Unitarity-based methods and Integration-by-parts identities for Feynman integrals

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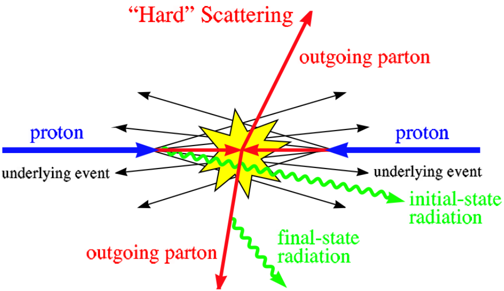
Overview

1. Scattering amplitudes
   - Phenomenology and theory
2. Feynman integrals
   - Integration-by-parts-identities
   - Current status
   - Open problems
     - Limitations and difficulties
3. A possible solution: PARSIVAL
   - Unitarity-based methods
   - Unitarity and Integration-by-parts-identities
   - New algorithm
     - PARSIVAL
     - Applications and Tests
4. Conclusions and Outlook

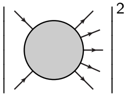
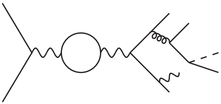
High-energy Particle Physics

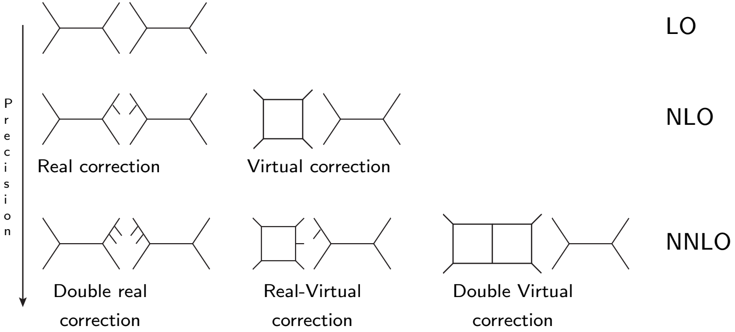


- Discovery of *Higgs boson* $\Longrightarrow$ Success of the *Standard Model* (SM)
- Evidences suggest that SM is not a complete theory (neutrino mass, Dark Matter etc.)
- New physics effects carried by massive particles $\Longrightarrow$ Precision measurements of *cross sections*
- High energies vs. High precisions
  - Direct production of new particles (+ jets) $\Longrightarrow$ many legs;
  - Indirect searches involving high precision measurements $\Longrightarrow$ many loops;

Scattering Amplitudes and Feynman diagrams

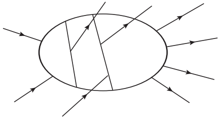
Cross section $\Longleftarrow$ Scattering amplitudes $\overset{QFT}{\Longleftarrow}$ **Feynman diagrams**

$\frac{d\sigma}{d\Omega}$ $\Longleftarrow$  $\overset{QFT}{\Longleftarrow}$ 



This thesis is focused on the *evaluation of virtual corrections*

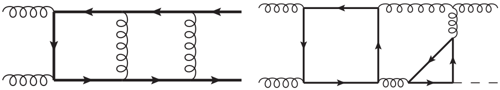
Evaluating Feynman Diagrams



$$\mathcal{F}(\mathbf{s}) = \quad = \int d^d k_1 \cdots d^d k_l \frac{\mathcal{N}(k_1, \ldots, k_l, \bar{p})}{\mathcal{D}(k_1, \ldots, k_l, \bar{p})}$$

Feynman Diagrams Evaluation: Problems

- Complexity of calculation depends on
  - number of legs;
  - number of free scale parameters;
  - number of loops;
- Evaluate a large number of Feynman integrals;
- Virtual corrections are the realm of integral calculus.



