

Top quark pair production at threshold and four-loop on-shell renormalization

Peter Marquard

DESY

in collaboration with

M. Beneke, P. Nason, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, D. Wellmann



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Outline

- 1 Top pair production at threshold
- 2 $\overline{\text{MS}}$ – on-shell relation
- 3 wave function renormalization: Z_2^{OS}
- 4 Renormalon ambiguity of the pole mass
- 5 Conclusions

Framework

Physics of bound states of heavy particles and threshold phenomena best described within an effective field theory:

Non-Relativistic QCD (NRQCD)

[Caswell,Lepage'86; Bodwin,Braaten,Lepage'95]

and potential Non-Relativistic QCD (pNRQCD)

[Beneke,Smirnov'97; Pineda,Soto'98; Brambilla,Pineda,Soto,Vairo'00]

or vNRQCD

[Luke,Manohar,Rothstein'00; Hoang,Stewart'03]

Prominent applications are

- production of $t\bar{t}$ pairs
- decays of $b\bar{b}$ bound states
- $b\bar{b}$ sum rules
- positronium spectra

Master Formula

$$\begin{aligned}
 R &= \frac{\sigma_{\text{tot}}(e^+ e^- \rightarrow t\bar{t})}{\sigma_0} \\
 &= \frac{18\pi}{m_t^2} \text{Im} \left\{ c_V \left[c_V - \frac{E}{m_t} \left(c_V + \frac{d_V}{3} \right) \right] G(E) + \dots \right\}
 \end{aligned}$$

Ingredients:

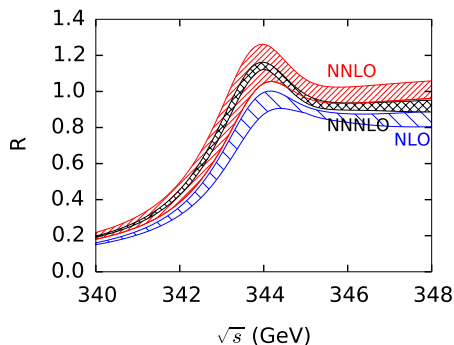
- 1 NRQCD matching coefficients: c_V, d_V

[Kallen,Sabry'55;Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97;Marquard,Piclum,Seidel,Steinhauser'06'08'14]

- 2 pNRQCD Green function: $G(E)$

[Beneke,Kiyo,Penin'07; Beneke,Kiyo'08; Beneke,Kiyo,Schuller'13]

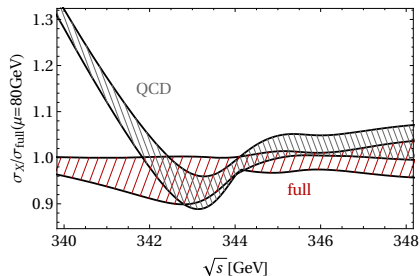
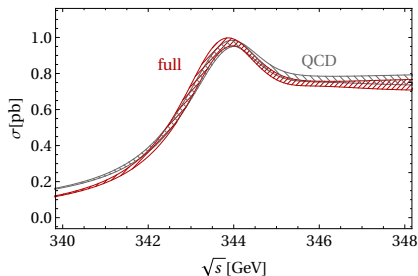
Total cross section – pure QCD



$m_t = 171.5 \text{ GeV}$, $\Gamma_t = 1.33 \text{ GeV}$,
 $50 \text{ GeV} \leq \mu \leq 350 \text{ GeV}$

- very good convergence of perturbative series below threshold
- above threshold -8% shift driven by large negative corrections to c_V

QCD + EW



[Beneke, Maier, Rauh, Ruiz-Femenia '17]

Prelim. studies indicate an error $\pm 45\text{ MeV}$ from the theory prediction.

