Towards alternative approach for top pair production at NNLO

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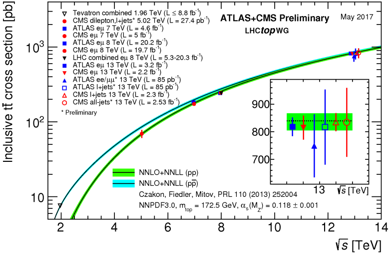
Top pair production: the status of QCD calculations

- A single *complete* NNLO result for total and differential cross section obtained with STRIPPER methodology [Czakon, Fiedler, Mitov '13; Czakon, Heymes, Mitov '16]



- Flavour off-diagonal channels at NNLO from q_T subtraction [Bonciani, Catani, Grazzini, Sargsyan, Torre '15]
- Approximate NNLO [Broggio, Papanastasiou, Signer '14] and N^{3}LO [Kidonakis '14]
- Soft and small-mass resummation at NNLL [Pecjak, Scott, Wang, Yang '16]
- Small-q_T resummation at NNLL [Li, Li, Shao, Yang, Zhu '13; Catani, Grazzini, Torre '14]

The q_T slicing method

[Catani, Grazzini '07, '15]

$$p + p \to F\,(q_T) + X$$

$$\sigma^{F}_{\mathrm{N}^{m}\mathrm{LO}} = \int_0^{q_{T,\mathrm{cut}}} dq_T \, \frac{d\sigma^{F}_{\mathrm{N}^{m}\mathrm{LO}}}{dq_T} + \int_{q_{T,\mathrm{cut}}}^{\infty} dq_T \, \frac{d\sigma^{F}_{\mathrm{N}^{m}\mathrm{LO}}}{dq_T}$$

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enough to know in
small-q_T approximation

known

Soft Collinear Effective Theory (SCET)

$$\text{SCET} \simeq \text{QCD}\Big|_{\text{IR limit}}$$

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QCD fields written as sums of collinear, anti-collinear and soft components:

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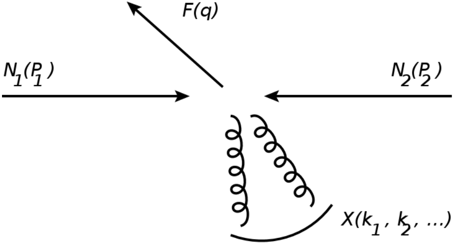
The new fields decouple in the Lagrangian

$$\mathcal{L}_{\text{SCET}} = \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_s$$

The above separation facilitates proofs of factorization theorems.

Small-q_T factorization in SCET



where $F = H, V, VV, t\bar{t}$

$$\frac{d\sigma}{d\Phi} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{H} \otimes \mathcal{S} + \mathcal{O}\left(\frac{q_T^2}{q^2}\right)$$

Small-q_T factorization in SCET

Gluons' momenta in light-cone coordinates

$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right)$$

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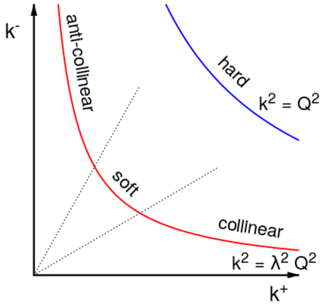
Regions

collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$	$\mathcal{B}_1$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$	$\mathcal{B}_2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$	$\mathcal{H}$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$	$\mathcal{S}$



Top pair production at small-q_T through NNLO

$$\frac{d\sigma^{\text{NNLO}}}{dq_T\, dy\, dM\, d\cos\theta} = \sum_{i,\bar{i}} \mathcal{B}_{i/N_1} \otimes \mathcal{B}_{\bar{i}/N_2} \otimes \text{Tr}\left[\mathcal{H}_{i\bar{i}} \otimes \mathcal{S}_{i\bar{i}}\right]$$

where

q_T, y, M : transverse momentum, rapidity, mass of top quark pair

θ : scattering angle of the top quark in $t\bar{t}$ rest frame

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$\mathcal{B}$ - known up to NNLO [Gehrmann, Lübbert, Yang '12, '14]

$\mathcal{H}$ - known up to NNLO [Czakon '08; Baernreuther, Czakon, Fiedler '13]

