Resummation of Threshold, Low- and High-Energy Expansions

An Application to Heavy Quark Correlators

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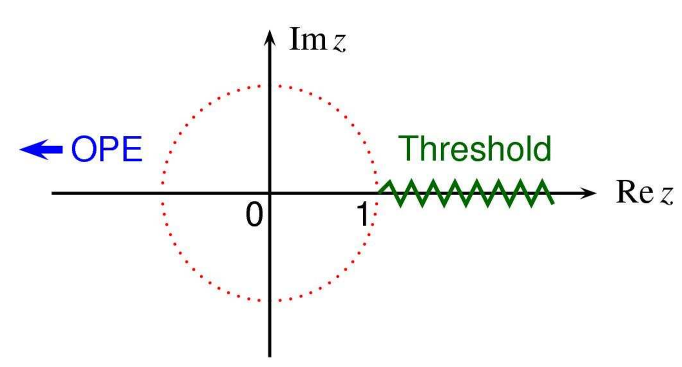
Hypothesis

Let consider of a 2 points Green function, a form factor or more generally a complex function Π :



- Π is analytic on a disk $|z|<1$: $\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{\infty} C(n)\, z^n$
- Π admits a branching cut $[1,\infty[$ and has the threshold expansion :

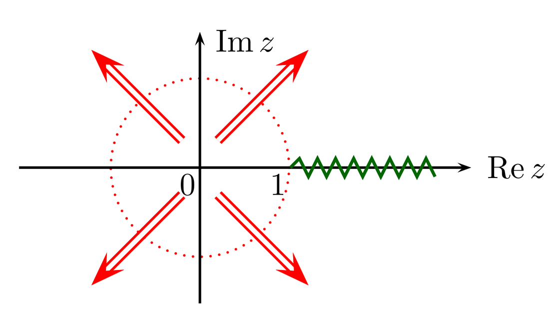
$$\Pi(z) \underset{z\to 1}{\sim} \sum_{p,k} A^{TH}(p,k)\,(1-z)^p \log^k(1-z)$$

- Π has the OPE expansion :

$$\Pi(z) \underset{z\to-\infty}{\sim} \sum_{p,k} B^{OPE}(p,k)\,\frac{1}{z^p}\log^k(-4z)$$

How to do analytic continuation ?

- **From the infinite number of Taylor coefficients : this is mathematics**



$$\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{\infty} C(n)\, z^n$$

By resumation or by construction order by order, with an infinite number of $C(n)$ the analytic continuation can be easily obtained.

- **From the finite number of Taylor coefficients**

$$\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{N^*} C(n)\, z^n$$

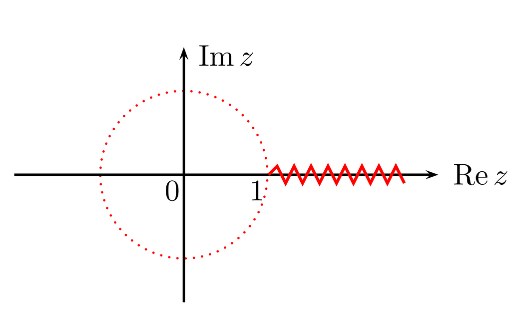
There is no way for a finite number N^* of $C(n)$, one needs the threshold and/or the OPE. Two approaches exist

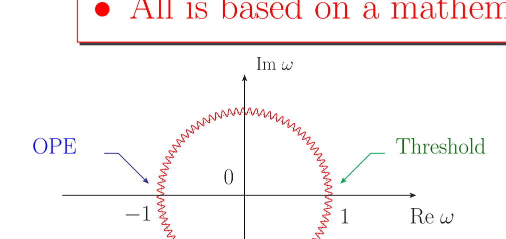
– Padé approximants under certain conditions (Π must be *Stieltjes*).
– Mellin-Barnes reconstruction, the method that we propose.

New method with Mellin transform

- It is an analytic reconstruction of the function.
- It is systematic and convergent order by order.
- It is a controlled approximation.
- All is based on a mathematical theorem.

