

# Towards analytic local sector subtraction at NNLO

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# Status

- ▶ **Considerable and successful** activity in NNLO subtraction in recent years. Different schemes face the problem in different ways (local vs non-local, analytic vs numerical, ...)
  - ▶ Antenna subtraction [Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.]
  - ▶ Sector-improved residue subtraction [Czakon, Mitov, et al.], [Boughezal, Caola, Melnikov, Petriello, et al.]
  - ▶ Colourful subtraction [Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.]
  - ▶ See also several NNLO computations with qT-slicing [Catani, Grazzini, et al.], N-jettiness slicing [Boughezal, Petriello, et al.], [Gaunt, Tackmann, et al.], projection to Born [Cacciari, Salam, Zanderighi, et al.], sector decomposition [Anastasiou, et al.], [Binoth, et al.],  $\mathcal{E}$ -prescription [Frixione, Grazzini], see also unsubtraction [Rodrigo, Sborlini, et al.].

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- ▶ Complexity of **subtraction structures** increases substantially with respect to NLO.
- ▶ Problem most of the times tackled **introducing various new elements/techniques with respect to commonly used analytic NLO solutions.**

# Motivations

- ▶ Investigation on **possible alternative NNLO schemes** motivated by questions like:
  - ▶ what are the **simplest possible** objects to achieve **local analytic subtraction at NNLO**?
  - ▶ are we maximally **exploiting the freedom** in defining the subtraction procedure?
  - ▶ what are NLO-subtraction ideas useful for NNLO, what bottlenecks? Can we cure them?
  - ▶ can we still hope to manage extensions (masses, higher orders) analytically?

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  - ▶ can we still hope to manage extensions (masses, higher orders) analytically?
- ▶ Won't provide final answers to all these questions, but this study is showing promising directions to be investigated.
- ▶ In the following, preliminary results on **massless and final-state-only** QCD partons.

## Subtraction at NLO

- ▶ Schematic structure of NLO correction ( $X = \text{IRC safe}$  observable,  $\delta_{X_i} \equiv \delta(X - X_i)$ ),  $X_i =$  observable computed with  $i$ -body kinematics.

$$\frac{d\sigma_{\text{NLO}} - d\sigma_{\text{LO}}}{dX} = \int d\Phi_n V \delta_{X_n} + \int d\Phi_{n+1} R \delta_{X_{n+1}} = \text{finite.}$$

$V$  diverges as  $\epsilon^{-2}$  (with  $d = 4 - 2\epsilon$ ),  $R$  diverges (doubly) in the radiation phase space.

- ▶ Add and subtract the counterterm

$$\int d\Phi_{n+1} K \delta_{X_n}.$$

$K$  with the same singularities as  $R$ , but simple to be integrated in  $d$  dimensions.

Possible thanks to the universality of IRC singularities.

- ▶ Subtracted cross section ( $I = \int \frac{d\Phi_{n+1}}{d\Phi_n} K$ ) becomes

$$\frac{d\sigma_{\text{NLO}} - d\sigma_{\text{LO}}}{dX} = \int d\Phi_n (V + I) \delta_{X_n} + \int d\Phi_{n+1} (R \delta_{X_{n+1}} - K \delta_{X_n})$$

- ▶  $V + I$  is finite for  $\epsilon = 0$ ,  $R - K$  has no phase-space singularities. Everything integrable numerically in  $d = 4$ .

## Subtraction at NLO: sectors

- ▶ All well established. Most used schemes Catani-Seymour (CS) dipole subtraction [Catani, Seymour, 9605323], [Catani et al., 0201036], FKS residue subtraction [Frixione, Kunszt, Signer, 9512328], [Frixione, 9706545], Nagy-Soper [Nagy, Soper, 0308127].
- ▶ FKS introduces a partition of the phase spaces into **sectors**, minimising the complexity of the subtraction problem.
  - ▶ Transparent singularity structure: only a limited number of identified partons can go soft/collinear in a given sector.
  - ▶ Help disentangling overlapping singularities.
  - ▶ Each sector can be treated separately, intrinsic parallelisation.

## ‘Standard’ sector subtraction at NLO: FKS

- ▶ Divide phase space with sector functions  $\mathcal{W}_{ij}$  dampening **all real singularities but** single soft ( $\xi_i \equiv 2E_i/\sqrt{s} \rightarrow 0$ ), and single collinear ( $y_{ij} \equiv \cos\theta_{ij} \rightarrow 1$ ).
- ▶ Properties:

$$\lim_{\xi_i \rightarrow 0} \sum_j \mathcal{W}_{ij} = 1, \quad \lim_{y_{ij} \rightarrow 1} (\mathcal{W}_{ij} + \mathcal{W}_{ji}) = 1.$$

- ▶ By summing over the sectors that do not vanish in the IRC limits, **the  $\mathcal{W}$ 's disappear.** **Key for the analytic integration of the counterterm:** integrating  $\mathcal{W}$ 's would be very challenging analytically.
- ▶ Each sector parametrised differently, in terms of  $\xi_i$  and  $y_{ij}$ . **Local** counterterm defined **after the parametrisation has been chosen:**

$$K_{ij} \sim R\mathcal{W}_{ij}\Big|_{\xi_i=0} + R\mathcal{W}_{ij}\Big|_{y_{ij}=1} - R\mathcal{W}_{ij}\Big|_{\substack{\xi_i=0 \\ y_{ij}=1}},$$

namely residues in the Laurent expansion around the IRC limits.