$\mathcal{S}$ - known up to NLO in small-q_T limit [Li, Li, Shao, Yan, Zhu '13; Catani, Grazzini, Torre '14] (up to NNLO in the threshold limit with massless tops [Ferroglia, Pecjak, Yang '12])

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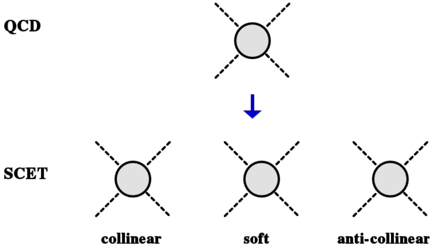
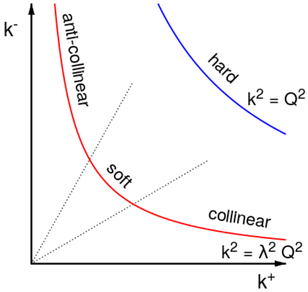
Calculating the missing NNLO correction to the soft function in the small-q_T limit, $\mathcal{S}$, is the aim of our work.

Rapidity divergences and analytic regulator





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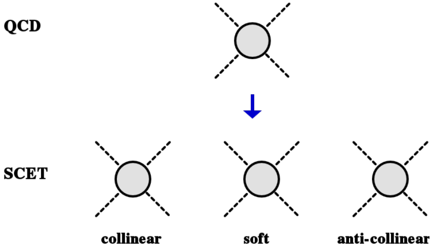

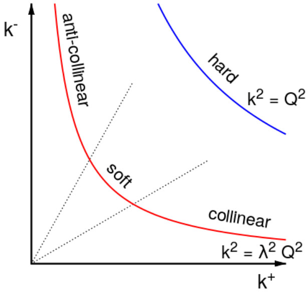

Modification of the measure [Becher, Bell '12]

$$\int d^d k\, \delta^+(k^2) \to \int d^d k \left(\frac{\nu_+}{k_+}\right)^{\alpha} \delta^+(k^2)$$

- The regulator is necessary at intermediate steps of the calculation.
- Rapidity divergences do not appear in QCD, hence, the complete SCET result has to stay finite in the limit $\alpha \to 0$.

Kinematics and notation

Partonic process

$$q(p_1) + \bar{q}(p_2) \;\to\; t(p_3) + \bar{t}(p_4) + \textstyle\sum_i g(k_i)$$

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Invariants

$$\hat{s} = (p_1 + p_2)^2 \qquad M^2 = (p_3 + p_4)^2 \qquad \beta = \sqrt{1 - \frac{4m_t^2}{M^2}}$$

$$t_1 = (p_1 - p_3)^2 - m_t^2 \qquad u_1 = (p_1 - p_4) - m_t^2$$

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Small-q_T limit

$$\hat{s}, M^2, t_1^2, u_1^2, m_t^2 \gg q_T^2 = (p_3 + p_4)_T^2 \gg \Lambda_{\mathrm{QCD}}^2$$

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$$\hat{s}, M^2, t_1^2, u_1^2, m_t^2 \gg q_T^2 = (p_3 + p_4)_T^2 \gg \Lambda_{\mathrm{QCD}}^2$$

Momenta

$$n = (1, 0, 0, 1)\,, \qquad \bar{n} = (1, 0, 0, -1)$$

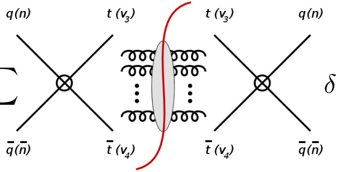
$$k_i^\mu = (n \cdot k_i)\,\frac{\bar{n}^\mu}{2} + (\bar{n} \cdot k_i)\,\frac{n^\mu}{2} + k_{i\perp}^\mu$$

$$p_1^\mu = m_t\, n\,, \qquad p_2^\mu = m_t\, \bar{n}\,, \qquad p_{3,4}^\mu = m_t\, v_{3,4}^\mu + \lambda_{3,4}^\mu$$

Soft function

- Represents corrections coming from exchanges of real, soft gluons, whose transverse momenta sum up to a fixed value q_T.

$$S(q_T, v_3, v_4) \propto \sum$$



$$\delta\left(q_T - \left|\sum_i k_{i\perp}\right|\right) \prod_i \delta^+(k_i^2)$$

