(Humboldt-Universität zu Berlin)

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Outline:

- Analytic continuation of the sunrise and the kite integral

  (with A. Schweitzer and S. Weinzierl)

- Feynman integral relations from parametric annihilators

  (with T. Bitoun, R.P. Klausen and E. Panzer)

Analytic continuation and numerical evaluation of the kite integral and the equal mass sunrise integral

with A. Schweitzer and S. Weinzierl

Nucl.Phys. B922 (2017) 528-550 [arXiv:1705.08952]

Families of scalar Feynman integrals in d spacetime dimensions:

$$\mathcal{I}(\nu_1, ..., \nu_N) = \left(\prod_{j=1}^{L} \int \frac{d^d l_j}{i\pi^{\frac{d}{2}}} \right) \prod_{i=1}^{N} D_i^{-\nu_i} \text{ with the } \nu_i \in \mathbb{C},$$

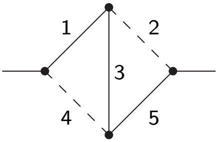
given by the $D_1, ..., D_N$: the usual inverse propagators and irreducible scalar products, at most quadratic in the loop-momenta $l_1, ..., l_L$.

Alternative representations using **Feynman parameters** x_i:

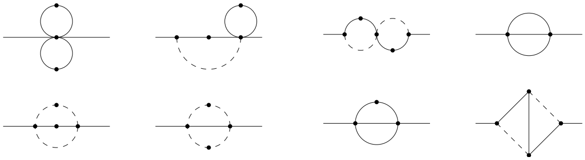
$$\begin{aligned} \mathcal{I}(\nu_1, ..., \nu_N) &= \Gamma(\omega) \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) \frac{\delta\left(1 - \sum_{j=1}^{N} x_j\right)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^{\omega}} \\ &= \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) \mathcal{G}^{-\frac{d}{2}} \end{aligned}$$

where $\omega = -L\frac{d}{2} + \sum_{i=1}^{N} \nu_i$; $\mathcal{U}$, $\mathcal{F}$ are the Symanzik polynomials and $\mathcal{G} = \mathcal{U} + \mathcal{F}$. (Lee, Pomeransky 2013)

Consider integrals $\mathcal{I}(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$ of the **kite family**



with a choice of 8 master integrals, e.g.:



including the **sunrise** and the **kite integral**.

Approach to compute the of master integrals $\mathcal{I}_1, ..., \mathcal{I}_8$: With respect to $t = p^2$ consider **differential equations**

$$\frac{\partial}{\partial t}\mathcal{I}_i = \sum_{j=1}^{n} A_{ij}\mathcal{I}_j.$$

Solve the system for the coefficients of

$$\mathcal{I}_j = \sum_{i=a}^{\infty} \mathcal{I}_j^{(i)}\epsilon^i, \text{ with } d = 4 - 2\epsilon.$$

In general, one often uses multiple polylogarithms

$$\mathrm{Li}_{n_1,\ldots,n_r}(z_1, ..., z_r) = \sum_{0<j_1<\ldots<j_r} \frac{z_1^{j_1} \ldots z_r^{j_r}}{j_1^{n_1} \ldots j_r^{n_r}} \text{ for } |z_i| < 1.$$

For the kite family these functions are **not sufficient** and we need **elliptic generalizations** of polylogarithms.

Various classes of such functions were introduced in the last few years.

Bloch, Vanhove 2013: Result for the sunrise integral in terms of an **elliptic dilogarithm**.

Adams, CB, Schweitzer, Weinzierl 2016: Results for the **kite master integrals** involving elliptic generalizations of polylogarithms

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} \frac{y^k}{k^m} \mathrm{Li}_n(q^k x),$$

and multi-variable generalizations

$$\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}\left(x_1,\dots,x_l;y_1,\dots,y_l;q\right)$$

$$= \sum_{j_1=1}^{\infty} \dots \sum_{j_l=1}^{\infty} \sum_{k_1=1}^{\infty} \dots \sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}} \dots \frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}} \dots \frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1 k_1 + \dots + j_l k_l}}{\prod_{i=1}^{l-1} \left(j_i k_i + \dots + j_l k_l\right)^{o_i}}$$

Here $q = q(t)$ is the nome of an underlying **family of elliptic curves**.

These results are valid in the region $t = p^2 < 0$. This talk: **analytic continuation** to arbitrary $t \in \mathbb{R}$.

The **family of elliptic curves** is given by $\mathcal{F} = 0$ for the second Symanzik polynomial of the sunrise graph (see Bloch, Vanhove 2013)

$$\mathcal{F} = -x_1 x_2 x_3 t + m^2 (x_1 + x_2 + x_3)(x_1 x_2 + x_2 x_3 + x_1 x_3).$$

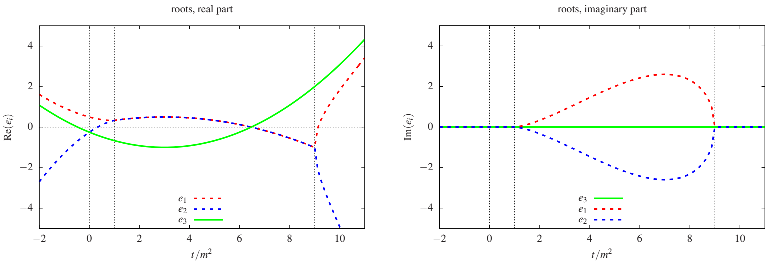
We change variables to **Weierstrass normal form** (in the chart $z = 1$):

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

with the three roots

$$\begin{aligned}
e_1 &= \frac{1}{24}\left(-t^2 + 6m^2 t + 3m^4 + 3\left(m^2 - t\right)^{\frac{3}{2}}\left(9m^2 - t\right)^{\frac{1}{2}}\right)\\
e_2 &= \frac{1}{24}\left(-t^2 + 6m^2 t + 3m^4 - 3\left(m^2 - t\right)^{\frac{3}{2}}\left(9m^2 - t\right)^{\frac{1}{2}}\right)\\
e_3 &= \frac{1}{24}\left(2t^2 - 12m^2 t - 6m^4\right).
\end{aligned}$$

For every value $t \in \mathbb{R}$ this defines an elliptic curve, except for values where two of the roots coincide (i.e. the family degenerates).