## Potential bottlenecks in counterterm analytic integration

- ▶ **Very successful and natural** subtraction method, but a couple of suboptimal features.
- ▶ By defining the counterterm **after parametrisation**, flexibility in its integration is lost: e.g. soft limit ( $\xi_i = 0$ ) features an eikonal sum  $\sum_{kl} \frac{s_{kl}}{s_{ik}s_{il}}$  that leaves

$$\int \frac{d\Phi_{n+1}}{d\Phi_n} \sum_j K_{ij}^{(\text{soft})} \sim \sum_{kl} \int d\bar{\Omega}_i \frac{1 - \cos \bar{\theta}_{kl}}{(1 - \cos \bar{\theta}_{ki})(1 - \cos \bar{\theta}_{il})}.$$

**Simplifications** (reparametrisations) after the energy variable is pulled out **are limited**.

- ▶ Non-invariant parametrisation: potentially complicated expressions in view of NNLO.
- ▶ Radiation phase space

$$\frac{d\Phi_{n+1}}{d\Phi_n} \sim d\xi_i dy_{ij} \left( \frac{1}{2 - \xi_i(1 - y_{ij})} \right)^{-2\epsilon} f(\xi_i)g(y_{ij})$$

easily integrated in  $d$  dimensions **only** because denominator ‘accidentally’ trivial **in all IRC limits**.

## ‘Modified’ sector subtraction at NLO (as a laboratory for NNLO)

- ▶ Singularity structure known in advance: build the easiest possible function with all real poles in sector  $ij$  in terms of dot products  $s_{ab} = 2p_a \cdot p_b$ , before parametrising.
- ▶ Action of singular limits on dot products
  - ▶ Soft limit  $\mathbf{S}_i$  ( $p_i^\mu \rightarrow 0$ ):  $s_{ia}/s_{ib} \rightarrow \text{constant}$ ,  $s_{ia}/s_{bc} \rightarrow 0$ ,  $\forall a, b, c \neq i$ .
  - ▶ Collinear limit  $\mathbf{C}_{ij}$  ( $k_\perp \rightarrow 0$ ):  $s_{ij}/s_{ia} \rightarrow 0$ ,  $s_{ij}/s_{jb} \rightarrow 0$ ,  $s_{ij}/s_{ab} \rightarrow 0$ ,  $\forall a, b \neq i, j$ .  
 $s_{ia}/s_{ja} \rightarrow \text{independent of } a$ .

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 $s_{ia}/s_{ja} \rightarrow \text{independent of } a$ .
- ▶  $\mathbf{S}_i R$  ( $\mathbf{C}_{ij} R$ ) are the most singular terms in  $R$  as  $p_i^\mu$  ( $k_\perp$ ) goes to 0.  
Limits commute:  $\mathbf{S}_i \mathbf{C}_{ij} R = \mathbf{C}_{ij} \mathbf{S}_i R$ .
- ▶ Function  $K_{ij} = (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) R \mathcal{W}_{ij} = [\mathbf{1} - (\mathbf{1} - \mathbf{S}_i)(\mathbf{1} - \mathbf{C}_{ij})] R \mathcal{W}_{ij}$  (limits applied both to  $R$  and to  $\mathcal{W}_{ij}$ ) contains all singularities of  $R \mathcal{W}_{ij}$  in sector  $ij$ , and  $R_{ij}^{\text{sub}} = R \mathcal{W}_{ij} - K_{ij}$  is finite everywhere in the phase space.
- ▶ Structure of the local counterterm  $K_{ij}$  in each sector is as minimal as in FKS (one-to-one correspondence), but parametrisation independent.

## Mapping to Born kinematics

- ▶ Momentum mapping  $p_1, \dots, p_{n+1} \rightarrow \bar{p}_1, \dots, \bar{p}_n$  necessary to factorise Born phase space from radiation phase space, and integrate the counterterm only in the latter.
- ▶ Born (and colour-correlated Born) amplitudes implicitly appearing in the counterterm  $(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) R \mathcal{W}_{ij}$  written in terms of mapped  $\bar{p}_i$  momenta.
- ▶ Convenient **Catani-Seymour** mappings

$$p_i + p_j + p_k = \bar{p}_{[ij]} + \bar{p}_k,$$
$$\bar{p}_k = \frac{1}{1-y} p_k, \quad \bar{p}_{[ij]} = p_i + p_j - \frac{y}{1-y} p_k,$$

with  $i, j, k$  chosen to simplify as much as possible counterterm integration. At variance with FKS, where the sector defines the mapping.