- Conformal mapping





$$z = \frac{4\omega}{(1+\omega)^2} \qquad \omega = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$$

$$\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{\infty} C(n)\, z^n \qquad \hat{\Pi}(\omega) \underset{|\omega|<1}{=} \sum_{n=0}^{\infty} \Omega(n)\, \omega^n$$

$$\Omega(n) \underset{n\to\infty}{\sim} \Omega^{AS}(n) = \alpha_{0,0} + \alpha_{0,1}\log n + \cdots + \frac{1}{n}(\alpha_{1,0}+\alpha_{1,1}\log n + \cdots) + \cdots + \mathbf{(-1)^n}\,[\alpha \longrightarrow \beta]$$

- Main result provided by the Converse Mapping Theorem

$\boldsymbol{\alpha_{k,\ell}}$ **is only a linear function of** $A^{TH}(p,k)$ threshold coefficients	
$\boldsymbol{\beta_{k,\ell}}$ **is only a linear function of** $B^{OPE}(p,k)$ OPE coefficients	

- The reconstruction : $\hat{\Pi}(\omega) = \underbrace{\sum_{n=0}^{N^*} \Omega(n)\, \omega^n}_{\text{EXACT}} + \underbrace{\sum_{n=N^*+1}^{\infty} \Omega^{AS}(n)\, \omega^n}_{\text{APPROXIMATION}} + \underbrace{\sum_{n=N^*+1}^{\infty} \left[\Omega(n) - \Omega^{AS}(n)\right] \omega^n}_{\text{ERROR}}$

The reconstruted expression is then a polynom in ω and a finite sum of polylogaritms $\mathrm{Li}^{(n)}(s,\omega) \doteq \frac{d^n}{ds^n}\sum_{k=1}^{\infty} k^{-s}\omega^k$,

$$\sum_{n=0}^{\infty} \Omega^{AS}(n)\, \omega^n = \alpha_{0,0}\,\frac{\omega}{1-\omega} + \alpha_{0,1}\,\mathrm{Li}^{(1)}(0,\omega) + \cdots + \begin{bmatrix} \alpha \longrightarrow \beta \\ \omega \longrightarrow -\omega \end{bmatrix}$$

In practice $\Omega^{AS}(n)$ are really efficient.

A perfect application example : Heavy-Quark Correlators

Let us recall the vacuum vector-vector polarization $\Pi(q^2)$ (in the massive case)

$$\left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)\Pi(q^2) = -i\int \mathrm{d}^4x\, \mathrm{e}^{iqx}\,\left\langle 0 \left| \mathsf{T}\, j_\mu(x)\, j_\nu(0) \right| 0 \right\rangle ,$$

with $j^\mu(x) \doteq \bar{q}(x)\gamma^\mu q(x)$, where $q(x)$ is a heavy quark. It may be decomposed as

$$\Pi(q^2) = \Pi^{(0)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)\Pi^{(1)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^2\Pi^{(2)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^3\Pi^{(3)}(q^2) + \mathcal{O}(\alpha_s^4)$$

we choose α_s in the $\overline{\mathrm{MS}}$ scheme at the scale $\mu = m_{pole}$.

- $\boldsymbol{\Pi^{(2)}}$: $\Pi^{(2)}(z) \underset{|z|<1}{=} \sum_{n=1}^{\mathbf{30}} C(n)\, z^n \qquad \boldsymbol{N^* = 30}$ and $z \doteq \frac{q^2}{4m^2}$

Threshold : $\Pi^{(2)}(z) \underset{z\to 1}{\sim} \frac{A(-\frac{1}{2},0)}{\sqrt{1-z}} + \left\{A(0,2)\,\log^2(1-z) + A(0,1)\,\log(1-z) + K^{(2)}\right\} + \left\{A(\tfrac{1}{2},1)\,\log(1-z) + A(\tfrac{1}{2},0)\right\}\sqrt{1-z} + \ldots$

OPE : $\Pi^{(2)}(z) \underset{z\to-\infty}{\sim} \left\{B(0,2)\,\log^2(-4z) + B(0,1)\,\log(-4z) + B(0,0)\right\} + \left\{B(-1,2)\,\log^2(-4z) + B(-1,1)\,\log(-4z) + B(-1,0)\right\}\frac{1}{z}$
$+ \left\{B(-2,3)\,\log^3(-4z) + B(-2,2)\,\log^2(-4z) + B(-2,1)\,\log(-4z) + B(-2,0)\right\}\frac{1}{z^2} + \ldots$