Roots coincide at

$$
\begin{aligned}
t = 0 \quad &: \quad e_2 = e_3, \\
t = m^2 \quad &: \quad e_1 = e_2, \\
t = 9m^2 \quad &: \quad e_1 = e_2, \\
t = \infty \quad &: \quad e_1 = e_3.
\end{aligned}
$$

Remark: These are the singularities of the second order Picard-Fuchs differential operator of the sunrise.

For $t < 0$ we have $e_2 < e_3 < e_1$. Here we define the **period integrals**

$$\psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y} = \int_{\delta_1} \frac{dx}{y},\ \psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y} = \int_{\delta_2} \frac{dx}{y},$$

with x, y of the Weierstrass normal form.

One can derive explicitly

$$\psi_1 = \frac{4}{(m^2-t)^{\frac{3}{4}}\,(9m^2-t)^{\frac{1}{4}}} K(k),\ \psi_2 = \frac{4i}{(m^2-t)^{\frac{3}{4}}\,(9m^2-t)^{\frac{1}{4}}} K(k')$$

with the complete elliptic integral of the first kind

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

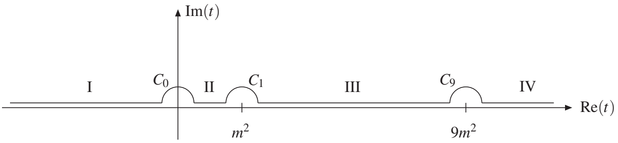
with moduli

$$k = \frac{e_3-e_2}{e_1-e_2},\ k'^2 = 1-k^2 = \frac{e_1-e_3}{e_1-e_2}.$$

With these periods, we define

$$q = e^{i\pi\frac{\psi_2}{\psi_1}}.$$

We set $t \to t + i\delta$ and consider the variation:



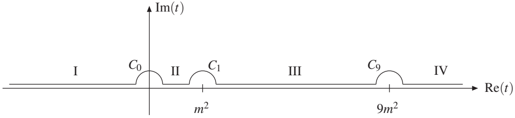
Our goal: Explicit expressions for $q = e^{i\pi \frac{\psi_2}{\psi_1}}$ $\Rightarrow$ numerical evaluation of the Feynman integrals in all regions

Branch cuts for the periods ψ_1, ψ_2 are coming from the complete elliptic integral

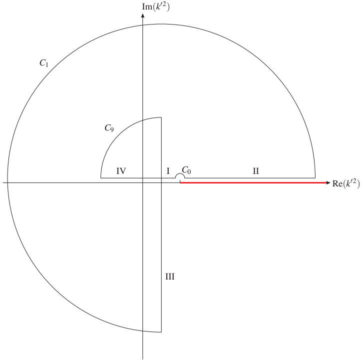
$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Simplification: Consider $\tilde{K}\left(k^2\right) = K(k)$ as a function of k^2.

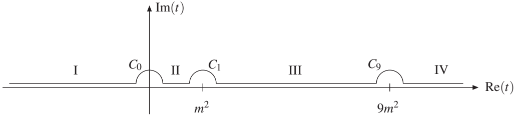
This function has **only one branch cut** in the k^2-plane at $[1, \infty[$.



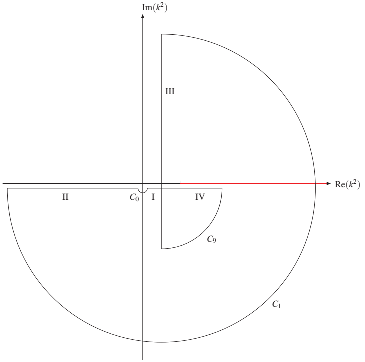
Along the path of t we have for k'^2 :



Branch-cut is **not** crossed. $\Rightarrow \psi_2\left(t + i\delta\right) = \frac{4i}{\left(m^2 - t\right)^{\frac{3}{4}}\left(9m^2 - t\right)^{\frac{1}{4}}} K\left(k'\left(t + i\delta\right)\right)$ for all

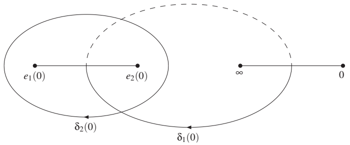


Along the path of t we have for k^2 :

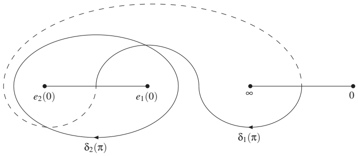


$\Rightarrow$ We only have to study the monodromy behaviour of ψ_1 along C_1.