- ▶ Modified sector subtraction at NLO is like a bridge between FKS and CS, **retaining the strengths of both** (sector approach, and minimal structure from FKS; Lorentz invariance, and phase-space mappings from CS).

## Advantages in counterterm integration

- ▶ In a given sector, there is **freedom to choose mappings and parametrisations differently for different contributions to the counterterm.**
- ▶ E.g. in the soft **each term in the eikonal sum  $\sum_{kl} \frac{s_{kl}}{s_{ik}s_{il}}$  is mapped and parametrised differently**, yielding straightforward analytic integration

$$\int \frac{d\Phi_{n+1}}{d\Phi_n} \frac{s_{kl}}{s_{ik}s_{il}} \propto (\bar{\mathbf{p}}_{[ik]} \cdot \bar{\mathbf{p}}_l)^{-\epsilon} \int_0^1 dy dz \left( y(1-y)^2 z(1-z) \right)^{-\epsilon} \frac{(1-y)(1-z)}{yz}$$
$$\propto (\bar{\mathbf{p}}_{[ik]} \cdot \bar{\mathbf{p}}_l)^{-\epsilon} B(-\epsilon, 2-\epsilon) B(-\epsilon, 2-2\epsilon).$$

- ▶  $y$  and  $z$  Catani-Seymour variables **for dipole  $ikl$** , in terms of which the phase space is simple (factorised) also far from the singular limits.
- ▶ ‘Modified’ sector subtraction has been **successfully applied at NLO: integrated counterterm shown analytically to reproduce all virtual poles.**
- ▶ Method works as well as FKS and CS, but **seems more easily exportable to NNLO** (still, study limited to massless and final-state-only QCD partons).

## Subtraction at NNLO

- Schematic structure of NNLO cross section ( $X = \text{IRC safe}$  observable,  $\delta_{X_i} \equiv \delta(X - X_i)$ )

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n VV \delta_{X_n} + \int d\Phi_{n+1} RV \delta_{X_{n+1}} + \int d\Phi_{n+2} RR \delta_{X_{n+2}}$$

$VV$  diverges as  $\epsilon^{-4}$  (with  $d = 4 - 2\epsilon$ ),  $RR$  diverges (quadruply) in the radiation phase space,  $RV$  diverges as  $\epsilon^{-2}$  and (doubly) in the radiation phase space.

- Add and subtract the counterterms

$$\int d\Phi_{n+2} \left( K^{(1)} \delta_{X_{n+1}} + K^{(2)} \delta_{X_n} \right), \quad \int d\Phi_{n+2} K^{(\text{RV})} \delta_{X_n}.$$

- Subtracted cross section ( $I^{(j)} = \int \frac{d\Phi_{n+2}}{d\Phi_{n+2-j}} K^{(j)}$ ,  $I^{(\text{RV})} = \int \frac{d\Phi_{n+1}}{d\Phi_n} K^{(\text{RV})}$ ) becomes

$$\begin{aligned} \frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} &= \int d\Phi_n (VV + I^{(2)} + I^{(\text{RV})}) \delta_{X_n} \\ &+ \int d\Phi_{n+1} \left( (RV + I^{(1)}) \delta_{X_{n+1}} - K^{(\text{RV})} \delta_{X_n} \right) \\ &+ \int d\Phi_{n+2} \left( RR \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - K^{(2)} \delta_{X_n} \right). \end{aligned}$$

Everything integrable numerically in  $d = 4$ .

## Sector functions at NNLO

- ▶ Divide phase space with sector functions  $\mathcal{W}_{abcd}$  dampening **all double-real singularities** **but** single soft and single collinear ( $\mathbf{S}_i, \mathbf{C}_{ij}$ ), double soft and double collinear ( $\mathbf{S}_{i,k}, \mathbf{C}_{ikj}, \mathbf{C}_{ij,kl}$ ), and soft collinear  $\mathbf{SC}_{i,kl}$ .

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- ▶ Required properties:
  - ▶ In the double-unres. limits, upon summing over the sectors that do not vanish,  $\mathcal{W}$ 's must disappear  $\implies$  **analytic integration of the  $K^{(2)}$  counterterms.**

$$\mathbf{S}_{i,k} \sum_{\text{perm } ik} \sum_j (\mathcal{W}_{ijkj} + \mathcal{W}_{ikkj} + \sum_l \mathcal{W}_{ijkl}) = 1, \quad \mathbf{C}_{ikj} \sum_{\text{perm } ijk} (\mathcal{W}_{ijjk} + \mathcal{W}_{ijkj}) = 1.$$