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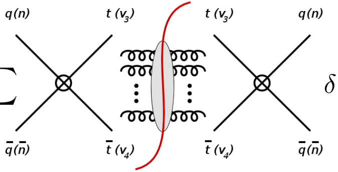

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- external momenta: Born kinematics
- eikonal Feynman rules

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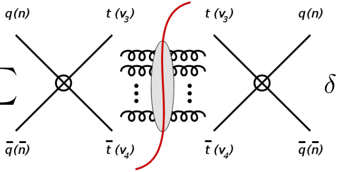

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- external momenta: Born kinematics
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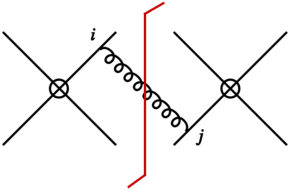
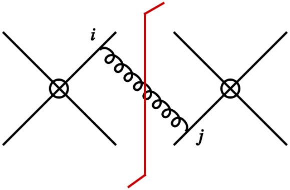
$$\boldsymbol{S}_{i\bar{i}} = \sum_{n=0}^{\infty} \boldsymbol{S}_{i\bar{i}}^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^n \qquad \boldsymbol{S}_{i\bar{i}}^{(n)} = \sum_{\{j\}} \boldsymbol{w}_{\{j\}}^{i\bar{i}} I_{\{j\}}$$

colour matrices ⤴ ⮤ phase space integrals

Soft function at NLO



Soft function at NLO



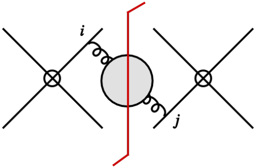
- Known in analytic form
  [Li, Li, Shao, Yan, Zhu '13; Catani, Grazzini, Torre '13]

$$
\begin{aligned}
\boldsymbol{S}^{(1)}_{i\bar{i}} &= 4L_\perp \left(2\boldsymbol{w}^{13}_{i\bar{i}} \ln\frac{-t_1}{m_t M} + 2\boldsymbol{w}^{23}_{i\bar{i}} \ln\frac{-u_1}{m_t M} + \boldsymbol{w}^{33}_{i\bar{i}}\right) \\
&\quad - 4\left(\boldsymbol{w}^{13}_{i\bar{i}} + \boldsymbol{w}^{23}_{i\bar{i}}\right) \mathrm{Li}_2\left(1 - \frac{t_1 u_1}{m_t^2 M^2}\right) + 4\boldsymbol{w}^{33}_{i\bar{i}} \ln\frac{t_1 u_1}{m_t^2 M^2} \\
&\quad - 2\boldsymbol{w}^{34}_{i\bar{i}} \frac{1+\beta_t^2}{\beta_t} \left[L_\perp \ln x_s - \mathrm{Li}_2\left(-x_s \,\mathrm{tg}^2 \frac{\theta}{2}\right) + \mathrm{Li}_2\left(-\frac{1}{x_s} \,\mathrm{tg}^2 \frac{\theta}{2}\right)\right. \\
&\quad \left. + 4 \ln x_s \ln\cos\frac{\theta}{2}\right], \qquad \text{where} \qquad L_\perp = \ln\frac{x_T^2 \mu^2}{4e^{-2\gamma_E}}
\end{aligned}
$$

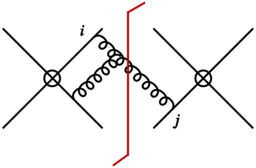
Soft function at NNLO

Three distinct groups of diagrams:

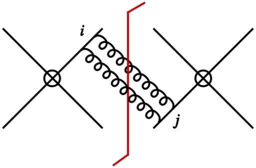
- bubble

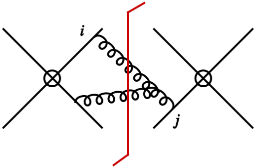


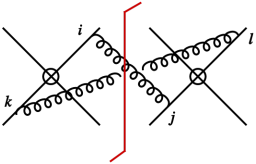
- single-cut



- double-cut







+ . . .