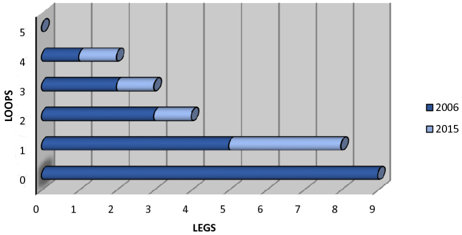
Evaluating Feynman Diagrams

Feynman Integrals Evaluation: Solution

- Passage from Integrals to Algebraic problems
- Automate the calculation

State of Automation

Tree-level $\Longrightarrow$ solved
1-loop $\Longrightarrow$ solved
2-loop $\Longrightarrow$ open problem



Automation of 1-loop diagram evaluation improved using *unitarity-based methods*.

Feynman diagrams: current solving strategy

Current evaluation approach

1. Amplitude decomposition
   - IBPs and generalized unitarity
     $\Longrightarrow$ Decomposition of Feynman diagrams in terms of *Master Integrals* (MI)
2. Master integrals evaluation
   - Differential and difference equations
   - Direct integration

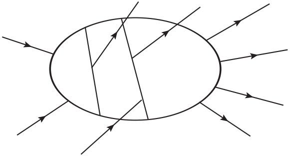
The core of this procedure are the **Integration-by-parts identities** (IBPs)

[K. G. Chetyrkin and F.V. Tkachov (1981)]
[S. Laporta (2001)]
[A. V. Kotikov (1991)]
[Remiddi (1997)]
[Gehrmann Remiddi (1999-2000)]
[Bonciani Mastrolia Remiddi (2003)]
[Bern Dixon Dunbar Kosower (1996)]
[Britto Cachazo Feng (2004)]
[Ossola Papadopoulos Pittau (2006)]

Tensor Decomposition and Feynman Integrals

Factorize the tensorial dependence of Feynman diagrams $\Longrightarrow$ Preliminary decomposition in *Feynman Integrals*

$$\mathcal{F}(\mathbf{s}) = $$



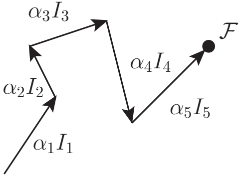
$$= \sum_i \alpha_i I_i(\mathbf{s})$$

Feynman integrals

$$I(\mathbf{s}) = \int d^d k_1 \cdots d^d k_l \frac{\mathcal{N}(k_1, \dots, k_l, \bar{p})}{D_1^{a_1} \cdots D_t^{a_t}}$$

Observation

- Integrals $I_i(\mathbf{s})$ are related by symmetries
- Evaluate the smallest set of Feynman integrals required by the full calculation



Integration-by-parts identities

Integration-by-parts identity

$$\int d^d k_1 \cdots d^d k_l \frac{\partial}{\partial k_i^\mu} \left[q_j^\mu \frac{\mathcal{N}(k_1, \ldots, k_l, \mathbf{s})}{D_1^{a_1} \cdots D_t^{a_t}} \right] = 0$$

$$q_j^\mu = \{k_j^\mu, p_j^\mu\}$$

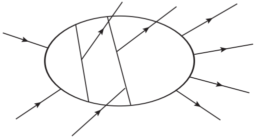
[K. G. Chetyrkin and F.V. Tkachov (1981)]

- Comes from:
  - Dimensional regularization;
  - Invariance of Feynman integrals with respect to shifts of loop momenta $k_i \to A_{ij} k_j + B_{ij} p_j$ $(|\det(A)| = 1)$;
- Generates a linear system of equations;
- IBP generation is performed by computer programs (Reduze, Kira, Fire, etc.)