Along C_1, the integration cycles of the periods ψ_1 and ψ_2 change twice from



to



$$\begin{aligned} \delta_1(\pi) &= \delta_1(0) - \delta_2(0), \\ \delta_2(\pi) &= \delta_2(0). \end{aligned}$$

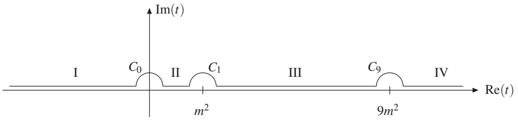
After a full rotation of e_1, e_2 around each other (i.e. of k^2 around 1) we have

$$\begin{aligned} \delta_1(2\pi) &= \delta_1(0) - 2\delta_2(0) \\ \delta_2(2\pi) &= \delta_2(0). \end{aligned}$$

$\Rightarrow$ As t varies along C_1 the periods change as

$$\begin{aligned} \psi_1 &\to \psi_1 - 2\psi_2, \\ \psi_2 &\to \psi_2. \end{aligned}$$

Result: To obtain a result for the eq. mass sunrise $S(2-2\epsilon, t)$ and kite integral $I(4-2\epsilon, t)$ for any t on the path



evaluate our results of the Euclidean region at

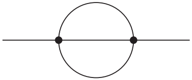
$$q(t+i\delta) = e^{i\pi \frac{\psi_2(t+i\delta)}{\psi_1(t+i\delta)}}$$

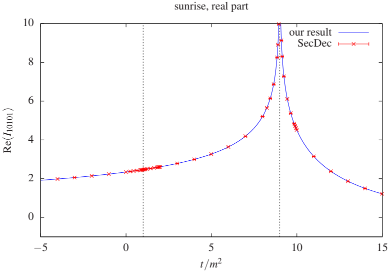
with

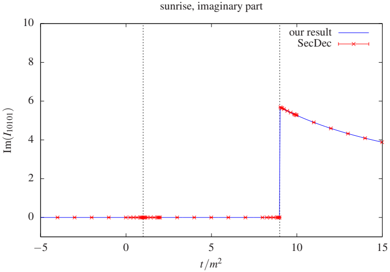
$$\begin{pmatrix} \psi_2(t+i\delta) \\ \psi_1(t+i\delta) \end{pmatrix} = \frac{4}{(m^2-t-i\delta)^{\frac{3}{4}}\,(9m^2-t-i\delta)^{\frac{1}{4}}} M_t \begin{pmatrix} iK\left(k'\left(t+i\delta\right)\right) \\ K\left(k\left(t+i\delta\right)\right) \end{pmatrix}$$

$$\text{and } M_t = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } -\infty < t < m^2, \\ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} & \text{for } m^2 < t < \infty. \end{cases}$$

Numerical evaluation: The ϵ^0-term of the equal mass sunrise integral

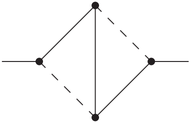


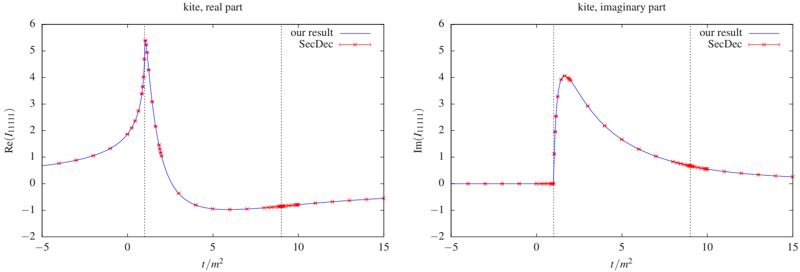




blue line: our result, red dots: SecDec (Borowka et at 2015)

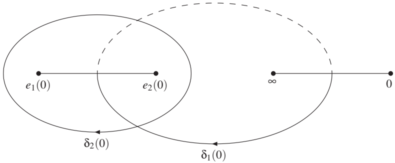
Numerical evaluation: The ϵ^0-term of the kite integral





blue line: our result, red dots: SecDec (Borowka et at 2015)

Remark: The perspective of Picard-Lefschetz theory



- The situation at $t = m^2$ is known as **pinch singularity**: here $e_1 = e_2$ and the contour δ_1 is trapped
- Let s be the oriented line (1-simplex) from e_1 to e_2. It is called a **vanishing cycle** w.r.t. the pinch singularity. Its boundary is $\partial s = e_2 - e_1$.
- The **Leray co-boundary** $\delta(e)$ of a point e is an oriented circle around e.
  $\Rightarrow \delta\left(\partial s\right) = \delta e_2 - \delta e_1$ are two circles with opposite orientations

The **Picard-Lefschetz** theorem states **how cycles change**, when a parameter is sent in a circle around a pinch singularity. (see the book of Pham)

Feynman integral relations from parametric annihilators

with Thomas Bitoun, René Pascal Klausen and Erik Panzer

[arXiv:1712.09215]

Back to the **families** of Feynman integrals

$$\mathcal{I}(\nu_1, ..., \nu_N) = \left(\prod_{j=1}^{L} \int \frac{d^d l_j}{i\pi^{\frac{d}{2}}}\right) \prod_{a=1}^{N} D_a^{-\nu_a} \text{ with the } \nu_i \in \mathbb{C}.$$

Alternative representations using **Feynman parameters** x_i:

$$\begin{aligned} \mathcal{I}(\nu_1, ..., \nu_N) &= \Gamma(\omega) \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} dx_i}{\Gamma(\nu_i)}\right) \frac{\delta\left(1 - \sum_{j=1}^{N} x_j\right)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^\omega} \\ &= \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} dx_i}{\Gamma(\nu_i)}\right) \mathcal{G}^{-\frac{d}{2}} \end{aligned}$$

One defines **shift operators** $\hat{\mathbf{i}}^+$ and $\mathbf{i}^-$ and the operator $\mathbf{n}_i := \hat{\mathbf{i}}^+\mathbf{i}^-$, acting as

$$\begin{aligned}
\hat{\mathbf{i}}^+\mathcal{I}(\nu_1,...,\nu_N) &= \nu_i\mathcal{I}(\nu_1,...,\nu_i+1,...,\nu_N),\\
\mathbf{i}^-\mathcal{I}(\nu_1,...,\nu_N) &= \mathcal{I}(\nu_1,...,\nu_i-1,...,\nu_N).\\
\mathbf{n}_i\mathcal{I}(\nu_1,...,\nu_N) &= \nu_i\mathcal{I}(\nu_1,...,\nu_N).
\end{aligned}$$

