Elliptic Functions for Feynman Integrals

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Based on collaborations with *J. Brödel*, *C. Duhr*, *F. Dulat*, *B. Penante*, *A. von Manteuffel*, *A. Primo*, *E. Remiddi*

Computation of Feynman Integrals unavoidable to extract physical predictions from Quantum Field Theory.

1. At the **LHC**, thorough investigation of EW-symmetry breaking mechanism requires precise theoretical control of different $2 \to 2$ and $2 \to 3$ processes, with very high precision (requires at least 2-loop calculations)

$$VV, \quad Hjet, \quad HH, \quad 2\,jets, \quad t\bar{t}, \quad 3\,jets, \quad V + 2\,jets\ldots$$

2. With **massive loops** and/or **"high" multiplicities** (already $2 \to 3$!), many of these calculations are almost impossible (algebraic complexity, new mathematical structures)

$$\Downarrow$$

Many results in both math and physics literature (*ELi-functions, elliptic polylogarithms, iterated integrals over products of elliptic integrals, iterated integrals over modular forms, twisted elliptic zeta values...*)

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Are these all **different sides of the same story**?

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One thing is for sure: **These functions appear everywhere**!

- Already in **QED** at two/three loops (form factors, electron self-energy, ...)
- **QCD** Two-loop corrections to H+jet, (similarly V+jet production)
- **QCD** Two-loop corrections to HH (similarly VV production)
- **QCD** Two-loop corrections to $t\bar{t}$
- N^3LO corrections to Higgs production in **QCD**
- More or less any two-loop amplitude in the **EW theory**

Precision physics at the **LHC** seems to require “elliptic functions”[1]

[1] Or, at least, reliable numerical results for scattering amplitudes that would evaluate to elliptic generalizations of MPLs...

One possible point of view – **Differential equations method**
[Kotikov '90, Remiddi '97, **Gehrmann-Remiddi '00**,..., **J. Henn '13**; C. Papadopoulos '14]

$\Downarrow$

Direct consequence of **Integration-by-parts (IBPs)** identities in d-dimensions!

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^{\mu}} \, v_{\mu} \, \frac{S_1^{\sigma_1} \ldots S_s^{\sigma_s}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} \right) = 0 \,, \qquad v^{\mu} = k_j^{\mu}, p_k^{\mu}$$

Reduced to N master integrals, $I_i(d; x_k)$ with $i = 1, \ldots, N$.

$\Downarrow$

Differentiating the masters and using the **IBPs** we get a system of
N coupled differential equations

$$\frac{\partial}{\partial\, x_k} I_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, I_j(d; x_k)\,.$$

Let's look more in detail - *we should recall* that equations are in block form

$$I_j(d;x_k) = (m_j(d;x_k)\,,\ \ sub_j(d;x_k))$$

$$\Downarrow$$

$$\frac{\partial}{\partial x_k} m_i(d;x_k) = \sum_{j=1}^{N} h_{ij}(d;x_k)\, m_j(d;x_k) + \sum_{j=1}^{M} nh_{ij}(d;x_k)\, sub_j(d;x_k)\,.$$

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homogeneous piece is MAIN source of complexity

$$\Downarrow$$

Loosely speaking:

- if decoupled as $d \to 4$, **MPLs**

- if coupled as $d \to 4$, **Elliptic...?**

How do we solve them? In general:

1- Solve the **homogeneous equations** in the limit $d \to 4$

$$\frac{d}{dx}\vec{I}(d;x) = A(x)\vec{I}(d;x) + (d-4)B(x)\vec{I}(d;x) + \mathcal{O}(d-4)^2\,,$$

with $A(x)$ $n \times n$, depending on how many eqs are coupled

Find $n \times n$ matrix homogeneous solutions $G(x)$, with

$$\frac{d}{dx}G(x) = A(x)G(x)\,, \qquad \to \qquad \vec{I}(d;x) = G(x)\vec{m}(d;x)$$

then

$$\frac{d}{dx}\vec{m}(d;x) = (d-4)G^{-1}(x)\,B(x)G(x)\,\vec{m}(d;x) + \mathcal{O}(d-4)^2\,,$$

2- Solution given by iterative integrals over kernels that contain **homogeneous solutions**, and previous orders

By expanding in $(d-4)$:

$$\vec{m}^{\,[n]}(x) = \int^x dy\, G^{-1}(y)\, B(y) G(y)\, \vec{m}^{\,[n-1]}(y) \quad + \quad \text{simpler terms}\,,$$

Or equivalently for the original functions

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Two Questions:

a) How do I get the matrix $G(x)$ if the system is coupled?
b) What are the functions defined by the integrals above?