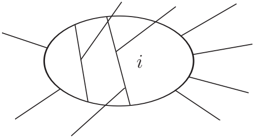
[S. Laporta (2001)]
[A.V. Smirnov (2008)]
[A. von Manteuffel and C. Studerus (2012)]
[P. Maierhoefer, J. Usovitsch and P. Uwer (2017)]

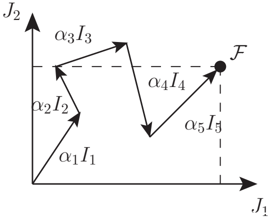
Reduction of Feynman integrals

$$\mathcal{F}(\mathbf{s}) = \sum_i \beta_i J_i(\mathbf{s}) \Longrightarrow$$



$$= \sum_i \beta_i$$





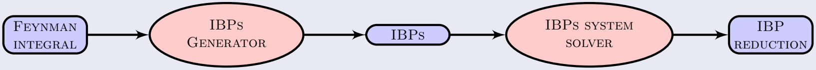
Reduction:

Decomposition of a Feynman integral into a basis of *master integrals*

- Generation of IBPs $\Longrightarrow$ Identification of MIs, $\bar{J}(\mathbf{s})$;
- MIs $\Longrightarrow$ Reduction of Feynman integrals

Automation of IBP reduction

Automating IBP reduction



Issue

The current IBP reduction strategy is *not optimized for NNLO calculations*:

- $2 \to 3$ NNLO corrections;
- $2 \to 2$ NNLO corrections with more than four scale parameters;

Idea

Exploit *Unitarity of Scattering Amplitudes* in order to develope an algorithm aimed at ameliorating the current IBP reduction stategy

Unitarity-based methods

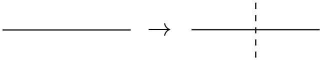
Unitarity of the S matrix

$$S^{+}S = \mathbb{I} \quad \Longrightarrow \quad i(T^{+} - T) = T^{+}T$$

Consequences:

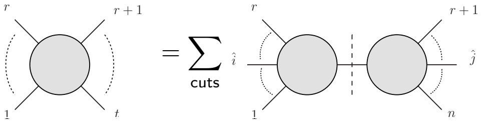
- Generalized Unitarity Cuts

$$\frac{1}{q^2 - m^2 + i\epsilon} \to 2\pi i \delta(q^2 - m^2),$$



[Z. Bern, L. Dixon, D.C. Dunbar, D.A. Kosower (1994)]

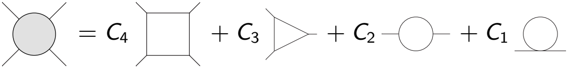
- BCFW tree-level recurrence relation



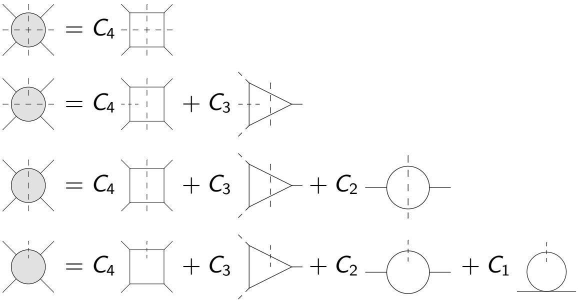
[R. Britto, F. Cachazo and B. Feng, (2005)]
[R. Britto, F. Cachazo, B. Feng, and E. Witten (2005)]

Unitarity-Based Methods and Reduction

4-point function reduction:



Multiple cuts on 4-point function:



Generalized unitarity

Use of unitarity cuts $\Longrightarrow$ Reconstruction of Feynman integrals reductions

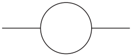
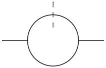
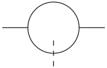
Unitarity-based methods and Partial fractioning: BCFW

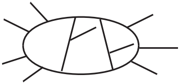
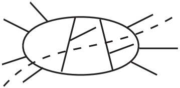
BCFW recurrence relation on a two propagator tree-level amplitude:

BCFW tree-level recurrence relation

$$\frac{1}{D_1 D_2} = \frac{1}{D_1(D_2 - D_1)} + \frac{1}{(D_1 - D_2)D_2}$$

This is the *Partial fractioning decomposition formula*: $\frac{1}{D_1 D_2} = \frac{A}{D_1} + \frac{B}{D_2}$.
Comparing with unitarity cuts:

1-loop  $= \int d^d k \frac{1}{D_1 D_2} = \int d^d k \frac{1}{D_1}$  $+ \int d^d k \frac{1}{D_2}$ 

2-loop  $= \sum_{i,j,k}^{\text{cuts}} \int d^d k_1 d^d k_2 \frac{1}{D_i D_j D_k}$  $_{ijk}$

Both cuts and partial fractioning generate *differences between denominators*.