And the shift algebra

$$S^N[d] := \mathbb{C}[d]\left\langle \hat{\mathbf{1}}^+,...,\hat{\mathbf{N}}^+,\mathbf{1}^-,...,\mathbf{N}^- \;\middle|\; \left[-\mathbf{j}^-,\hat{\mathbf{i}}^+\right] = \delta_{ij}\right\rangle.$$

On every given family of Feynman integrals there are **shift relations**

$$\mathbf{s}\mathcal{I}(\nu_1,...,\nu_N) = 0 \text{ with } \mathbf{s}\in S^N[d],$$

i.e. linear relations

$$\sum_i c_i\mathcal{I}_i = 0$$

between integrals $\mathcal{I}_i$ differing from each other by shifts of the ν_j with rational coefficients in d (and in kinematic invariants and squared particle masses).

The usual way to generate shift relations is by the **IBP method** (Chetyrkin, Tkachov 1981).

The **insertion of differential operators** $\frac{\partial}{\partial l_i} q_j$ (with q_j a loop-momentum or external momentum) under the integral sign of $\mathcal{I}(\nu_1, ..., \nu_N) = \left(\prod_{j=1}^{L} \int \frac{d^d l_j}{i\pi^{\frac{d}{2}}}\right) \prod_{a=1}^{N} D_a^{-\nu_a}$ gives

$$\left(\prod_{j=1}^{L} \int \frac{d^d l_j}{i\pi^{\frac{d}{2}}}\right) \frac{\partial}{\partial l_i} q_j \prod_{a=1}^{N} D_a^{-\nu_a} = 0.$$

Evaluating the differentiations leads to **shift relations**.

These are combined by **Laporta's algorithm** (Gauß-elimination) (Laporta 2001), implemented in several programs (Anastasiou, Lazopoulos 2004, Smirnov 2008, Studerus 2009, Studerus, von Manteuffel 2012, Maierhöfer, Usovitsch, Uwer 2017).

$\Longrightarrow$ Every ingtegral of the family is expressed as linear combination in a basis of **"master integrals"**.

Alternative approaches in Feynman parameters $x_1, ..., x_N$:

Tkachov's idea (Tkachov 1996, also see: Sabbah 1987 and Gyoja 1993): Utilize **Bernstein's equation**

$$Pf^{s+1} = b(s)f^s$$

for f a (Symanzik) polynomial in $x_1, ..., x_N$, $b(s)$ the Bernstein-Sato polynomial and P a polynomial differential operator.

Insertion of operators into $\mathcal{I}(\nu_1, ..., \nu_N) = \Gamma(\omega) \left(\prod_{i=1}^N \int_0^\infty \frac{x_i^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) \frac{\delta\left(1 - \sum_{j=1}^N x_j\right)}{\mathcal{U}^{d/2-\omega} \mathcal{F}^\omega}$ led to some numerical results at low loop orders (Bardin et al 2000, Passarino et al 2001-2002) but to **no sustainable method** to derive shift relations for the general case.

Lee's idea (Lee 2014): Start from

$$\mathcal{I}(\nu_1,...,\nu_N)=\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}-\omega\right)}\left(\prod_{i=1}^{N}\int_0^\infty\frac{x_i^{\nu_i-1}dx_i}{\Gamma(\nu_i)}\right)\mathcal{G}^{-\frac{d}{2}}\text{ with }\mathcal{G}=\mathcal{U}+\mathcal{F}$$

and insert *s*-**parametric annihilators** P of $\mathcal{G}$, defined by

$$P\mathcal{G}^s=0\quad(s=-d/2),$$

belonging to the Weyl algebra

$$A^N[s]:=\mathbb{C}[s]\left\langle x_1,...,x_N,\partial_1,...,\partial_N\,|[\partial_i,x_j]=\delta_{ij}\right\rangle.$$

The annihilators of $\mathcal{G}$ form the **ideal** $\mathrm{Ann}_{A^N[s]}\left(\mathcal{G}^s\right)$.

The generators of this ideal can be derived by algorithmically, e.g. using SINGULAR

(Andres et al 2010).

Definition: The **twisted (multi-dimensional) Mellin transform** of a function $f : \mathbb{R}_+^N \to \mathbb{C}$ is

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{i=1}^{N} \int_0^\infty \frac{x_i^{\nu_i - 1} dx_i}{\Gamma(\nu_i)} \right) f(x_1, ..., x_N) \text{ with } \nu = (\nu_1, ..., \nu_N).$$

Feynman integrals:

$$\mathcal{I}(\nu) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \mathcal{M}\left\{\mathcal{G}^{-\frac{d}{2}}\right\}(\nu) = \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} - \omega\right)} \tilde{\mathcal{I}}(\nu).$$

The correspondence between annihilators and shift relations is given by:

$$\begin{aligned} \mathcal{M}\{\alpha f + \beta g\}(\nu) &= \alpha \mathcal{M}\{f\}(\nu) + \beta \mathcal{M}\{g\}(\nu), \\ \mathcal{M}\{x_i f\}(\nu) &= \nu_i \mathcal{M}\{f\}(\nu + e_i) = \left(\hat{\mathbf{i}}^+ \mathcal{M}\{f\}\right)(\nu), \\ \mathcal{M}\{\partial_i f\}(\nu) &= -\mathcal{M}\{f\}(\nu - e_i) = \left(-\mathbf{i}^- \mathcal{M}\{f\}\right)(\nu). \end{aligned}$$

$$\mathcal{M}\{\bullet\} : A^N \overset{\cong}{\to} S^N,\ P \mapsto \mathcal{M}\{P\} := P|_{x_i \mapsto \hat{\mathbf{i}}^+, \partial_i \mapsto -\mathbf{i}^- \text{ for all } 1 \leq i \leq N}$$