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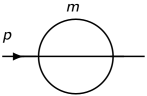
Solution:

- Take an older idea by [S.Laporta, E.Remiddi '04]

- Generalize it to all cases [A.Primo, L.Tancredi '16, '17]

$$\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s) \right)$$



$$+ G(d;s)\,\mathrm{Tad}(d;m^2) = 0$$

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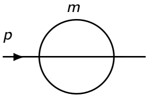

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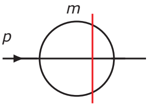
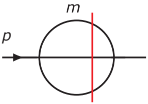
Cut $\rightarrow$ $\left(\frac{d^2}{d\,s^2} + A(d;s)\,\frac{d}{d\,s} + B(d;s) \right)$



$= 0$

<u>Maximal cut</u> solves homogeneous differential equations

[A.Primo, L.Tancredi '16, '17]



$$= \frac{1}{\sqrt{(3m-\sqrt{s})(\sqrt{s}+m)^3}}\,\mathrm{K}\left(\frac{16m^3\sqrt{s}}{(3m-\sqrt{s})(\sqrt{s}+m)^3}\right)$$

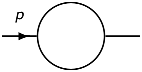
where $\mathrm{K}(x)$ is the complete elliptic integral of the first kind.

$$\mathrm{K}(x) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-x\,t^2)}}$$

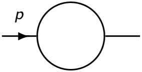
Computation of the maximal cut can be simplified in **Baikov representation**
[Papadopoulos, Frellesvig '17; Bosma, Sogaard, Zhang '17;
Harley, Moriello, Schabinger '17]

This is clearly true also when we have only **first order differential equations**!

$$\frac{\partial}{\partial p^2}$$



$$= \frac{1}{2}\left[(d-3)\left(\frac{1}{s}+\frac{1}{s-4m^2}\right)-\frac{d-2}{s}\right]$$



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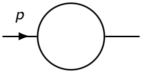

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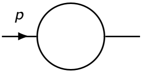

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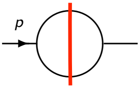
$$\Rightarrow$$



$$= \frac{\sqrt{s-4m^2}}{\sqrt{s}}\,, \qquad \text{in } d=4$$

Rational or **algebraic** function

In this case, it can be rationalized by $s = m^2(1+x)^2/x$

What are the functions defined by these **iterated integrals**?

Natural that solution expressed in terms of so-called **multiple polylogarithms**?
Scattering amplitudes have logarithmic singularities...

$$G(0;x) = \ln(x)\,, \qquad G(a;x) = \ln\left(1 - \frac{x}{a}\right) \quad \text{for} \quad a \neq 0$$

$$G(\underbrace{0,\ldots,0}_{n};x) = \frac{1}{n!}\ln^n(x)\,, \qquad G(a,\vec{w};x) = \int_0^x \frac{dy}{y-a}\,G(\vec{w};y)\,.$$

[E.Remiddi, J.Vermaseren '99; T. Gehrmann, E.Remiddi '00; Goncharov et al '00; Duhr, Gangl, Rhodes '13; ...]

A bit more "mathematically":

- Space of functions generated by integrating **rational functions** on the **Riemann sphere** $\mathbb{CP}^1$ $\sim$ complex plane plus infinity $\mathbb{C} \cup \{\infty\}$

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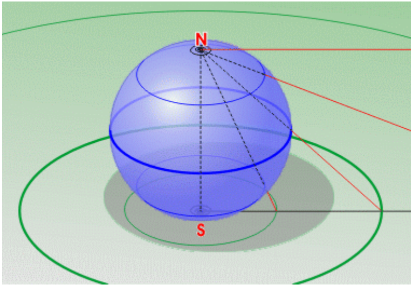
Riemann Sphere ($\mathbb{CP}^1$) is a surface of genus 0, it has no **holes**



Rational functions with poles at

$$x = c_i, \quad c_i \in \mathbb{C}$$

$$\int \frac{dx}{(x-c_i)^k} = -\frac{1}{k-1}\frac{1}{(x-c_i)^{k-1}}, \quad k > 1$$

$$\int \frac{dx}{(x-c_i)} = \log(x-c_i) \quad \rightarrow \quad \textbf{simple poles!}$$

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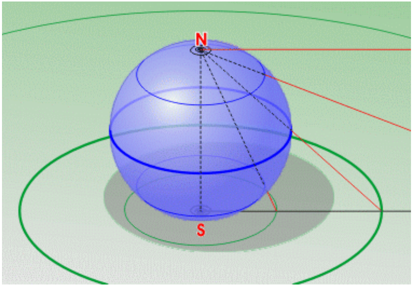

