The status of expansion by regions

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In collaboration with Tatiana Semenova and Alexander Smirnov [arXiv:1809.04325 [hep-th]]

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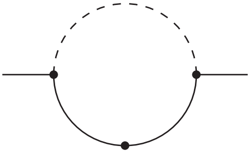
- Integrate the integrand, expanded in this way in each region, over the *whole integration domain* of the loop momenta.
- Set to zero any scaleless integral.

A simple example



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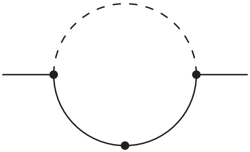

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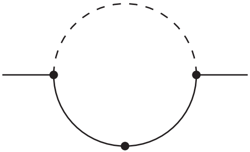

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with $d = 4 - 2\varepsilon$ in the limit $m^2/q^2 \to 0$.

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Expansion by subgraphs [K.G. Chetyrkin'88, S. Gorishny'89].

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and U and V are two basic functions (Symanzik polynomials, or graph polynomials).

One can consider quite general limits for a Feynman integral which depends on external momenta q_i and masses and is a scalar function of kinematic invariants and squares of masses, s_i, and assume that each s_i has certain scaling ρ^{κ_i} where ρ is a small parameter.

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A region $\to$ scaling, i.e. $x_i \to \rho^{r_i} x_i$ where ρ is a small parameter connected with a given limit.

A systematical procedure to find regions based on geometry of polytopes and implemented as a public computer code `asy.m` [A. Pak & A.V. Smirnov'10] which is now included in the code `FIESTA` [A.V. Smirnov'09-16]

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Generalizations of this procedure to some cases where terms of the function F are negative
[B. Jantzen, A. Smirnov & V.S.'12]

Let us use the parametric representation of Lee and Pomeransky [R.N. Lee and A.A. Pomeransky'17]

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \ ,$$

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Feynman parametric representation can be obtained from it by inserting $1 = \int \delta(\sum_i x_i - \eta)\mathrm{d}\eta$, scaling $x \to \eta x$ and integrating over η.

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Let P be a polynomial with positive coefficients,

$$P(x_1, \ldots, x_n, t) = \sum_{w \in S} c_w x_1^{w_1} \ldots x_n^{w_n} t^{w_{n+1}} \ ,$$

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The Newton polytope $\mathcal{N}_P$ of P is the convex hull of the set S in the $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ equipped with the scalar product $v \cdot w = \sum_{i=1}^{n+1} v_i w_i$.

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A facet of P is a face of maximal dimension, i.e. n.

The main conjecture.

The asymptotic expansion of

$$G(t,\varepsilon) = \int_0^\infty \ldots \int_0^\infty P^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \, ,$$

in the limit $t \to +0$ is given by

$$G(t,\varepsilon) \sim \sum_\gamma \int_0^\infty \ldots \int_0^\infty \left[M_\gamma \left(P(x_1,\ldots,x_n,t) \right)^{-\delta} \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n \, ,$$

where the sum runs over facets of the Newton polytope $\mathcal{N}_P$ of P, for which the normal vectors $r^\gamma = (r_1^\gamma, \ldots, r_n^\gamma, r_{n+1}^\gamma)$, oriented inside the polytope have $r_{n+1}^\gamma > 0$. Let us call these facets *essential*.

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Let us normalize these vectors by $r_{n+1}^\gamma = 1$.

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$$P^\gamma(x_1, \ldots, x_n, t) = P(t^{r_1^\gamma} x_1, \ldots, t^{r_n^\gamma} x_n, t) \equiv \sum_{w \in S} c_w x_1^{w_1} \ldots x_n^{w_n} t^{w \cdot r^\gamma}$$

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The scalar product $w \cdot r^\gamma$ is proportional to the projection of the point w on the vector r^γ. For $w \in S$, it takes a minimal value for all the points belonging to the considered facet $w \in S \cap \gamma$. Let us denote it by $L(\gamma)$.