Unitarity Cuts $\sim$ Partial fractioning

New idea: Partial fractioning for Feynman integrals

Observation

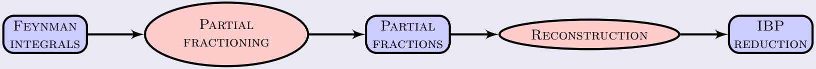
Partial fractioning $\sim$ Unitarity cuts $\implies$ IBP Reconstruction

Goal of our work

Develope a *new reduction strategy inspired by unitarity* of Scattering Amplitudes

$\implies$ *Partial fractioning* of integrands of Feynman integrals

Algorithm



Result 1: Partial Fractioning for Feynman integrals

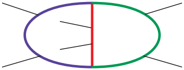
Result 1: We found a **symbolical expression** of the partial fractioning identity

Generalization of Partial Fractioning

$$\frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_t^{a_t}} = \sum_{p=1}^{t} \sum_{i=0}^{a_p - 1} \sum_{\{j_1,\dots,\hat{j}_p,\dots,j_t\}=0}^{j_1+\dots+\hat{j}_p+\dots+j_t=i} \frac{\prod_{k=1,k\neq p}^{t} (-1)^{j_k} \binom{a_k+j_k-1}{a_k-1}}{(D_1 - D_p)^{a_1+j_1} \cdots D_p^{a_p-i} \cdots (D_p - D_t)^{a_t+j_t}}$$

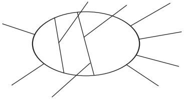
Properties of partial fractions:

- $D_j - D_i$ is *linear in loop momenta*;
- the $j-$th propagator is represented by a dashed line

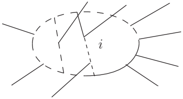


$$D_j \to D_j - D_i \implies \text{———} \to \text{- - - - - - -}$$

- Integrals of partial fractions dubbed *linearized Feynman integrals*;
- Multi-loop extension $\implies$ Partial fractioning for every *Branch*.



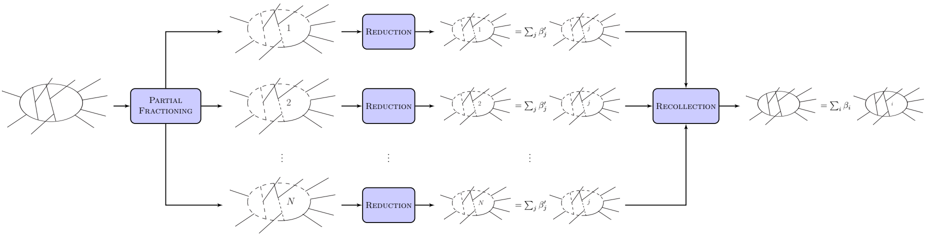
$$= \sum_{\bar{i}} B_{\bar{i}}$$



New idea: Developments of a novel algorithm

Observations

- degree of partial fractions denominators $<$ degree of starting integrand denominators
- The IBP reductions of linearized Feynman integrals can be parallelised

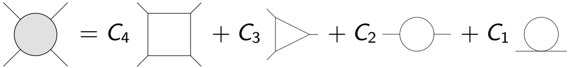


Recollection

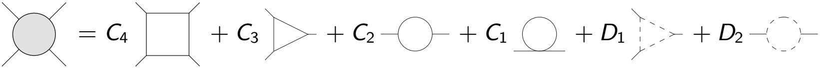
Combining Master Integrals found in the reductions of linearized Feynman Integrals

Result 2: Reconstruction of IBP Reduction

IBP Reduction of 1-loop Feynman Integral:



IBP Reconstruction of 1-loop Feynman Integral with our algorithm:



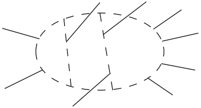
Spurious terms appear

Linearized Feynman Integrals with no quadratic propagators. In general $D_1 \neq 0$ and $D_2 \neq 0$.