NNLO integrals

Example:

$$I_{3gv,ij} = \int \frac{d^d k_1\, d^d k_2\, \delta^+(k_1^2)\, \delta^+(k_2^2)\, \delta((k_1+k_2)_T^2 - q_T^2)}{(n\cdot k_1)^\alpha\, (n\cdot k_2)^\alpha\, (n_i\cdot k_1)\, (n_j\cdot(k_1+k_2))\, (k_1+k_2)^2} + \cdots$$

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- a range of overlapping singularities
- complication introduced by $\delta((k_1+k_2)_T^2 - q_T^2)$ which additionally couples gluon's momenta
- divergent in the limits $\epsilon \to 0$ and $\alpha \to 0$

NNLO integrals

Our strategy:

1. Represent each integral as

$$I_G = \int \mathcal{I}_G \times \mathcal{W}_G$$

boundary integral
simpler but encoding
all $\alpha, \epsilon \to 0$ singularities

weight
complicated but finite

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2. Use *sector decomposition* to calculate series expansion of $\mathcal{I}_G$

$$\mathcal{I}_G = \sum_{m,n=-1} a_{mn}\, \alpha^m \epsilon^n$$

3. Convolute the above with $\mathcal{W}_G$ and perform numerical integration of the coefficients

$$I_G = \sum_{m,n=-1} \left(\int a_{mn} \times \mathcal{W}_G \right) \alpha^m \epsilon^n = \sum_{m,n=-1} c_{mn}\, \alpha^m \epsilon^n$$

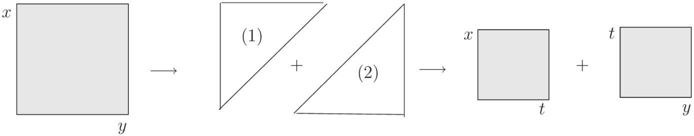
Sector decomposition

- A method to disentangle overlapping singularities [Binoth, Heinrich, '00; Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke '17]

$$\int_0^1 dx\, dy \frac{\mathcal{W}(x,y)}{(x+y)^{2+\epsilon}} = \int_0^1 dx\, dy \frac{\mathcal{W}(x,y)}{(x+y)^{2+\epsilon}} \left[\overbrace{\Theta(x-y)}^{(1)} + \overbrace{\Theta(y-x)}^{(2)} \right]$$

$$(1) \quad y = x\, t \qquad\qquad (2) \quad x = y\, t$$

$$= \int_0^1 dx\, dt \frac{\mathcal{W}(x,tx)}{(1+t)^{2+\epsilon} x^{1+\epsilon}} + \int_0^1 dt\, dy \frac{\mathcal{W}(ty,y)}{(1+t)^{2+\epsilon} y^{1+\epsilon}}$$



Sector decomposition

Two types of singularities

- Endpoint, *e.g.* soft:

$$\left(k_1^+, k_1^-, k_1^\perp\right) \to 0$$

Sector decomposition

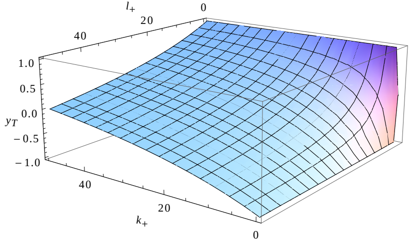
Two types of singularities

- Endpoint, *e.g.* soft:

$$\left(k_1^+, k_1^-, k_1^\perp\right) \rightarrow 0$$

- Manifold, *e.g.* collinear

$$k_1 \cdot k_2 \rightarrow 0$$



The strategy

Given the integral:

$$I_G = \int d^d k_1 d^d k_2 \, \mathcal{I}_G \times \mathcal{W}_G$$

- Analytically integrate 3 out of $2d$ dimensions

$$= \sum_j \int \prod_{i=1}^{2d-3} dx_i \left(\mathcal{I}_G \times \mathcal{W}_G\right)_j$$

- Map the remaining variables to a unit hypercube (split the original integral into a sum if necessary)
- Apply sector decomposition to disentangle overlapping singularities

$$= \sum_{j,k} \int \prod_{i=1}^{2d-3} dx_i \left(\mathcal{I}_G \times \mathcal{W}_G\right)_{jk}$$

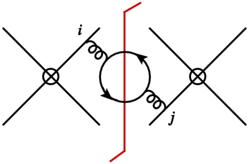
- Expand the result in Laurent series in ϵ and α

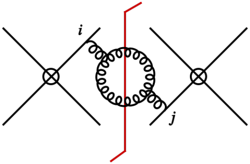
$$= \sum_{j,k} \sum_{m,n=-1} \left(\int a_{mn} \times \mathcal{W}_G \right) \alpha^m \epsilon^n$$