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Integration by parts to reduce iterated integrals

$$\int \frac{dx_1}{(x_1-c_1)^{k_1}} \int^{x_1} \frac{dx_2}{(x_2-c_2)^{k_2}} \cdots \int^{x_{n-1}} \frac{dx_n}{(x_n-c_n)^{k_n}}$$

to rational functions and MPLs

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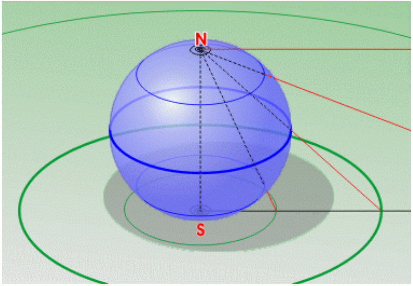

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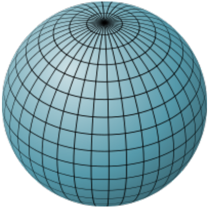
Generalization:

Iterated integrals of **rational functions** on surfaces of higher genus



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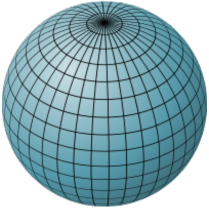

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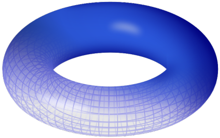

Genus 1

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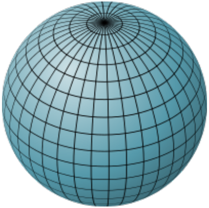

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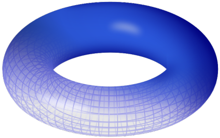

Genus 1

Genus 4

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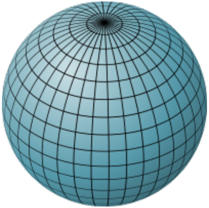

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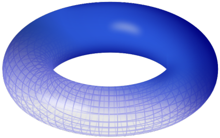
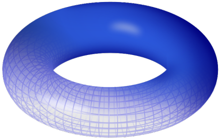


Genus 1

Genus 4

We want to attempt a similar construction but on these more general surfaces
Based on [F. Brown, A. Levin '11; F. Brown 2014, ...]

Genus 1 - *Complex* Torus equivalent to an Elliptic curve!



Take a complex lattice

$$\Lambda = \{\omega_1\, m + \omega_2\, n \quad : \quad m, n \in \mathbb{Z}\}$$

$\omega_{1,2}$ are called the *periods* on the lattice

Complex Torus $\sim \mathbb{C}/\Lambda$

Weierstrass $\wp(z)$ function, *doubly periodic* on the torus

$$\wp(z) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0}\left[\frac{1}{(z+n\omega_1+m\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2}\right]$$

One finds $z \to [x, y, 1] \equiv [\wp(z), \wp'(z), 1]$

$$[\wp'(z)]^2 = 4\wp(z)^3 + g_2\wp(z)^2 - g_3 \quad \Rightarrow \quad y^2 = 4x^3 + g_2x^2 - g_3$$

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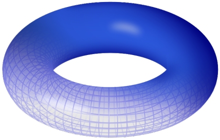

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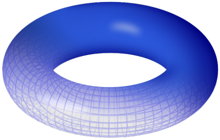

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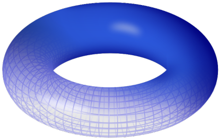
Weierstrass $\wp(z)$ function, *doubly periodic* on the torus

$$\wp(z) = \frac{1}{z^2} + \sum_{n^2+m^2\neq 0} \left[\frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right]$$

One finds $z \to [x, y, 1] \equiv [\wp(z), \wp'(z), 1]$

$$[\wp'(z)]^2 = 4\wp(z)^3 + g_2\wp(z)^2 - g_3 \quad \Rightarrow \quad y^2 = 4x^3 + g_2x^2 - g_3$$

Genus 1 - *Complex* Torus equivalent to an Elliptic curve!



Take a complex lattice

$$\Lambda = \{\omega_1\, m + \omega_2\, n \quad : \quad m, n \in \mathbb{Z}\}$$

$\omega_{1,2}$ are called the *periods* on the lattice

Complex Torus $\sim \mathbb{C}/\Lambda$

Any generic elliptic curve $\mathcal{E}$

$$y^2 = P(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

can be put in this standard form!

We are "just" dealing with a **more complicated geometry**, with one hole!

Geometry suggests us to consider integrals of "*rational functions*" on the torus!

What is a rational function on the elliptic curve?