Result 2: Classification of linearized Feynman integrals lead us to the identification of vanishing integrals

Classification

We proven that every *Linearized Feynman Integral made solely of linear denominators* **vanishes**.



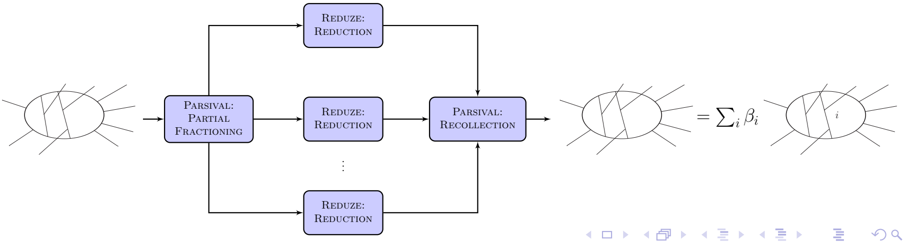
$= 0$

PARSIVAL: PARtial fraction-based Strategy for Feynman Integral eVALuation

Result 3: We implemented our new algorithm in a MATHEMATICA code, called PARSIVAL.

It automates:

- the decomposition Feynman integrals in linearized Feynman Integrals;
- the parallelisation of IBP reductions of linearized Feynman Integrals;
- the reconstruction of the IBPs of the starting Feynman integral

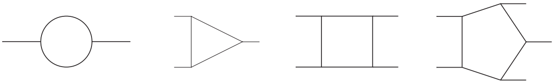


Strenght of PARSIVAL

- **Generality**: partial fractioning can be applied to any multi-loop Feynman integral;
- **Flexibility**: it can be interfaced with any available IBP generator.

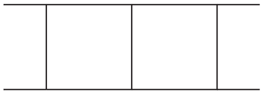
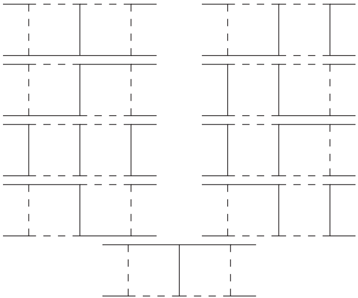
Tests

- 1-loop Feynman diagrams



- 2-loop Feynman diagrams

Massless planar box 2-loop diagram

Test with existing routine: REDUZE	Test with our code: PARSIVAL+REDUZE
	

RUNNING TIME
20399s

RUNNING TIMES	
6936s	14080s
5897s	5448s
5497s	16150s
15001s	9842s
6275s	

Conclusions and outlines

Conclusions:

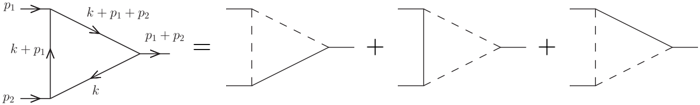
- Novel algorithm
  - The improvement of Feynman integrals evaluation passes through unitarity-based methods;
  - BCFW tree-level recurrence inspired the use of partial fractioning;
  - Partial fractions shares properties with unitarity cuts;
  - From IBP reconstruction we get the right IBP reduction;
  - The procedure exposed is completely general, applyable to any Feynman integral.
- Automation
  - Implementation of the novel algorithm in Parsival;
  - 1-loop and simple 2-loop tests shown an improvement of IBP generation time;

Outlook:

- Possibility to improve Parsival (e.g. adding symmetries of the diagram);
- Improvements on our new algorithm passes through the understanding of *linearized Feynman integrals*;
- Use of Parsival in the evaluation of $2 \to 2$ and $2 \to 3$ 2-loop Feynman integrals.