For annihilators P , $P\mathcal{G}^s = 0$, we have

$$\mathcal{M}\left\{P\mathcal{G}^{-\frac{d}{2}}\right\}(\nu) = \mathcal{M}\{P\}\mathcal{M}\left\{\mathcal{G}^{-\frac{d}{2}}\right\}(\nu) = 0.$$

Example: For $\mathcal{G} = x_1 + x_2 - p^2 x_1 x_2$ we find an annihilator

$$P = -p^2\left(-\frac{d}{2} - x_1\partial_1 + 1\right)x_1 + \left(-\frac{d}{2} - x_1\partial_1 - x_2\partial_2\right)$$

$$\mapsto \mathcal{M}\{P\} = -p^2\left(-\frac{d}{2} + \mathbf{n}_1 + 1\right)\hat{\mathbf{1}}^+ + \left(-\frac{d}{2} + \mathbf{n}_1 + \mathbf{n}_2\right)$$

$\Rightarrow$ Shift relation:

$$-p^2\nu_1\tilde{\mathcal{I}}(\nu_1+1,\nu_2) = -\frac{-\frac{d}{2}+\nu_1+\nu_2}{-\frac{d}{2}+\nu_1+1}\tilde{\mathcal{I}}(\nu_1,\nu_2).$$

Re-introducing the gamma-factors we obtain:

$$-p^2\nu_1\mathcal{I}(\nu_1+1,\nu_2) = \frac{\left(-\frac{d}{2}+\nu_1+\nu_2\right)(2s+\nu_1+\nu_2+1)}{-\frac{d}{2}+\nu_1+1}\mathcal{I}(\nu_1,\nu_2).$$

The Mellin transform $\mathcal{M}$ is invertible and defines a bijection between annihilators and shift relations.

$\Rightarrow$ **Every shift relation** comes from an annihilator.

Open questions:

- Can annihilators improve **reductions** to master integrals?
- Do the annihilators of **classical IBP relations** generate the ideal $\mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$? In other words: Does the IBP method include every shift relation?
- Is $\mathrm{Ann}_{A^N[s]}(\mathcal{G}^s)$ generated by **linear operators**?

After some tests at low loop-numbers we have no counter examples so far.

Definition: Let $V_{\mathcal{G}}$ be the **vector space** of all Feynman integrals of the family with given $\mathcal{G}$ over rational functions in d and the indices ν_i, that is

$$V_{\mathcal{G}} := \sum_{n\in\mathbb{Z}^N} \mathbb{C}(s,\nu)\cdot\mathcal{M}\left\{\mathcal{G}^s\right\}(\nu+n) = \mathbb{C}(s,\nu)\otimes_{\mathbb{C}[s,\nu]}\left(S^N[s]\cdot\tilde{\mathcal{I}}\right)\ \text{(with } s=-d/2).$$

The **number of master integrals** of the family is the dimension of this vector space:

$$\mathfrak{C}\left(\mathcal{G}\right) := \dim_{\mathbb{C}(s,\nu)} V_{\mathcal{G}}.$$

First observation: By the inverse Mellin transform $\mathcal{M}^{-1}$ we can write $\mathfrak{C}\left(\mathcal{G}\right)$ as the dimension of a corresponding space of **integrands**:

$$\mathfrak{C}\left(\mathcal{G}\right) = \dim_{\mathbb{C}(s,\theta)}\left(\mathbb{C}(s,\theta)\otimes_{\mathbb{C}[s,\theta]} A^N[s]\cdot\mathcal{G}^s\right)$$

with $\theta = (\theta_1,...,\theta_N)$, $\theta_i := x_i\partial_i = \mathcal{M}^{-1}\{-\nu_i\}$.

$\mathfrak{C}\left(\mathcal{G}\right)$ will be an **upper bound for other countings** of master integrals in the literature which restrict to $\nu_i\in\mathbb{Z}$, regard symmetries and may discard simple graphs.

$A^N[s] \cdot \mathcal{G}^s$ is a **holonomic D-module**. Therefore a theorem of Loeser and Sabbah (1991, 1992) applies, which relates such dimensions to the **Euler characteristic** χ of a de Rham complex. We show that this is the topological Euler characteristic of the **complement of the zero-set of** $\mathcal{G}$.

Our main result is:

$$(-1)^N \mathfrak{C}(\mathcal{G}) = \chi\left((\mathbb{C}^\star)^N \setminus \mathbb{V}(\mathcal{G})\right)$$

with the variety $\mathbb{V}(\mathcal{G}) = \{\mathcal{G} = 0\}$ and the Euler characteristic

$$\chi(X) = \sum_i (-1)^i \dim H^i(X).$$

$\Rightarrow$ The result implies that the number of master integrals is **always finite**. (cf. Smirnov, Petukhov 2010)

It can furthermore be **exactly computed**.

Tools for the computation:

- Instead of $\mathbb{V}(\mathcal{G})$ we can consider the equivalence class in the **Grothendieck ring**.

  $\Rightarrow \chi(X) = \chi(X \backslash Z) + \chi(Z)$ where Z is a closed subvariety of X.

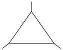
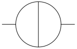
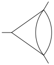
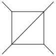
  $\Rightarrow$ We obtain $\chi\left((\mathbb{C}^\star)^N \backslash \mathbb{V}(A + x_N B)\right) = -\chi\left((\mathbb{C}^\star)^{N-1} \backslash \mathbb{V}(A \cdot B)\right)$ for linear polynomials $A + x_N B$ and other useful relations, reducing the computation to **subgraphs**.