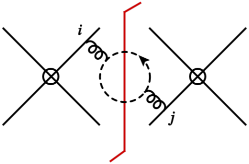
- Numerically integrate series coefficients

$$= \sum_{j,k} \sum_{m,n=-1} c^{jk}_{mn} \, \alpha^m \epsilon^n$$

Validation with the bubble







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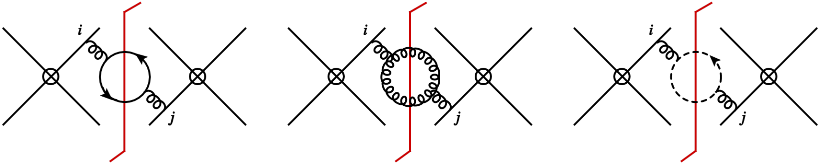

- Laboratory to stress-test sector decomposition-based methodology
- Non-trivial tensor structure → challenging numerators
- Solvable analytically: direct cross check of our sector decomposition-based implementation
- Comparable with n_f part of Renormalization Group prediction

Renormalization

separately divergent

finite

$$\frac{d\sigma}{d\Phi} = \mathcal{B}_1^{(0)} \otimes \mathcal{B}_2^{(0)} \otimes \mathrm{Tr}\left[\mathcal{H}^{(0)} \otimes \mathcal{S}^{(0)}\right]$$

$$= Z_B \mathcal{B}_1^{(0)} \otimes Z_B \mathcal{B}_2^{(0)} \otimes \mathrm{Tr}\left[\boldsymbol{Z}_H^{\dagger} \mathcal{H}^{(0)} \boldsymbol{Z}_H \otimes \boldsymbol{Z}_S^{\dagger} \mathcal{S}^{(0)} \boldsymbol{Z}_S\right]$$

$$= \mathcal{B}_1(\mu) \otimes \mathcal{B}_2(\mu) \otimes \mathrm{Tr}\left[\mathcal{H}(\mu) \otimes \mathcal{S}(\mu)\right]$$

separately finite

$$\frac{d}{d\mu}\frac{d\sigma}{d\Phi} = 0 \quad \rightarrow \quad \text{Renormalization Group Equations for } \mathcal{B}, \mathcal{H} \text{ and } \mathcal{S}$$

Renormalization

$$\boldsymbol{S}(\mu) = \boldsymbol{Z}_s^\dagger(\mu,\epsilon)\boldsymbol{S}_{\text{bare}}(\epsilon)\boldsymbol{Z}_s(\mu,\epsilon)$$

- RG equation

$$\frac{d}{d\ln\mu}\boldsymbol{S}_{i\bar{i}}(\mu) = -\gamma^{s\dagger}_{i\bar{i}}\,\boldsymbol{S}_{i\bar{i}}(\mu) - \boldsymbol{S}_{i\bar{i}}(\mu)\,\gamma^{s}_{i\bar{i}}$$

- Soft anomalous dimension

$$\gamma^s = -\boldsymbol{Z}_s^{-1}\frac{d\boldsymbol{Z}_s}{d\ln\mu}$$

Renormalization

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- Soft anomalous dimension

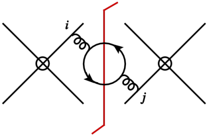
$$\gamma^s = -\boldsymbol{Z}_s^{-1}\frac{d\boldsymbol{Z}_s}{d\ln\mu}$$

Specifically, at the order α_s^2, we get

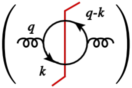
$$\underbrace{\boldsymbol{S}^{(2)}}_{\text{finite part only}} = \overbrace{\boldsymbol{Z}_s^{\dagger(2)}\tilde{\boldsymbol{S}}_{\text{bare}}^{(0)} + \tilde{\boldsymbol{S}}_{\text{bare}}^{(0)}\boldsymbol{Z}_s^{(2)} + \boldsymbol{Z}_s^{\dagger(1)}\tilde{\boldsymbol{S}}_{\text{bare}}^{(0)}\boldsymbol{Z}_s^{(1)}}^{\text{pole part only}}$$

$$+ \underbrace{\boldsymbol{Z}_s^{\dagger(1)}\tilde{\boldsymbol{S}}_{\text{bare}}^{(1)} + \tilde{\boldsymbol{S}}_{\text{bare}}^{(1)}\boldsymbol{Z}_s^{(1)} + \tilde{\boldsymbol{S}}_{\text{bare}}^{(2)} - \frac{\beta_0}{\epsilon}\,\tilde{\boldsymbol{S}}_{\text{bare}}^{(1)}}_{\text{finite + pole part}}$$