- ▶ In the single-unres. limits, NNLO sector functions must factorise NLO ones  $\implies$  combination of the single-unresolved  $I^{(1)}$  with the subtracted  $R\bar{V}$  **NLO sector by NLO sector.**

$$\begin{aligned} \mathbf{C}_{ij} \mathcal{W}_{ijkj} &\sim \bar{\mathcal{W}}_{k[ij]} \mathbf{C}_{ij} \mathcal{W}_{ij}, & \mathbf{S}_i \mathcal{W}_{ijkj} &\sim \bar{\mathcal{W}}_{kj} \mathbf{S}_i \mathcal{W}_{ij}, \\ \mathbf{C}_{ij} \mathcal{W}_{ijkl} &\sim \bar{\mathcal{W}}_{kl} \mathbf{C}_{ij} \mathcal{W}_{ij}, & \mathbf{S}_i \mathcal{W}_{ijkl} &\sim \bar{\mathcal{W}}_{kl} \mathbf{S}_i \mathcal{W}_{ij}. \end{aligned}$$



## Structure of the double-real counterterms

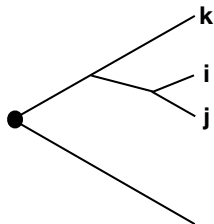
- Different combinations of indices in the sector functions select different singular limits:

sector  $\mathcal{W}_{ijkj} : \mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{SC}_{i,jk}, \mathbf{C}_{ikj}, \mathbf{S}_{i,k},$

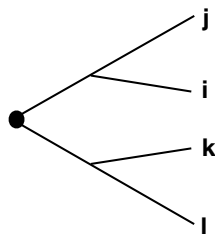
sector  $\mathcal{W}_{ijkl} : \mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{SC}_{i,kl}, \mathbf{C}_{ij,kl}, \mathbf{S}_{i,k},$

(sector  $\mathcal{W}_{ijjk}$  fully analogous to  $\mathcal{W}_{ijkj}$  with  $\mathbf{S}_{i,k} \leftrightarrow \mathbf{S}_{i,j}$ ).

- Roughly, they correspond to two topologies



$\mathcal{W}_{ijkj}$



$\mathcal{W}_{ijkl}$

## Structure of the double-real counterterms

- ▶ Local counterterm in sector  $ijkj$  built from the singularities  $\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{SC}_{i,jk}, \mathbf{C}_{ikj}, \mathbf{S}_{i,k}$  that are not killed by function  $\mathcal{W}_{ijkj}$ :

$$K_{ijkj}^{(1)} + K_{ijkj}^{(2)} = [\mathbf{1} - (\mathbf{1} - \mathbf{S}_i)(\mathbf{1} - \mathbf{C}_{ij})(\mathbf{1} - \mathbf{S}_{i,k})(\mathbf{1} - \mathbf{C}_{ikj})(\mathbf{1} - \mathbf{SC}_{i,jk})]RR\mathcal{W}_{ijkj}.$$

- ▶ Analogously for sector  $ijkl$ :

$$K_{ijkl}^{(1)} + K_{ijkl}^{(2)} = [\mathbf{1} - (\mathbf{1} - \mathbf{S}_i)(\mathbf{1} - \mathbf{C}_{ij})(\mathbf{1} - \mathbf{S}_{i,k})(\mathbf{1} - \mathbf{C}_{ij,kl})(\mathbf{1} - \mathbf{SC}_{i,kl})]RR\mathcal{W}_{ijkl}.$$

- ▶  $\mathbf{S}_{i,k}RR$ ,  $\mathbf{C}_{ikj}RR$ , and  $\mathbf{SC}_{i,jk}RR$  are universal kernels [Catani, Grazzini, 9810389, 9908523], [Campbell, Glover, 9710255], [Berends, Giele, 1989].
- ▶ **All limits commute**: the above expressions collect all double-real poles in sectors  $ijkj$  and  $ijkl$  (and similarly for  $ijjk$ ).
- ▶ See also [Frisoni, Grazzini, 0411399] for a discussion on commutation and counterterm minimality.

## Simplification of the double-real counterterms

- ▶ Out of five singular limits, **only four are truly ‘irreducible’** (no more than four poles per sector in  $RR$ ).
- ▶ Idempotency of IRC projectors eliminates redundancies. In sector  $ijkj$ :

$$\begin{aligned}
 \mathbf{S}_i \mathbf{S} \mathbf{C}_{i,jk} \{ \mathbf{1}, \mathbf{C}_{ij}, \mathbf{S}_{i,k}, \dots \} RR &= \mathbf{S} \mathbf{C}_{i,jk} \{ \mathbf{1}, \mathbf{C}_{ij}, \mathbf{S}_{i,k}, \dots \} RR \\
 \implies K_{ijkj}^{(2)} &= (\mathbf{S}_{i,k} + \mathbf{C}_{ikj} - \mathbf{S}_{i,k} \mathbf{C}_{ikj}) RR \mathcal{W}_{ijkj}, \\
 K_{ijkj}^{(1)} &= (\mathbf{1} - \mathbf{S}_{i,k}) (\mathbf{1} - \mathbf{C}_{ikj}) (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) RR \mathcal{W}_{ijkj}.
 \end{aligned}$$