  $\Rightarrow$ Particularly useful for **linearly reducible** $\mathcal{G}$.

- Algorithms for arbitrary polynomials are available in SINGULAR and Macaulay2.

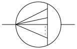
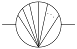
- An elegant computation via **Newton polygons** would be possible for the (rare) case of non degenerate $\mathcal{G}$. (Kouchnirenko 1976)

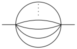
Examples:

Graph G				
$\mathfrak{C}(G)$ massless	4	3	4	20
$\mathfrak{C}(G)$ massive	7	30	19	55

computed with Macaulay2's command `Euler` and checked with Azurite (Georgoudis, Larsen, Zhang 2016)

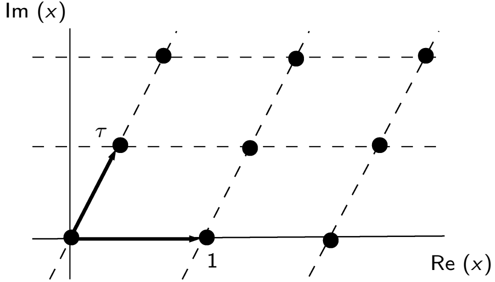
Results for infinite classes of L-loop graphs:

Wheels  and  : $\mathfrak{C}(G) = \frac{L(L+1)}{2}$

Bananas  : $\mathfrak{C}(G) = 2^{L+1} - 1$ agrees with (Kalmykov, Kniehl 2016)

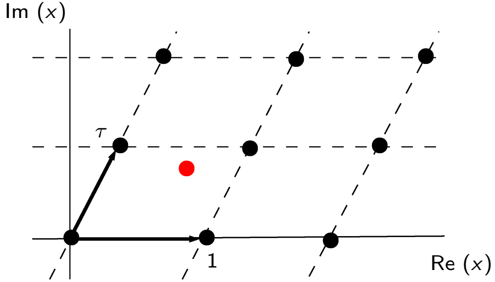
Summary:

- The **analytic continuation** of the kite family is controlled by a variation of a **family of elliptic curves**.
- We only need to know how **periods** of the elliptic curve change. This can be understood from **Picard-Lefschetz theory**.
- **All shift relations** between Feynman integrals can be obtained from the insertion of **parametric annihilators**.
- The **number of master integrals**, defined as the dimension of the corresponding vectorspace, is an **Euler characteristic**.



Consider the **lattice** $L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$.

Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.



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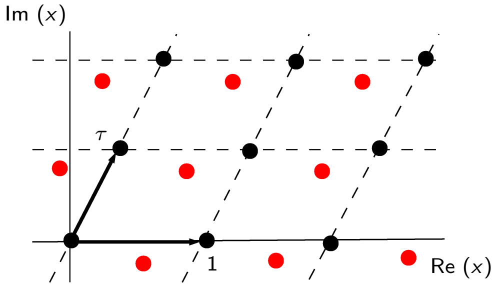

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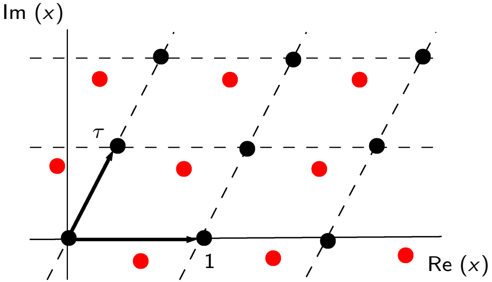


Consider the **lattice** $L = \mathbb{Z} + \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$.

Elliptic functions f with respect to L: $f(x) = f(x + \lambda)$ for $\lambda \in L$.

Let$\tau = \frac{\psi_1}{\psi_2}$ with ψ_1, ψ_2 the **periods of an elliptic curve** E.

$\Rightarrow$ E is isomorphic to a cell of L. $\Rightarrow$Consider f as a function defined on E.

Change variables to $z = e^{2\pi i x} \in \mathbb{C}^\star$

$\Rightarrow$Ellipticity $f(x) = f(x+\lambda)$ means $\tilde{f}(z) = \tilde{f}(z \cdot q_\lambda)$, $q_\lambda \in e^{2\pi i \lambda}$ for $\lambda \in L$.

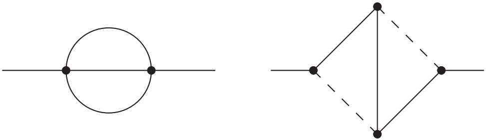
Particularly: $q = e^{2\pi i \tau}$.

Basic concept: For some function g **construct** an elliptic function of the type

$f(z, q) = \sum_{n\in\mathbb{Z}} g(z \cdot q^n)$

E.g. Brown, Levin 2011 consider **elliptic polylogarithms** $\sum_{n\in\mathbb{Z}} u^n \mathrm{Li}_m(z \cdot q^n)$,

elliptic multiple polylogarithms and a framework of **iterated integrals**



We have obtained results for

- the **equal mass sunrise** $S(2-2\epsilon, t)$ in 2 dimensions (Adams, CB, Weinzierl 2015)
- the **kite** integral $I(4-2\epsilon, t)$ in 4 dimensions (Adams, CB, Schweitzer, Weinzierl 2016)

Many other people have provided results. (Laporta, Remiddi, Tancredi, Bloch, Vanhove, Sabry, Bauberger, Berends, Böhm, Buza, Weiglein, Scharf, Broadhurst, Fleischer, Tarasov, Groote, Körner, Pivovarov, ...)

Using our framework of **ELi-functions** as elliptic generalizations of polylogarithms,

we can recursively compute arbitrary orders in ϵ.