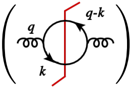
Bubble part of the soft function from differential equations



$$\propto \int \frac{d^d q\, \delta(q_T - 1)\, \theta^+(q^2)\, n_i^\mu n_j^\nu}{q^4\, (n_i \cdot q)\, (n_j \cdot q)} \left(\qquad \right)_{\mu\nu}$$



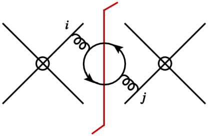
where



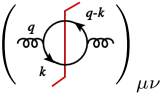
$$\left(\qquad \right)_{\mu\nu} = \int \frac{d^d k\, N_{\mu\nu}\, \delta^+(k^2)\, \delta^+((q-k)^2)}{(n \cdot k)^\alpha\, (n \cdot (q - k))^\alpha k^2 (q-k)^2}$$

$$= T_{00}\, g^{\mu,\nu} + T_{qq}\, q^\mu q^\nu + T_{nn}\, n^\mu n^\nu + T_{qn}\, (n^\mu q^\nu + q^\mu n^\nu)$$

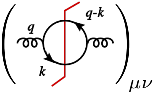
Bubble part of the soft function from differential equations



$$\propto \int \frac{d^d q\, \delta(q_T - 1)\, \theta^+(q^2)\, n_i^\mu n_j^\nu}{q^4\,(n_i\cdot q)\,(n_j\cdot q)}$$



where



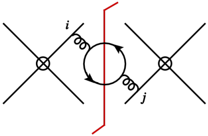
$$= \int \frac{d^d k\, N_{\mu\nu}\, \delta^+(k^2)\, \delta^+((q-k)^2)}{(n\cdot k)^\alpha\,(n\cdot(q-k))^\alpha k^2 (q-k)^2}$$

$$= T_{00}\, g^{\mu,\nu} + T_{qq}\, q^\mu q^\nu + T_{nn}\, n^\mu n^\nu + T_{qn}\, (n^\mu q^\nu + q^\mu n^\nu)$$

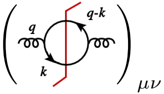
- Topology:

$$\int \frac{d^d k}{(n\cdot k)^{a_1+2\alpha}\ (\bar{n}\cdot k)^{a_2}\ (v_3\cdot k)^{a_3}\ (v_4\cdot k)^{a_4}\ (k^2-m^2)^{a_5}\ ((n\cdot k)\,(\bar{n}\cdot k)-m^2-1)^{a_6}}$$

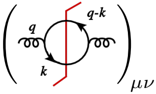
Bubble part of the soft function from differential equations



$$\propto \int \frac{d^d q\, \delta(q_T - 1)\, \theta^+(q^2)\, n_i^\mu n_j^\nu}{q^4\, (n_i \cdot q)\, (n_j \cdot q)}$$



where



$$= \int \frac{d^d k\, N_{\mu\nu}\, \delta^+(k^2)\, \delta^+((q-k)^2)}{(n \cdot k)^\alpha\, (n \cdot (q-k))^\alpha k^2 (q-k)^2}$$

$$= T_{00}\, g^{\mu,\nu} + T_{qq}\, q^\mu q^\nu + T_{nn}\, n^\mu n^\nu + T_{qn}\, (n^\mu q^\nu + q^\mu n^\nu)$$

- Topology:

$$\int \frac{d^d k}{(n \cdot k)^{a_1 + 2\alpha}\ (\bar{n} \cdot k)^{a_2}\ (v_3 \cdot k)^{a_3}\ (v_4 \cdot k)^{a_4}\, (k^2 - m^2)^{a_5}\, ((n \cdot k)\, (\bar{n} \cdot k) - m^2 - 1)^{a_6}}$$

- IBPs $\rightarrow$ reduction $\rightarrow$ DE $\rightarrow$ solutions $\rightarrow \int dm^2 \rightarrow I_{jk}(\beta, \theta)$