- ▶ Analogously for sector  $ijkl$ :

$$\begin{aligned}
 \mathbf{S}_i \mathbf{S} \mathbf{C}_{i,kl} \{ \mathbf{1}, \mathbf{C}_{ij}, \mathbf{S}_{i,k}, \dots \} RR &= \mathbf{S} \mathbf{C}_{i,kl} \{ \mathbf{1}, \mathbf{C}_{ij}, \mathbf{S}_{i,k}, \dots \} RR, \\
 \mathbf{C}_{ij} \mathbf{C}_{ij,kl} \{ \mathbf{1}, \mathbf{S}_{i,k}, \dots \} RR &= \mathbf{C}_{ij,kl} \{ \mathbf{1}, \mathbf{S}_{i,k}, \dots \} RR \\
 \implies K_{ijkl}^{(2)} &= \mathbf{S}_{i,k} RR \mathcal{W}_{ijkl}, \\
 K_{ijkl}^{(1)} &= (\mathbf{1} - \mathbf{S}_{i,k}) (\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) RR \mathcal{W}_{ijkl}.
 \end{aligned}$$

- ▶ Limits  $\mathbf{C}_{ij,kl}$  and  $\mathbf{S} \mathbf{C}_{i,jk}$  **completely disappear from the definition of the counterterm**, accounted for by the overlap of other limits (see also [\[Caola, Melnikov, Roentsch, 1702.01352\]](#) about the redundancy of  $\mathbf{S} \mathbf{C}$ ).

## Integration of the double-unresolved counterterm $K^{(2)}$

$$\begin{aligned} I^{(2)} &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \left\{ \sum_{ijk} (\mathbf{S}_{i,k} + \mathbf{C}_{ijk} - \mathbf{S}_{i,k} \mathbf{C}_{ijk})_{RR} (\mathcal{W}_{ijkj} + \mathcal{W}_{ikkj}) + \sum_{ijkl} \mathbf{S}_{i,k}{}_{RR} \mathcal{W}_{ijkl} \right\} \\ &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \left\{ \sum_{ik} \mathbf{S}_{i,k} + \sum_{ijk} (\mathbf{C}_{ijk} - \mathbf{S}_{i,k} \mathbf{C}_{ijk}) \right\}_{RR} \end{aligned}$$

- All sector functions are gone owing to their sum rules in the IRC limits. Integration of kernels manageable analytically.

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 &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \left\{ \sum_{ik} \mathbf{S}_{i,k} + \sum_{ijk} (\mathbf{C}_{ijk} - \mathbf{S}_{i,k} \mathbf{C}_{ijk}) \right\} RR
 \end{aligned}$$

- All sector functions are gone owing to their sum rules in the IRC limits. Integration of kernels manageable analytically.
- Examples: double-soft  $q\bar{q}$  (with two partons at Born) and double collinear  $qq'\bar{q}'$ . Use Catani-Seymour-like variables  $y, z, y', z'$ . Phase-space measure factorises  $d\mu = dx' dy' dz' dy dz [x'(1-x')y'(1-y')^2 z'(1-z')y^2(1-y)^2 z(1-z)]^{-\epsilon} [x'(1-x')]^{-1/2} (1-y')y(1-y)$

$$\begin{aligned}
 \int \frac{d\Phi_{n+2}}{d\Phi_n} \mathbf{S}_{i,k} RR &= (4\pi\alpha_S^u \mu_0^{2\epsilon})^2 T_R \sum_{l,m=1}^2 \tilde{B}_{lm} \int \frac{d\Phi_{n+2}}{d\Phi_n} \frac{4(s_{il}s_{km} + s_{im}s_{kl} - s_{ik}s_{lm})}{s_{ik}^2 (s_{il} + s_{kl})(s_{im} + s_{km})} \\
 &\propto \int_0^1 d\mu \frac{z'(1-z')}{y^2 y'^2} \frac{y'(1-z)}{y'(1-z) + z} \\
 &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \left( \frac{\mu^2}{s} \right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{17}{9\epsilon^2} + \frac{1}{\epsilon} \left( \frac{7}{18}\pi^2 - \frac{232}{27} \right) + \left( \frac{38}{9}\zeta_3 + \frac{131}{54}\pi^2 - \frac{2948}{81} \right) \right] + \mathcal{O}(\epsilon),
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{d\Phi_{n+2}}{d\Phi_n} \mathbf{C}_{ikj} RR &= (4\pi\alpha_S^u \mu_0^{2\epsilon})^2 T_R C_F \tilde{B} \int \frac{d\Phi_{n+2}}{d\Phi_n} \frac{4}{s_{ijk}^2} \frac{1}{2} \frac{s_{ijk}}{s_{ik}} \left[ -\frac{t_{ik,j}^2}{s_{ik}s_{ikj}} + \frac{4z_j + (z_i - z_k)^2}{z_i + z_k} + (1-2\epsilon) \left( z_i + z_k - \frac{s_{ik}}{s_{ikj}} \right) \right] \\
 &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \left( \frac{\mu^2}{s} \right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{31}{18\epsilon^2} + \frac{1}{\epsilon} \left( \frac{1}{2}\pi^2 - \frac{889}{108} \right) + \left( \frac{80}{9}\zeta_3 + \frac{31}{12}\pi^2 - \frac{23941}{648} \right) \right] + \mathcal{O}(\epsilon),
 \end{aligned}$$