Open problem so far: Our results are given only for $t < 0$ where $t = p^2$.

As a generalization of classical polylogarithms $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}$ we define **basic ELi-functions**

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} \frac{y^k}{k^m} \mathrm{Li}_n(q^k x),$$

and **multi-variable generalization**:

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q)$$

$$= \sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \sum_{k_1=1}^{\infty} \cdots \sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}} \cdots \frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}} \cdot\cdot \frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1 k_1 + \ldots + j_l k_l}}{\prod_{i=1}^{l-1} (j_i k_i + \ldots + j_l k_l)^{o_i}}$$

Dependences in our results in the equal mass case:

q is a function of t and of m^2; $y_i \in \{-1, 1\}$

Multiplication property:

$$\mathrm{ELi}_{\tilde{n},\tilde{m}}\left(\tilde{x};\tilde{y};q\right)\cdot\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}\left(x_1,\ldots,x_l;y_1,\ldots,y_l;q\right)$$

$$=\mathrm{ELi}_{n_1,\ldots,n_l,\tilde{n};m_1,\ldots,m_l,\tilde{m};2o_1,\ldots,2o_{l-1},0}\left(x_1,\ldots,x_l,\tilde{x};y_1,\ldots,y_l,\tilde{y};q\right)$$

Integration property:

$$\int^q \frac{dq'}{q'}\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}\left(x_1,\ldots,x_l;y_1,\ldots,y_l;q'\right)$$

$$=\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}+1}\left(x_1,\ldots,x_l;y_1,\ldots,y_l;q\right)$$

$\Rightarrow$We can **multiply** with $\mathrm{ELi}_{n;m}(x;y;q)$ and **integrate** over $\frac{dq}{q}$ **without leaving this class of functions**.

Our results involve **combinations** like

$$\mathrm{E}_{n;m}(x;\,y;\,q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x;\,y;\,q) - \mathrm{ELi}_{n;m}(x^{-1};\,y^{-1};\,q)\right) & ,\, n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_n(x) + \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x;\,y;\,q) + \mathrm{ELi}_{n;m}(x^{-1};\,y^{-1};\,q) & ,\, n+m \text{ odd}. \end{cases}$$

These are closer to elliptic polylogarithms of the mathematical literature, such as

$$E^{BL}_{n;m}(z;u;q) = \sum_{n\in\mathbb{Z}} u^n \mathrm{Li}_m\left(z\cdot q^n\right)$$

of Brown, Levin 2011.

For the **sunrise**

$$S(2-2\epsilon,\, t) = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + ...$$

we **re-write the differential equations** in terms of $q(t)$.

$\Rightarrow$ We obtain for $S^{(j)} = \frac{\psi_1}{\pi} E^{(j)}$ a **recursive relation** of the form

$$E^{(j)} = \int_0^q \frac{dq_1}{q_1} \int_0^{q_1} \frac{dq_2}{q_2} \left(a_j + b \cdot E^{(j-2)} \right).$$

- From previous work we know $E^{(0)}$ and $E^{(1)}$ in terms of $\mathrm{ELi}_{n_1,\ldots;m_1,\ldots;2o_1,\ldots}$.
- We can express all a_j and b in terms of $\mathrm{ELi}_{n;m}$.

$\Rightarrow$ Using the multiplication- and integration-properties, we can compute $E^{(j)}$ to

arbitrary j in terms of $\mathrm{ELi}_{n_1,\ldots;m_1,\ldots;2o_1,\ldots}$.

In a **similar** way, we obtain a result for the **kite** integral to arbitrary order.

Example: For the equal mass sunrise $S(2-2\epsilon,\ t) = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + ...$

we have with $\psi_1 = \frac{4}{\left(m^2-t\right)^{\frac{3}{4}}\left(9m^2-t\right)^{\frac{1}{4}}} K(k)$ results

$$
\begin{aligned}
E^{(0)} &= 3\mathrm{E}_{2;0}\left(r_3; -1; -q\right), \\
E^{(1)} &= 3\mathrm{E}_{3;1}\left(r_3; -1; -q\right) + 3\mathrm{E}_{0,1;-2,0;4}\left(r_3, r_3; -1, -1; -q\right) \\
&\quad -9\mathrm{E}_{0,1;-2,0;4}\left(r_3, r_3; -1, 1; -q\right) + 18\mathrm{E}_{0,1;-2,0;4}\left(r_3, -1; -1, 1; -q\right) \\
&\quad + \frac{3}{2i}\left(-2\mathrm{Li}_{2,1}\left(r_3, 1\right) - 2\mathrm{Li}_3\left(r_3\right) + 2\mathrm{Li}_{2,1}\left(r_3^{-1}, 1\right)\right. \\
&\quad \left. + 2\mathrm{Li}_3\left(r_3^{-1}\right) + 6\mathrm{Li}_1\left(-1\right)\left(\mathrm{Li}_2\left(r_3\right) - \mathrm{Li}_2\left(r_3^{-1}\right)\right)\right) + L_{1;0} E^{(0)}_{111}
\end{aligned}
$$

where $r_3 = e^{2\pi i/3}$.

What happens when we vary t?

$\Rightarrow$ The points e_1, e_2, e_3 will will move around and the moduli $k = \frac{e_3 - e_2}{e_1 - e_2}, \; k' = \frac{e_1 - e_3}{e_1 - e_2}$ will change.

$\Rightarrow q = e^{i\pi \frac{\psi_2}{\psi_1}}$ will change.

Important detail: Segments of the real t-axis correspond to **branch-cuts** of these functions.

We always need to control "**on which side** of the cut we are"

and how to evaluate **on** the cut.