Renormalization constants and Quark masses

- Quark masses can be measured with high precision using different renormalization schemes
 - ⇒ have to be able to translate between them
- pole masses suffer from (infrared) renormalon problems
 - ⇒ try to quantify the effect
- Renormalization constants are on their own fundamental quantities of the given theory
- Central role: $Z_m^{\overline{\text{MS}}} / Z_m^{\text{OS}}$, most threshold masses derived from pole mass

Setup of the calculation

- Need to calculate mass renormalization constant Z_m^{OS} by calculating four-loop on-shell integrals
- Together with the renormalization constant in the $\overline{\text{MS}}$ -scheme $Z_m^{\overline{\text{MS}}}$

[Chetyrkin '97; Larin, van Rittbergen, Vermaseren '97; Baikov,Chetyrkin,Kühn '14]

we get for the relation between the two schemes

$$\left. \begin{aligned} m_{\text{bare}} &= Z_m^{\text{OS}} M \\ m_{\text{bare}} &= Z_m^{\overline{\text{MS}}} m \end{aligned} \right\} \Rightarrow m = M \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}}$$

- 1-loop
- 2-loop
- 3-loop

[Tarrach'81]

[Gray,Broadhurst,Grafe,Schilcher'90]

[Chetyrkin,Steinhauser'99; Melnikov, v. Rittbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]

Scheme definitions

One considers the renormalized quark propagator

$$S_F(q) = \frac{-i Z_2}{\not{q} - Z_m m + \Sigma(q, m)}$$

with $\Sigma(q, m)$ the quark two-point function.

For the $\overline{\text{MS}}$ scheme we require

$$S_F(q) \text{ finite}$$

and for the **on-shell scheme** we require a pole at the position of the mass

$$S_F(q) \xrightarrow{q^2 \rightarrow M^2} \frac{-i}{\not{q} - M}$$

Setup of the calculation

expand the quark two-point function

$$\Sigma(q, M) \approx M \Sigma_1(M^2, M) + (\not{q} - M) \left(2M^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} + \Sigma_2(M^2, M) \right) + \dots$$

renormalization constants are then given by

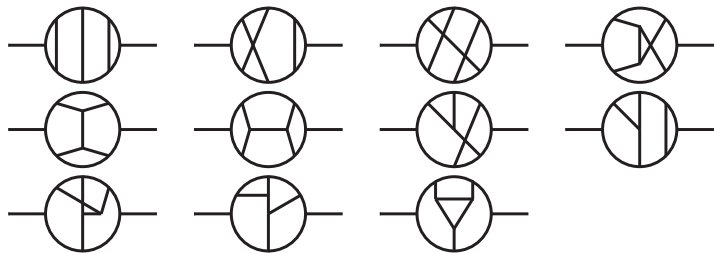
$$\begin{aligned} Z_m^{\text{OS}} &= 1 + \Sigma_1(M^2, M), \\ (Z_2^{\text{OS}})^{-1} &= 1 + 2M^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} + \Sigma_2(M^2, M). \end{aligned}$$

best to apply the projector

$$\begin{aligned} \text{Tr} \left\{ \frac{\not{Q} + M}{4M^2} \Sigma(q, M) \right\} &= \Sigma_1(q^2, M) + t \Sigma_2(q^2, M) \\ &= \Sigma_1(M^2, M) + \left(2M^2 \frac{\partial}{\partial q^2} \Sigma_1(q^2, M) \Big|_{q^2=M^2} + \Sigma_2(M^2, M) \right) t \end{aligned}$$

Setup of the calculation cont'd

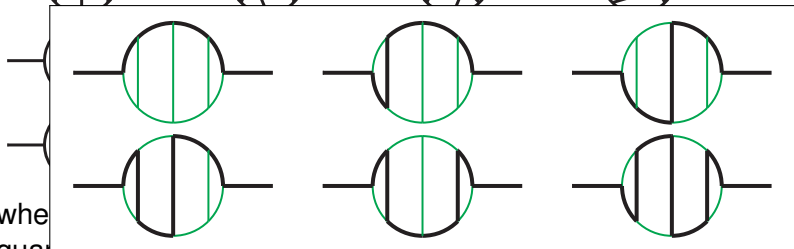
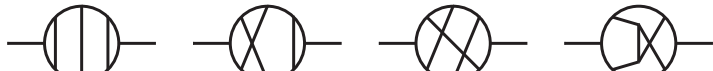
Need to calculate 4-loop on-shell diagrams of the form



where we have to consider all possible ways to route the external quark through the diagrams \Rightarrow 100 integral families.