## Real-virtual and integrated single-unresolved counterterms

- ▶ Real-virtual contribution has NLO kinematics, subtracted by counterterm

$$K_{kl}^{(\mathbf{RV})} = (\mathbf{S}_k + \mathbf{C}_{kl} - \mathbf{S}_k \mathbf{C}_{kl}) RV \bar{\mathcal{W}}_{kl}.$$

- ▶  $RV_{kl}^{\text{sub}} = RV \bar{\mathcal{W}}_{kl} - K_{kl}^{(\mathbf{RV})}$  finite in rad. phase space, but with  $1/\epsilon$  poles (one loop).
- ▶ Combined **sector by sector** with the integrated single-unresolved counterterm:

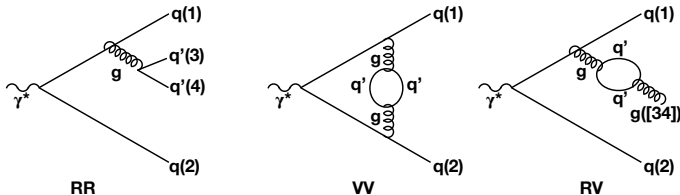
$$I_{kl}^{(1)} = \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} K_{kl}^{(1)}.$$

$K_{kl}^{(1)}$  is the collection of all terms in  $K^{(1)}$  featuring  $\bar{\mathcal{W}}_{kl}$ . **Possible due to the factorisation properties of the  $\mathcal{W}_{abcd}$  functions.**

- ▶  $RV \bar{\mathcal{W}}_{kl} + I_{kl}^{(1)}$  **finite in  $d = 4$**  (analogously to NLO subtraction, virtual plus integrated counterterm).

## A proof-of-concept example

- $T_{RCF}$  NNLO contribution to the total cross section for  $e^+e^- \rightarrow q\bar{q}$  (analytic matrix elements from [Hamberg, van Neerven, Matsuura, 1991], [Gehrmann De Ridder, Gehrmann, Glover, 0403057], [Ellis, Ross, Terrano, 1980])



$$VV = \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_{RCF} \left\{ \left( \frac{\mu^2}{s} \right)^{2\epsilon} \left[ \frac{1}{3\epsilon^3} + \frac{14}{9\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{11}{18}\pi^2 + \frac{353}{54} \right) + \left( -\frac{26}{9}\zeta_3 - \frac{77}{27}\pi^2 + \frac{7541}{324} \right) \right] \right. \\ \left. + \left( \frac{\mu^2}{s} \right)^\epsilon \left[ -\frac{4}{3\epsilon^3} - \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( \frac{7}{9}\pi^2 - \frac{16}{3} \right) + \left( \frac{28}{9}\zeta_3 + \frac{7}{6}\pi^2 - \frac{32}{3} \right) \right] \right\},$$

$$\int \frac{d\Phi_{n+1}}{d\Phi_n} RV = \frac{\alpha_S}{2\pi} \frac{2}{3} \frac{T_R}{\epsilon} \int \frac{d\Phi_{n+1}}{d\Phi_n} \bar{R} \\ = \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_{RCF} \left( \frac{\mu^2}{s} \right)^\epsilon \left[ \frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{7}{9}\pi^2 + \frac{19}{3} \right) + \left( -\frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + \frac{109}{6} \right) \right],$$

$$\int \frac{d\Phi_{n+2}}{d\Phi_n} RR = \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_{RCF} \left( \frac{\mu^2}{s} \right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon} \left( \frac{11}{18}\pi^2 - \frac{407}{54} \right) + \left( \frac{134}{9}\zeta_3 + \frac{77}{27}\pi^2 - \frac{11753}{324} \right) \right].$$

## Double-real integrated counterterms $I^{(2)}$ and $I^{(1)}$

- $I^{(2)}$ : in the case at hand only  $\mathbf{S}_{3,4}RR$ ,  $\mathbf{C}_{134}RR$ ,  $\mathbf{C}_{234}RR$  are non-zero, so

$$\begin{aligned}
 I^{(2)} &= \int \frac{d\Phi_{n+2}}{d\Phi_n} \left\{ \mathbf{S}_{3,4} + \mathbf{C}_{134} + \mathbf{C}_{234} - \mathbf{S}_{3,4}\mathbf{C}_{134} - \mathbf{S}_{3,4}\mathbf{C}_{234} \right\} RR \\
 &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \left( \frac{\mu^2}{s} \right)^{2\epsilon} \left[ -\frac{1}{3\epsilon^3} - \frac{14}{9\epsilon^2} + \frac{1}{\epsilon} \left( \frac{11}{18}\pi^2 - \frac{425}{54} \right) + \left( \frac{122}{9}\zeta_3 + \frac{74}{27}\pi^2 - \frac{12149}{324} \right) \right].
 \end{aligned}$$