$\Rightarrow$ We use **Feynman's prescription**: replace $t \to t + i\delta$ with a small, positive $\delta \in \mathbb{R}$ which is sent to zero.

Example:

$$\sqrt{-t} \to \sqrt{-t - i\delta} = \begin{cases} -i\sqrt{|t|} & \text{for } t > 0, \\ \sqrt{|t|} & \text{for } t \leq 0. \end{cases}$$

Observation: This **suffices for the analytic continuation** of the sunrise and kite results! (compare work of Tancredi et al)

Check of the observation:

The relation

$$t = -9m^2 \frac{\eta(\tau)^4 \eta\left(\frac{3\tau}{2}\right)^4 \eta(6\tau)^4}{\eta\left(\frac{\tau}{2}\right)^4 \eta(2\tau)^4 \eta(3\tau)^4}$$

with $\tau = \frac{\psi_2}{\psi_1}$ remains true for regions II, III and IV after these replacements.

Assume a **pinch singularity** at $t = t_0$ and let s be a corresponding **vanishing cycle**.

Consider a **variation** of t in a circle around $t = t_0$ such that s transforms into itself.

The **Picard-Lefschetz theorem states**: Any cycle c transforms under this variation like

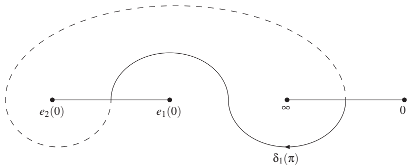
$$c \to c + n \cdot K(s, c) \cdot h.$$

- $n \in \{-1, 1\}$, depending on the dimension of the space and of c,
- $K(s, c) \in \{-1, 0, 1\}$: the intersection number or Kronecker index of s and c, depending on their relative orientation,
- h : a cycle defined by the Leray co-boundary of a vanishing cycle or its boundary. In our simple case: $h = \delta\,(\partial s)$.

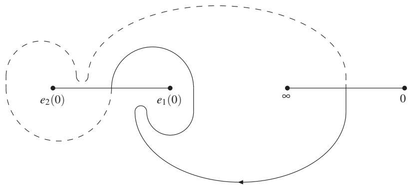
Our case: $c = \delta_1,\ h = \delta_2,\ n \cdot K(s, c) = -1$, and therefore $\delta_1 \to \delta_1 - \delta_2$. This reproduces our $\delta_1\,(\pi) = \delta_1(0) - \delta_2(0)$.

The theorem applies in **much greater generality**. (see work of Pham et al)

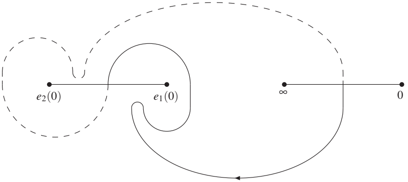
Consider the contour $\delta_1(\pi)$:



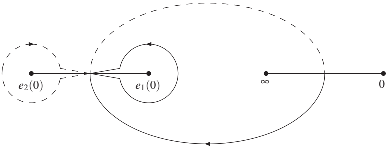
Without changing the integral, we may deform it to



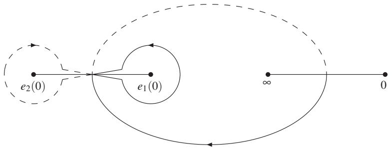
... and furthermore from



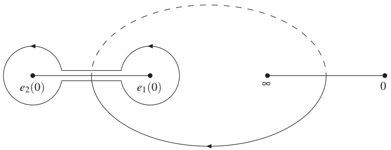
to



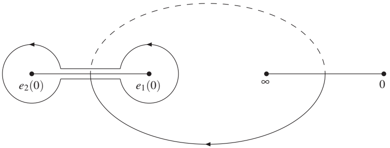
Notice: The two circles around e_1 and e_2 are in different Riemann sheets.



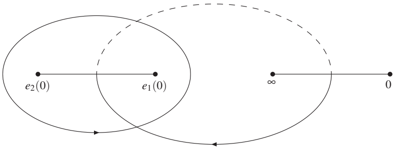
Now pull the circle around e_2 to the other sheet:



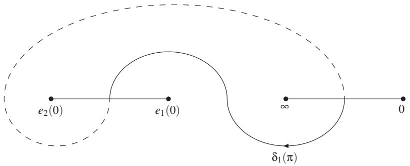
Notice: Both circles have the same orientation.

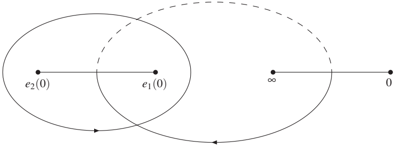


We can therefore merge the two circles to one contour:



The result of this operation is the contour $-\delta_2(0)$ (minus sign: changed orientation).





Fazit:

$$\begin{aligned}\delta_1(\pi) &= \delta_1(0) - \delta_2(0),\\ \delta_2(\pi) &= \delta_2(0).\end{aligned}$$

Consider the Legendre form

$$y^2 = x(x - \lambda)(x - 1) \text{ with } \lambda = k^2.$$

We have k^2 rotating (anti-clockwise) around 1.

Equivalently consider

$$y^2 = x(x - e_1(\phi))(x - e_2(\phi))$$

where two points $e_1(\phi) = 1 - re^{i\phi}$ and $e_2(\phi) = 1 + re^{i\phi}$ **rotate around each other** as

we send ϕ from 0 to 2π.

Furthermore: Consider **periods as integrals over cycles** $\delta_1,\ \delta_2$ on the elliptic curve:

$$\psi_1 = \int_{\delta_1} \frac{dx}{y},\ \psi_2 = \int_{\delta_2} \frac{dx}{y}$$

with $y = -\sqrt{x}\sqrt{x - e_1(\phi)}\sqrt{x - e_2(\phi)}$.