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The polynomial P_0^γ is independent of t while P_1^γ can be represented as a linear combination of positive rational powers of t with coefficients which are polynomials of x.

For a given facet γ, the operator M_γ acts on the integrand as follows

$$
\begin{aligned}
& M_\gamma \left(P(x_1,\ldots,x_n,t)\right)^{-\delta} \\
= \;& t^{\sum_{i=1}^n r_i^\gamma - L(\gamma)\delta} \mathcal{T}_t \left(P_0^\gamma(x_1,\ldots,x_n) + P_1^\gamma(x_1,\ldots,x_n,t)\right)^{-\delta} \\
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The LO term of a given facet γ

$$t^{-L(\gamma)\delta + \sum_{i=1}^n r_i^\gamma} \int_0^\infty \ldots \int_0^\infty \left(P_0^\gamma(x_1,\ldots,x_n)\right)^{-\delta} \mathrm{d}x_1 \ldots \mathrm{d}x_n \,.$$

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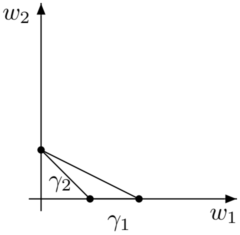
The Newton polytope (triangle)



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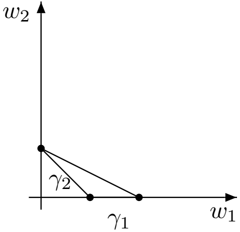

Two essential facets γ_1 and γ_2 with the corresponding normal vectors $r_1 = (0,1)$ and $r_2 = (1,1)$.

$\gamma_1 \to$ expanding the integrand in t. L0 is given by

$$\int_0^\infty (x^2 + x)^{\varepsilon - 1} \mathrm{d}x = \frac{\Gamma(1 - 2\varepsilon)\Gamma(\varepsilon)}{\Gamma(1 - \varepsilon)}$$

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$\gamma_2 \to t$ times the integral of the integrand with $x \to tx$ expanded in powers of t. L0 is given by

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The sum of the contributions in the LO:

$$G(t, \varepsilon) \sim -\log t + O(\varepsilon)$$

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We proved a similar result for LP integrals

$$\int_0^\infty \ldots \int_0^\infty P^{\varepsilon-2} \prod_i x_i^{\lambda_i} \, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

Let $\pi(K)$ be the projection of a set K on the hyperplane $w_{n+1} = 0$.

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Change the variables
$x_i \to x_i^{1/(\lambda_i+1)}$ to obtain $1/\prod_{i=1}^n (1+\lambda_i)$ times

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$$\bar{P}(x_1,\ldots,x_n,t) = \sum_{v\in\bar{S}} c_v x_1^{v_1} \ldots x_n^{v_n} t^{v_{n+1}} \; ,$$

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$\bar{P}$ is, generally, not a polynomial.

3. The leading contribution of a given essential facet.

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If the point $(\frac{1}{\delta}, \ldots, \frac{1}{\delta}) \in \mathbb{R}^n$ is inside $\pi(\Gamma)$ for some facet Γ then the leading asymptotics is given by

$$t^{-L(\Gamma)\delta+\sum_i r_i^\Gamma} \int_0^\infty \ldots \int_0^\infty \left(\sum_{w\in\Gamma\cap S} c_w y_1^{w_1} \ldots y_n^{w_n} \right)^{-\delta} \mathrm{d}y_1 \ldots \mathrm{d}y_n$$

when $t \to +0$.

One can adjust the analytic regularization parameters λ_i, i.e. to turn to the integral

$$\int_0^\infty \ldots \int_0^\infty P^{-\delta} \prod_i x_i^{\lambda_i} \, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

by satisfying the condition $\left(\frac{1+\lambda_1}{\delta}, \ldots, \frac{1+\lambda_n}{\delta}\right) \in \pi(\Gamma)$, so that the contribution of this facet will be leading.

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- *Divide et impera*