Setup of the calculation

Need to calculate 4-loop on-shell diagrams of the form



when a quark passes through the diagrams \Rightarrow 100 integral families. nal

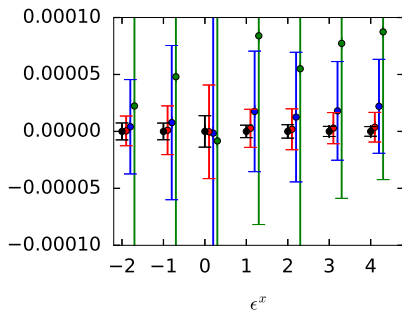
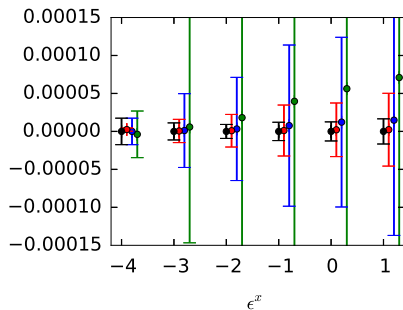
Setup of the calculation

Follow the *standard* procedure for multi-loop calculations

- exploit that the appearing integrals are not linear independent but related through Integration-By-Parts identities [Chetyrkin, Tkachov '81]
- reduce all appearing integrals to a small set of basis integrals using FIRE [Smirnov] or Crusher [PM,Seidel]
- evaluate the remaining basis integrals ($\mathcal{O}(350)$) using analytic or numerical techniques (Mellin-Barnes or Sector Decomposition)[MB.m [Czakon] FIESTA [Smirnov]]

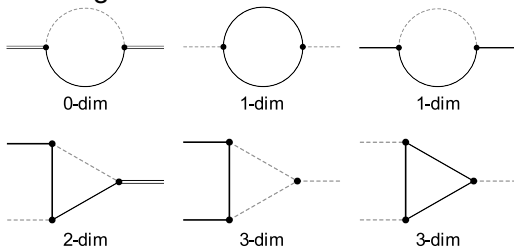
Numerics: Convergence of SD

Check the convergence of the results using Sector Decomposition when increasing statistics 500k \rightarrow 500M points



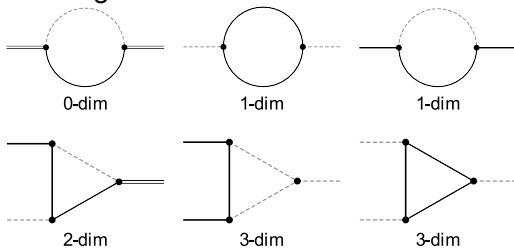
Numerics: Mellin-Barnes

- Building blocks

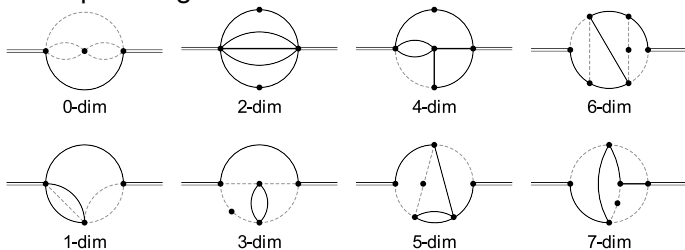


Numerics: Mellin-Barnes

● Building blocks

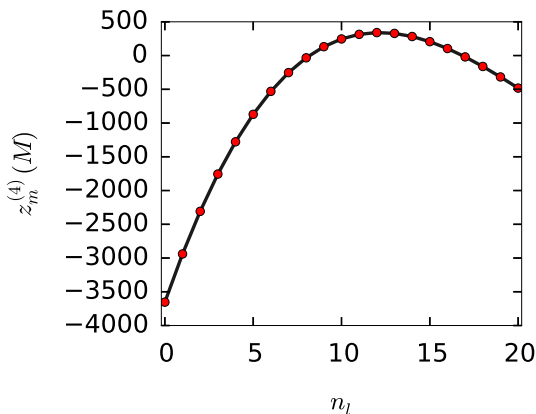


● Example Integrals



$\overline{\text{MS}}$ –on-shell relation at four-loop order

$$z_m^{(4)} = -3654.15 \pm 1.64 + (756.942 \pm 0.040)n_l - 43.4824n_l^2 + 0.678141n_l^3.$$