Validation with the bubble: results

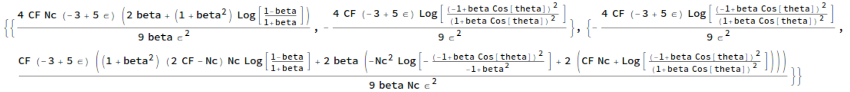
1. Cancellation of α poles
    - Analytically

$$\boldsymbol{w}_{13}^{q\bar{q}}\left(-\frac{8}{3\epsilon\alpha}+\frac{8}{3\epsilon\alpha}-\frac{8(5+3\gamma_E+2\ln 2)}{9\alpha}+\frac{8(5+3\gamma_E+2\ln 2)}{9\alpha}+\ldots\right)$$

    - Numerically

$$\boldsymbol{w}_{13}^{q\bar{q}}\left(-\frac{2.66597}{\epsilon\alpha}+\frac{2.66597}{\epsilon\alpha}-\frac{4.13986}{\alpha}+\frac{4.13986}{\alpha}+\ldots\right)$$

2. Agreement with the n_f part of RG prediction



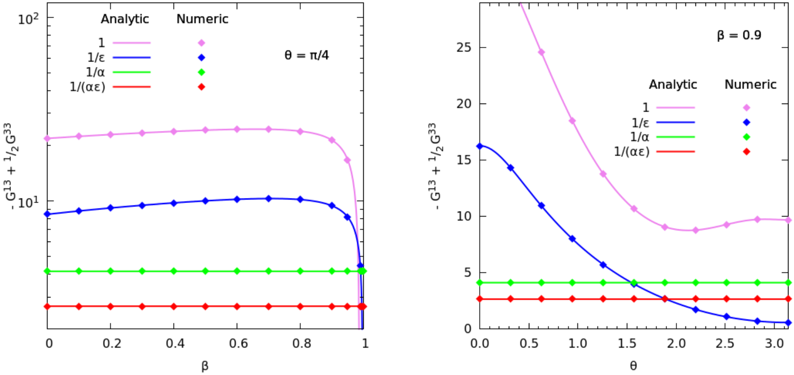
$$\Big\{\Big\{\frac{4\ \mathrm{CF}\ \mathrm{Nc}\ (-3+5\,\epsilon)\ \left(2\ \mathrm{beta}+\left(1+\mathrm{beta}^2\right)\ \mathrm{Log}\left[\frac{1-\mathrm{beta}}{1+\mathrm{beta}}\right]\right)}{9\ \mathrm{beta}\ \epsilon^2},-\frac{4\ \mathrm{CF}\ (-3+5\,\epsilon)\ \mathrm{Log}\left[\frac{(-1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}{(1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}\right]}{9\ \epsilon^2}\Big\},\Big\{-\frac{4\ \mathrm{CF}\ (-3+5\,\epsilon)\ \mathrm{Log}\left[\frac{(-1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}{(1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}\right]}{9\ \epsilon^2},$$

$$\frac{\mathrm{CF}\ (-3+5\,\epsilon)\ \left(\left(1+\mathrm{beta}^2\right)\ (2\ \mathrm{CF}-\mathrm{Nc})\ \mathrm{Nc}\ \mathrm{Log}\left[\frac{1-\mathrm{beta}}{1+\mathrm{beta}}\right]+2\ \mathrm{beta}\ \left(-\mathrm{Nc}^2\ \mathrm{Log}\left[-\frac{(-1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}{-1+\mathrm{beta}^2}\right]+2\ \left(\mathrm{CF}\ \mathrm{Nc}+\mathrm{Log}\left[\frac{(-1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}{(1+\mathrm{beta}\ \mathrm{Cos}[\mathrm{theta}])^2}\right]\right)\right)\right)}{9\ \mathrm{beta}\ \mathrm{Nc}\ \epsilon^2}\Big\}\Big\}$$

Validation with the bubble: results

3. Agreement with analytic result

$$-G^{13} + \frac{1}{2}G^{33}$$



Complete Soft Function at NNLO

- In momentum space

$$S^{(2)}(q_T, \beta, \theta) = \frac{1}{q_T^p}\left[S^{(2)}_{\text{bubble}}(\beta, \theta) + S^{(2)}_{\text{1-cut}}(\beta, \theta) + S^{(2)}_{\text{2-cut}}(\beta, \theta)\right]$$