Thank you for the attention!

Example: Partial fractioning for QED 1-loop vertex



$$\int \frac{d^d k}{D_1 D_2 D_3} = \int \frac{d^d k}{D_1(D_2 - D_1)(D_3 - D_1)} + \int \frac{d^d k}{(D_1 - D_2)D_2(D_3 - D_2)} + \int \frac{d^d k}{(D_1 - D_3)(D_2 - D_3)D_3}$$

Example

$$D_1 = k^2 - m_1^2, \qquad D_2 = (k + p_1)^2 - m_2^2, \qquad D_3 = (k + p_1 + p_2)^2 - m_3^2$$

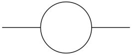
$$D_2 - D_1 = p_1 \cdot (2k + p_1) - (m_2^2 - m_1^2)$$

$$D_3 - D_1 = (p_1 + p_2) \cdot (2k + p_1 + p_2) - (m_3^2 - m_1^2)$$

$$D_3 - D_2 = p_2 \cdot (2k + 2p_1 + p_2) - (m_3^2 - m_2^2)$$

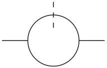
Action of generalized unitarity

1-loop bubble diagram:



$$= \int d^d k \frac{1}{D_1 D_2}$$

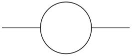
Single cut of 1-loop bubble diagram:



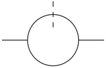
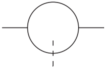
$$= \int d^d k \frac{\delta(D_1)}{D_2|_{D_1=0}} = \int d^d k \frac{\delta(D_1)}{D_2 - D_1}$$

Cuts generates *differences between denominators*

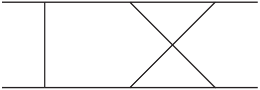
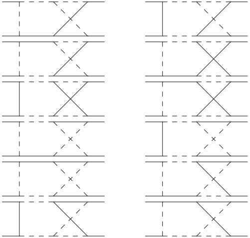
Using a little abuse of notation:
Partial fractioning of 1-loop bubble diagram:



$$= \int d^d k \frac{1}{D_1(D_2 - D_1)} + \int d^d k \frac{1}{(D_1 - D_2)D_2}$$

$$= \int d^d k \frac{1}{D_1}$$  $$+ \int d^d k \frac{1}{D_2}$$ 

Massless non-planar box 2-loop diagram

Test with existing routine: Reduze	Test with our code: Parsival+Reduze
	

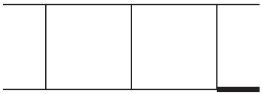
Running time
37235s

Running times	
3963s	2627s
11023s	6589s
17967s	7124s
4924s	12570s
4929s	18957s
4154s	7850s

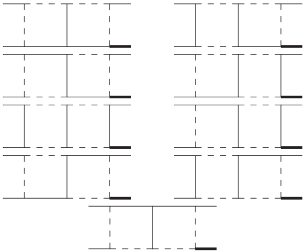
Parsival: running tests

Massless planar box 2-loop diagram - 1 massive external leg

Test with existing routine: Reduze	Test with our code: Parsival+Reduze



Running time
38722s

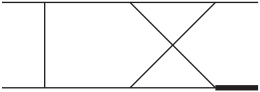


Running times	
5608s	11795s
6890s	61770s
79043s	73513s
44039s	54455s
74978s	

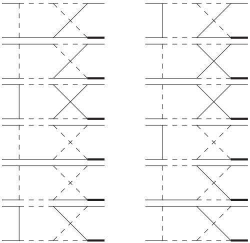
PARSIVAL: RUNNING TESTS

Massless non-planar box 2-loop diagram - 1 massive external leg

Test with existing routine: REDUZE	Test with our code: PARSIVAL+REDUZE



RUNNING TIME
107277s



RUNNING TIMES	
50571s	63103s
83474s	104584s
103021s	111266s
91578s	75797s
84110s	94146s
56200s	58803s