$\overline{\text{MS}}$ –on-shell relation at four-loop order

$\overline{\text{MS}} \rightarrow$ on-shell

$$\begin{aligned} m_t(m_t) &= M_t (1 - 0.4244 \alpha_s - 0.9246 \alpha_s^2 - 2.593 \alpha_s^3 \\ &\quad - (8.949 \pm 0.018) \alpha_s^4) \\ &= 173.34 - 7.924 - 1.859 - 0.562 \\ &\quad - (0.209 \pm 0.0004) \text{ GeV} \end{aligned}$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

$\overline{\text{MS}}$ –on-shell relation at four-loop order $\overline{\text{MS}} \rightarrow$ on-shell

$$\begin{aligned}
 m_t(m_t) &= M_t (1 - 0.4244 \alpha_s - 0.9246 \alpha_s^2 - 2.593 \alpha_s^3 \\
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 &= 173.34 - 7.924 - 1.859 - 0.562 \\
 &\quad - (0.209 \pm 0.0004) \text{ GeV}
 \end{aligned}$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

$$\begin{aligned}
 M_b &= m_b(m_b) (1 + 0.4244 \alpha_s + 0.9401 \alpha_s^2 + 3.045 \alpha_s^3 \\
 &\quad + (12.685 \pm 0.025) \alpha_s^4) \\
 &= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003) \text{ GeV}
 \end{aligned}$$

Light quark mass dependence

- starts at 2 loops
- available up to 3 loops
- for top quark
 - 11 MeV @ two loop
 - 16 MeV @ three loop
 - getting more important at higher orders due to renormalon enhancement

[Gray,Broadhurst,Grafe,Schilcher '90]

[Bekavac,Grozin,Seidel,Steinhauser '07]

[Hoang '17]

Threshold masses, e.g. PS mass

$$m^{\text{PS}} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2} \int_{|\vec{q}| < \mu_f} \frac{d^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi} (1 + \alpha_s \dots)$$

need static potential @ three loops

[Smirnov, Smirnov, Steihauser '09; Anzai, Kiyo, Sumino '09]

$$\begin{aligned}
 m_t^{\text{PS}}(\mu_f = 80 \text{ GeV}) &= 163.508 + (7.531 - 3.685) \\
 &\quad + (1.607 - 0.989) + (0.495 - 0.403) \\
 &\quad + (0.195 - 0.211 \pm 0.0004) \text{ GeV} \\
 &= 163.508 + 3.847 + 0.618 + 0.092 \\
 &\quad - (0.016 \pm 0.0004) \text{ GeV}
 \end{aligned}$$

- large cancellations between contributions from OS-MS and PS-OS
- good convergence

Z_2^{OS} Overview

- final missing building block to complete the on-shell renormalization procedure of QCD at four loops.
- gauge dependence starts only at 3 loops

[Broadhurst,Gray,Schilcher]

[Melnikov,van Ritbergen;Marquard,Mihaila,Piclum,Steinhauser]

- computationally more involved due to additional derivative and gauge dependence

Z_2^{OS} 4-loop results, QCD, i.e. $N_c = 3$

ξ^0	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^0
n_1^0	-1.7728 ± 0.00040	-27.666 ± 0.0041	-317.093 ± 0.029	-3142.15 ± 0.33	-28709.9 ± 3.2
n_1^1	0.460936 ± 0.000016	6.69143 ± 0.00023	74.6540 ± 0.0013	696.6612 ± 0.0076	6174.390 ± 0.084
n_1^2	-0.039931	-0.51572	-5.5055	-48.777	-418.93
n_1^3	0.0011574	0.012539	0.12676	1.0711	8.916

[PM,Smirnov,Smirnov,Steinhauser '18]

- Not showing $\mathcal{O}(\xi)$ terms
- As usual largish cancellations between hard 4-loop part and counter term contributions
- first application: four-loop massive form factor
- also allows for extraction of the corresponding HQET quantities

$\overline{\text{MS}}$ -on-shell relation beyond 4 loops

$$m_p = m(\mu_m) \left(1 + \sum_{n=1}^{\infty} c_n(\mu, \mu_m, m(\mu)) \alpha_s^n(\mu) \right)$$

for large n

$$c_n(\mu, \mu_m, m(\mu_m)) \xrightarrow{n \rightarrow \infty} N c_n^{(\text{as})}(\mu, m(\mu_m)) \equiv N \frac{\mu}{m(\mu_m)} \tilde{c}_n^{(\text{as})},$$

[Beneke, Braun '94; Beneke '94 '99]

where

$$\tilde{c}_{n+1}^{(\text{as})} = (2b_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)} \left(1 + \frac{s_1}{n+b} + \frac{s_2}{(n+b)(n+b-1)} + \dots \right).$$

b_0, b, s_1, s_2 : Combinations of coefficients of the β -function.