A rational function on the elliptic curve is a function $R(x, y)$ subject to the constraint $y = \sqrt{P(x)}$

$$R(x,y) = \frac{p_1(x) + p_2(x)\,y}{q_1(x) + q_2(x)\,y} = \frac{p_1(x) + p_2(x)\,\sqrt{P(x)}}{q_1(x) + q_2(x)\,\sqrt{P(x)}} = R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x)$$

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Given **elliptic curve** $y^2 = P(x)$, with $P(x)$ (cubic polynomial for simplicity), let us study iterated integrals of rational functions on the curve.

$$\int dx \left(R_1(x) + \frac{1}{\sqrt{P(x)}} R_2(x) \right) = \ ?$$

After **partial fractioning**, one clearly ends up with

$\int \frac{dx}{(x-c_i)^k}$	from $R_1(x)$
$\int \frac{dx}{y} x^k, \quad \int \frac{dx}{y(x-c_i)^k},$	from $\frac{1}{y} R_2(x)$

Integration by parts reduce everything to 4 kernels

$$\int \frac{dx}{(x-c_i)}, \quad \int \frac{dx}{y}, \quad \int \frac{x\,dx}{y}, \quad \int \frac{dx}{y(x-c_i)}$$

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MPLs have one more property: integration kernels with **simple poles**!

We could define iterated integrals over these four kernels

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$$\int \frac{x\,dx}{y} \sim -\int du \left(\frac{2}{u^2} + \mathcal{O}(u^0) \right) \quad \rightarrow \quad \text{double pole at infinity!}$$

Chose instead its **primitive**!

$$Z_3(x) \sim \int^x \frac{x\,dx}{y} \sim \frac{1}{u} \qquad u = \frac{1}{\sqrt{x}} \quad \rightarrow \quad \textit{Trascendental Kernel!}$$

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Trascendental Kernel!

Fundamental differences with MPLs:

-Impossible to find basis of kernels which are **rational** and with **simple poles**.
-We need **infinite tower** of integration kernels to span the whole space!

$$\begin{aligned}
\varphi_0(0,x) &= \frac{c_3}{y}\,, \\
\varphi_1(c,x) &= \frac{1}{x-c}\,, \qquad \varphi_{-1}(c,x) = \frac{y_c}{y(x-c)}\,, \qquad \varphi_1(\infty,x) = \frac{c_3}{y}\,Z_3(x)\,, \\
\varphi_n(c,x) &= \left(\frac{1}{x-c} + \frac{c_3}{2y}\,Z_3(x)\right)\,Z_3^{(n-1)}(x)\,, \\
\varphi_{-n}(c,x) &= \frac{y_c}{y(x-c)}\,Z_3^{(n-1)}(x)\,, \qquad \varphi_n(\infty,x) = \frac{c_3}{y}\,Z_3^{(n)}(x)\,.
\end{aligned}$$

$$\mathrm{E}_3\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix};x\right) = \int_0^x dt\,\varphi_{n_1}(c_1,t)\,\mathrm{E}_3\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix};t\right)\,,$$

With the usual properties (shuffle...)

$$\mathrm{E}_3(\vec{c};x)\,\mathrm{E}_3(\vec{d};x) = \sum_{\vec{w}\in\vec{c}\,\sqcup\!\sqcup\,\vec{d}} \mathrm{E}_3(\vec{w};x)\,, \qquad \vec{c} = \left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix}\right)\,, \quad \text{same for } \vec{d}$$

We can prove that our functions are equivalente to the **elliptic polylogarithms** introduced in the mathematical literature [F. Brown, A. Levin, '11]

1. Iterated integrals build on the **Torus**!

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_k \\ c_1 & \cdots & c_k \end{smallmatrix} ; x \right) = \int_0^x dt\, f^{(n_1)}(t - c_1)\, \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_k \\ c_2 & \cdots & c_k \end{smallmatrix} ; t \right)$$

2. $f^{(n)}$ have well defined properties on the Torus (double periodicity...) They are not holomorphic!

3. Our construction preserves **holomorphicity** (Feynman Integrals!)

4. It is straight-forward to go from the E_3 functions to the corresponding $\widetilde{\Gamma}$, algorithmically

Indeed, many examples both from the math and from the physics world turn out to be expressible in terms of this functions, including the (in-)famous **two-loop massive sunrise graph**

$$S_{111}(p^2, m^2) = \frac{1}{m^2 - p^2}\frac{1}{c_4}\left[\frac{1}{c_4}\,\mathrm{E}_4\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}; 1\right) - 2\mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix}; 1\right) + \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}; 1\right)\right.$$
$$\left. + \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix}; 1\right) - \mathrm{E}_4\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; 1\right)\right]$$

The E_4 are the same as the E_3, but starting from a different representation of the elliptic curve as a root of a **quartic polynomial**. The two can be converted to each other, but in some cases it might be more convenient one representation or the other...

Conclusions and Outlook

1. We can generalize construction of polylogs as iterated integrals of rational functions on surfaces of arbitrary genus
2. For **genus 1**, we obtain a class of functions which we can prove to be *equivalent to the elliptic polylogarithms introduced in the math literature*
3. Mathematicians proved these functions are all that there is, as long as integrals of rational functions on elliptic curves are concerned
4. Our **new representation** allows us to solve many physical problems in terms of these functions (the Sunrise)
5. Analytical and Algebraic properties under study, generalization of **symbol map**, **coaction**, etc etc...