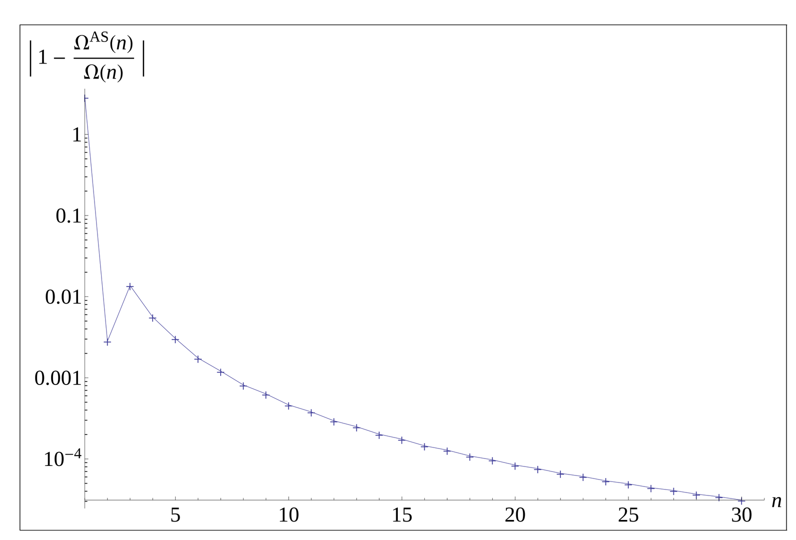
$$\begin{aligned}\Omega^{AS}(n) =& \alpha_{0,0} + \left\{\alpha_{1,0} + \alpha_{1,1}\,\log n\right\}\frac{1}{n} + \alpha_{2,0}\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\log^{\ell_1} n\right) \\ &+ (-1)^n\left[\left\{\beta_{1,0} + \beta_{1,1}\log n\right\}\frac{1}{n} + \left\{\beta_{3,0} + \beta_{3,1}\log n\right\}\frac{1}{n^3} + \left\{\beta_{5,0} + \beta_{5,1}\log n + \beta_{5,2}\log^2 n\right\}\frac{1}{n^5} + \mathcal{O}\left(\frac{1}{n^7}\log^{\ell_2} n\right)\right]\end{aligned}$$

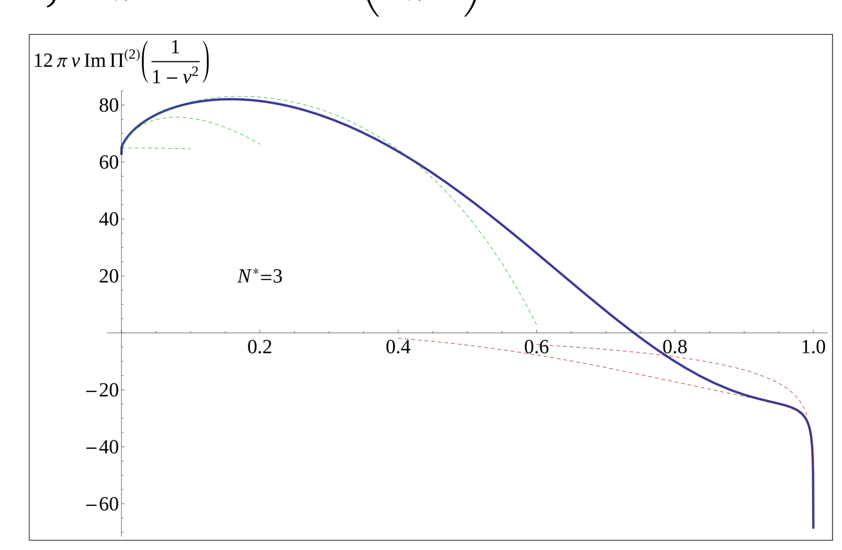
where the α's and the β's are known analytically by identification