- $I^{(1)}$ : in the case at hand only  $\mathbf{C}_{34}$  is non-zero, so

$$\begin{aligned}
 I^{(1)} &= I_{12}^{(1)} + I_{1[34]}^{(1)} + I_{2[34]}^{(1)}, \\
 I_{12}^{(1)} &= \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} \mathbf{C}_{34}RR (\bar{W}_{12} + \bar{W}_{21}), \\
 I_{i[34]}^{(1)} &= \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} \mathbf{C}_{34} \left[ RR (\bar{W}_{i[34]} + \bar{W}_{[34]i}) - \mathbf{C}_{i34}RR - \mathbf{S}_{3,4}RR \bar{W}_{[34]i} + \mathbf{S}_{3,4}\mathbf{C}_{i34}RR \right].
 \end{aligned}$$



## Real-virtual contribution

- ▶ Real virtual, subtracted in sector  $ij$ :

$$RV_{ij}^{\text{sub}} = RV\bar{\mathcal{W}}_{ij} - K_{ij}^{(\mathbf{RV})} = \frac{\alpha_S}{2\pi} \frac{2}{3} \frac{T_R}{\epsilon} \bar{R}_{ij}^{\text{sub}} \quad (\text{gluon self-energy contribution to } T_R C_F).$$

- ▶ Combination with the single-unresolved integrated counterterm  $I_{ij}^{(1)}$ :

$$\begin{aligned} RV_{12}^{\text{sub}} + I_{12}^{(1)} &= -\frac{\alpha_S}{2\pi} T_R \left( \frac{2}{3} \ln \frac{\mu^2}{\bar{s}_{1[34]}} + \frac{16}{9} \right) \bar{R}(\bar{\mathcal{W}}_{12} + \bar{\mathcal{W}}_{21}), \\ RV_{i[34]}^{\text{sub}} + I_{i[34]}^{(1)} &= -\frac{\alpha_S}{2\pi} T_R \left( \frac{2}{3} \ln \frac{\mu^2}{\bar{s}_{r[34]}} + \frac{16}{9} \right) [\bar{R}(\bar{\mathcal{W}}_{i[34]} + \bar{\mathcal{W}}_{[34]i}) - \mathbf{C}_{i[34]}\bar{R} - \mathbf{S}_{[34]}\bar{R}\bar{\mathcal{W}}_{[34]i} + \mathbf{S}_{[34]}\mathbf{C}_{i[34]}\bar{R}] \end{aligned}$$

$I^{(1)}$  **NLO sector by NLO sector** factorises the same structure of  $\mathcal{W}$ 's as the subtracted  $RV$  (in this case proportional to the full subtracted real).

**Sum finite in  $d = 4$  and integrated numerically.**

- ▶ Real-virtual integrated counterterm (summed over NLO sectors)

$$\begin{aligned} I^{(\mathbf{RV})} &= \sum_{ij} \int \frac{d\Phi_{n+1}}{d\Phi_n} K_{ij}^{(\mathbf{RV})} = \frac{\alpha_S}{2\pi} \frac{2}{3} \frac{T_R}{\epsilon} \int \frac{d\Phi_{n+1}}{d\Phi_n} (\mathbf{S}_{[34]} + \mathbf{C}_{1[34]} + \mathbf{C}_{2[34]} - \mathbf{S}_{[34]}\mathbf{C}_{1[34]} - \mathbf{S}_{[34]}\mathbf{C}_{2[34]}) \bar{R} \\ &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \left( \frac{\mu^2}{s} \right)^\epsilon \left[ \frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( -\frac{7}{9}\pi^2 + \frac{20}{3} \right) + \left( -\frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + 20 \right) \right] + \mathcal{O}(\epsilon). \end{aligned}$$

## Collection of results

- ▶ Example for  $\mu/\sqrt{s} = 0.35$ .