How well does the asypt. formula work?

Fit N to 1, ..., 4-loop term and compare

n_l	$c_1/c_1^{(as)}$	$c_2/c_2^{(as)}$	$c_3/c_3^{(as)}$	$c_4/c_4^{(as)}$	Δ_{34}
-1000000	0.6953	0.9624	0.9349	0.9714	0.038
-10	0.4744	0.7152	0.6898	0.7005	0.015
0	0.4377	0.6357	0.6130	0.5977	0.025
3	0.3954	0.6150	0.5723	0.5370	0.064
4	0.3633	0.6120	0.5522	0.5056	0.088
5	0.3143	0.6119	0.5244	0.4616	0.127
6	0.2436	0.6089	0.4818	0.3942	0.200
7	0.1474	0.5378	0.4084	0.2786	0.378
8	0.0098	0.0379	0.2719	0.0564	1.312
10	0.2684	-0.0916	-0.1108	-1.7228	1.758

$$\Delta_{34} = 2 \frac{|c_3/c_3^{(as)} - c_4/c_4^{(as)}|}{|c_3/c_3^{(as)} + c_4/c_4^{(as)}|}.$$

Beyond 4 loops

Fit N to 4-loop term and take higher orders from asymptotic formula

j	$\tilde{c}_j^{(\text{as})}$	$\tilde{c}_j^{(\text{as})} \alpha_s^j$
5	0.985499×10^2	0.001484
6	0.641788×10^3	0.001049
7	0.495994×10^4	0.000880
8	0.443735×10^5	0.000854
9	0.451072×10^6	0.000942
10	0.513535×10^7	0.001164

Asymptotic series \Rightarrow converges only up to ≈ 8 loop

5+ loops and remaining ambiguity

An asymptotic series f can (sometimes) be summed by using the Borel transform $B[f]$

$$f(\alpha_s) = \sum_{n=0}^{\infty} c_n \alpha_s^n \quad \Rightarrow \quad B[f](t) = \sum_{n=0}^{\infty} c_{n+1} \frac{t^n}{n!}$$

the Borel integral

$$\int_0^{\infty} dt e^{-t/\alpha_s} B[f](t)$$

has the same series expansion as $f(\alpha_s)$ and the same value.

5+ loops and remaining ambiguity cont'd

In our case we have

$$c_{n+1} = (2b_0)^n n! \quad \Rightarrow \quad \int_0^\infty dt e^{-t/\alpha_s} \frac{1}{1 - 2b_0 t}$$

Not integrable due to pole at $1 - 2b_0 t = 0$

Possible prescription for the integral:

Take principle value and assign ambiguity Im/π

$$\delta^{(5+)} m_p = 0.250_{-0.038}^{+0.015} (N) \pm 0.001 (c_4) \\ \pm 0.010 (\alpha_s) \pm 0.071 \text{ (ambiguity) GeV}$$

5+ loops and remaining ambiguity: light mass effects

- take 2-loop and 3-loop mass effects into account
- 4 loop: massless value using five-flavour theory
- 5 loop: massless value using four-flavour theory
- 6+ loops: massless value using three-flavour theory
- final estimate depends on $\Lambda_{\text{QCD}}^{(3)}$

$$\delta^{(5+)} m_p = 0.304_{-0.063}^{+0.012} (N) \pm 0.030 (m_{b,c}) \\ \pm 0.009 (\alpha_s) \pm 0.108 \text{ (ambiguity) GeV}$$

Cmp. analysis by [Hoang, Lepenik, Preisser '17] using different prescription

$$\frac{1}{2} \sum_{c_n < f c_n^{\min}} c_n \rightarrow 250 \text{ MeV} \quad \text{with} \quad f = 5/4$$

Conclusions

- Presented current status of the theory prediction for $t\bar{t}$ production @ threshold
- $\overline{\text{MS}}$ -on-shell relation
- renormalon ambiguity of the top quark
- on-shell wave function renormalization constant