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- In position space

$$S^{(2)}(L_\perp,\beta,\theta) = \left[\frac{1}{\epsilon} + L_\perp + L_\perp^2 + \ldots\right]$$

$$\times\left[S^{(2)}_{\text{bubble}}(\beta,\theta) + S^{(2)}_{\text{1-cut}}(\beta,\theta) + S^{(2)}_{\text{2-cut}}(\beta,\theta)\right]$$

$$= \underbrace{\frac{1}{\epsilon^2}S^{(2,-2)} + \frac{1}{\epsilon}S^{(2,-1)}}_{\text{can be cross-checked against RG, fixes many terms in } S^{(2,\text{ren})}} + S^{(2,\text{ren})}$$

Results: higher order poles

Even though the NNLO Soft Function exhibits at most $\frac{1}{\epsilon^2}$ singularity, higher order poles appear in individual contributions.

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- All α poles, as well as $\frac{1}{\epsilon^4}$ pole cancel within each colour structure, for example

$$\frac{1}{\epsilon^4}\begin{pmatrix} 0.000093\,N_c^{-1} - 0.000093\,N_c & -0.000023\,N_c^2 - 0.000093\,N_c^{-2} + 0.00012 \\ -0.000023\,N_c^2 - 0.000093\,N_c^{-2} + 0.00012 & 0.000081\,N_c^3 - 0.000058\,N_c + 0.00007\,N_c^{-3} - 0.000093\,N_c^{-1} \end{pmatrix}$$

Results: higher order poles

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- $\frac{1}{\epsilon^3}$ pole cancels between 1-cut and 2-cut contributions

$$\frac{1}{\epsilon^3}\begin{pmatrix} 0.00039\, N_c^3 - 0.00075\, N_c + 0.00035\, N_c^{-1} + 0. & 0.00036\, N_c^2 - 0.00035\, N_c^{-2} - 7.5 \times 10^{-6} \\ 0.00036\, N_c^2 - 0.00035\, N_c^{-2} - 7.5 \times 10^{-6} & -0.0004\, N_c^3 - 0.000012\, N_c + 0.00026\, N_c^{-3} + 0.00015\, N_c^{-1} \end{pmatrix}$$

[†] We used $\beta = 0.4$, $\theta = 0.5$.

Results: comparison with Renormalization Group

- Double pole

$$\left[S^{(2,-2)}_{\text{direct}} + S^{(2,-2)}_{\text{RGE}}\right]_{\beta=0.4,\theta=0.5} =$$

$$\begin{pmatrix} -0.00013\, N_c^3 - 0.0033\, N_c + 0.0034\, N_c{}^{-1} + 0. & -0.0011\, N_c^2 - 0.0042\, N_c{}^{-2} + 0.0053 \\ -0.0011\, N_c^2 - 0.0042\, N_c{}^{-2} + 0.0053 & -0.00096\, N_c^3 + 0.0023\, N_c + 0.0033\, N_c{}^{-3} - 0.0046\, N_c{}^{-1} \end{pmatrix}$$

- Single pole

$$\left[S^{(2,-1)}_{\text{direct}} + S^{(2,-1)}_{\text{RGE}}\right]_{\beta=0.0,\theta=0.0} =$$

$$\begin{pmatrix} 0.00031\, N_c^3 - 0.0058\, N_c + 0.0055\, N_c{}^{-1} + 0. & -0.0024\, N_c^2 - 0.0054\, N_c{}^{-2} + 0.0079 \\ -0.0024\, N_c^2 - 0.0054\, N_c{}^{-2} + 0.0079 & -0.0018\, N_c^3 + 0.0053\, N_c + 0.0041\, N_c{}^{-3} - 0.0076\, N_c{}^{-1} \end{pmatrix}$$

Conclusions

- Our goal: Use small-q_T factorization and q_T slicing to perform independent calculation of top pair production at NNLO

- The only missing component: the NNLO soft function

- We have developed a sector decomposition-based framework for calculation of the NNLO soft function integrals

- The framework has been extensively validated and cross-checked:
  1. Cancellation of α and ϵ poles beyond $1/\epsilon^2$
  2. Perfect agreement with analytic calculation for bubble graphs
  3. RG result for the complete NNLO soft function recovered

- Next and final step: the missing bit of the NNLO Soft Function for top pair production: the q_T-independent part of the ϵ^0 term.

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European Commission

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