- ▶ Subtracted double-virtual (fully analytic integrated counterterms):

$$\begin{aligned}VV + I^{(2)} + I^{(\mathbf{RV})} &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \left( \frac{8}{3} \zeta_3 - \frac{1}{9} \pi^2 - \frac{44}{9} - \frac{4}{3} \ln \frac{\mu^2}{s} \right) \\ &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \times 0.01949914.\end{aligned}$$

- ▶ Subtracted real-virtual and double-real (integrated numerically in  $d = 4$ ):

$$\begin{aligned}\int \frac{d\Phi_{n+1}}{d\Phi_n} (RV + I^{(1)} - K^{(\mathbf{RV})}) &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \times (-0.90635 \pm 0.00011), \\ \int \frac{d\Phi_{n+2}}{d\Phi_n} (RR - K^{(1)} - K^{(2)}) &= \tilde{B} \left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F \times (+2.29491 \pm 0.00038).\end{aligned}$$

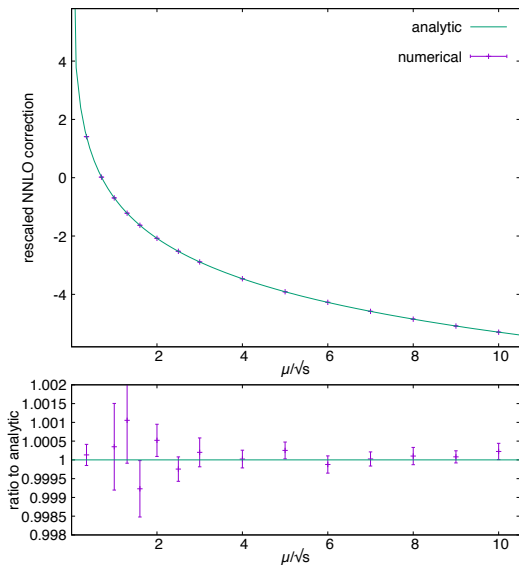
- ▶ NNLO correction, evaluated with subtraction method, is

$$\frac{1}{\left( \frac{\alpha_S}{2\pi} \right)^2 T_R C_F} \frac{\sigma_{\text{NNLO}} - \sigma_{\text{NLO}}}{\sigma_{\text{LO}}} = 1.40806 \pm 0.00040.$$

- ▶ Analytic result

$$-\frac{11}{2} + 4\zeta_3 - \ln \frac{\mu^2}{s} = 1.40787186.$$

## Renormalisation-scale dependence



## Conclusions

- ▶ Investigation on **possible simplifications** for local analytic subtraction at NNLO.
- ▶ Approach conjugates counterterm **minimality**, and integration **simplicity**.
- ▶ **Use of sector functions** like FKS but more flexible and parametrisation independent.
- ▶ Method works at NLO in general, and shown at NNLO in a proof-of-concept case.  
**Compact local counterterms integrable analytically** owing to sector-function sum rules.  
**No sector decomposition** involved.
- ▶ Preliminary study on massless and final-state-only QCD. More general cases not yet investigated; hopefully complexity scaling under control.
- ▶ Structural simplicity may display links to fundamental concepts, like factorisation, or may be considered for automation.

Thank you for your attention

## Backup: soft/collinear commutation at NLO

- ▶ Soft limit  $\mathbf{S}_i$  ( $p_i^\mu \rightarrow 0$ ):  $s_{ia}/s_{ib} \rightarrow \text{constant}$ ,  $s_{ia}/s_{bc} \rightarrow 0$ ,  $\forall a, b, c \neq i$ .
- ▶ Collinear limit  $\mathbf{C}_{ij}$  ( $k_\perp \rightarrow 0$ ):  $s_{ij}/s_{ia} \rightarrow 0$ ,  $s_{ij}/s_{jb} \rightarrow 0$ ,  $s_{ij}/s_{ab} \rightarrow 0$ ,  $\forall a, b \neq i, j$ .  
 $s_{ia}/s_{ja} \rightarrow \text{independent of } a$ .
- ▶ Commutation in case  $i = \text{gluon}$  and  $j = \text{quark}$ .
- ▶ Altarelli-Parisi collinear kernel involved is  $P(z_i) = [1 + (1 - z_i)^2]/z_i$ , with  $z_i = s_{ir}/(s_{ir} + s_{jr})$ , with arbitrary  $r \neq i, j$ .

$$\begin{aligned} \mathbf{S}_i R &= -8\pi\alpha_S \bar{\mu}^{2\epsilon} \sum_{l, k \neq l} \frac{s_{lk}}{s_{ik} s_{il}} B_{kl} \\ \implies \mathbf{C}_{ij} \mathbf{S}_i R &= -16\pi\alpha_S \bar{\mu}^{2\epsilon} \sum_{k \neq j} \frac{s_{jk}}{s_{ik} s_{ij}} B_{kj} = -16\pi\alpha_S \bar{\mu}^{2\epsilon} \frac{s_{jr}}{s_{ir} s_{ij}} (-C_j B), \\ \mathbf{C}_{ij} R &= 8\pi\alpha_S \bar{\mu}^{2\epsilon} \frac{1}{s_{ij}} C_j B \frac{1 + [1 - s_{ir}/(s_{ir} + s_{jr})]^2}{s_{ir}/(s_{ir} + s_{jr})} \\ \implies \mathbf{S}_i \mathbf{C}_{ij} R &= -16\pi\alpha_S \bar{\mu}^{2\epsilon} \frac{s_{jr}}{s_{ir} s_{ij}} (-C_j B). \end{aligned}$$