$$\begin{cases} \alpha_{0,0} = 2\,A(-\tfrac{1}{2},0) \simeq 3.44514 \\ \alpha_{1,0} \simeq -0.492936 \\ \alpha_{1,1} = 2.25 \\ \alpha_{2,0} \simeq 3.05433 \end{cases} \quad \text{and} \quad \begin{cases} \beta_{1,0} \simeq 0.33723 \\ \beta_{1,1} \simeq 0.211083 \\ \beta_{3,0} \simeq 0.183422 \end{cases} \begin{cases} \beta_{3,1} \simeq -0.620598 \\ \beta_{5,0} \simeq -1.89016 \\ \beta_{5,2} \simeq 1.38684 \end{cases}$$

and we estimate the error,

$$\left[\Omega(n) - \Omega^{AS}(n)\right]_{n>N^*(=30)} \cong \begin{Bmatrix} +1 \\ -0 \end{Bmatrix} \frac{\log^{1.5} n}{n^3} \pm (-1)^n\,\mathcal{O}\left(\frac{\log^{\ell_2} n}{n^7}\right)$$





- $\boldsymbol{\Pi^{(3)}}$: $\Pi^{(3)}(z) \underset{|z|<1}{=} \sum_{n=1}^{\mathbf{3}} C(n)\, z^n \qquad \boldsymbol{N^* = 3}$

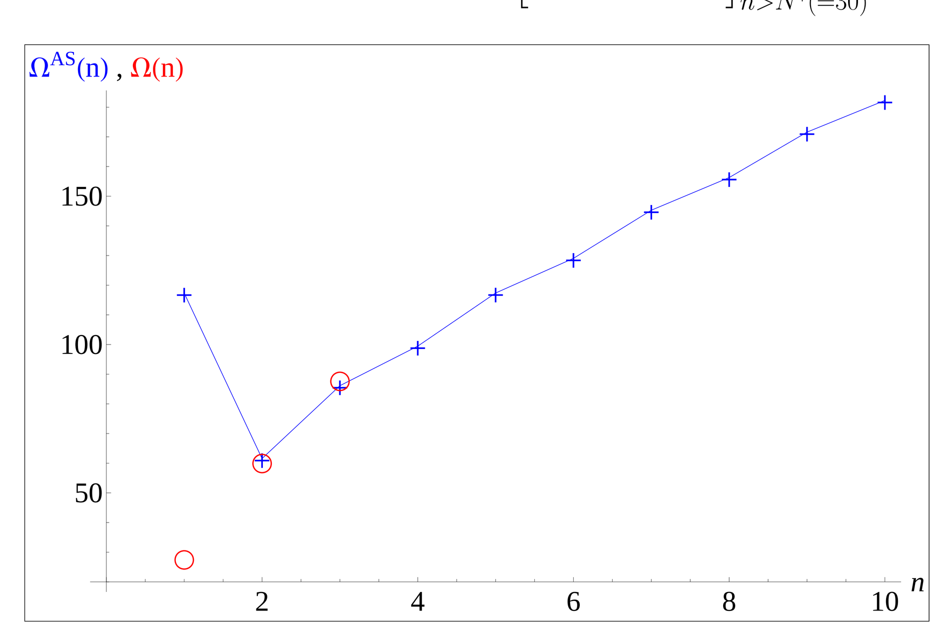
$$\begin{aligned}\Omega^{AS}(n) =& \alpha_{-1,0}\,n + \left\{\alpha_{0,0} + \alpha_{0,1}\ln n\right\} + \left\{\alpha_{1,0} + \alpha_{1,1}\ln n + \alpha_{1,2}\ln^2 n\right\}\frac{1}{n} + \mathcal{O}\left(\tfrac{1}{n^2}\log^{\ell_1} n\right) \\ &+ (-1)^n\left[\left\{\beta_{1,0} + \beta_{1,1}\ln n + \beta_{1,2}\ln^2 n\right\}\frac{1}{n} + \left\{\beta_{3,0} + \beta_{3,2}\ln^2 n\right\}\frac{1}{n^3} + \left\{\beta_{5,0} + \beta_{5,1}\ln n + \beta_{5,2}\ln^2 n + \beta_{5,3}\ln^3 n\right\}\frac{1}{n^5} + \mathcal{O}\left(\frac{1}{n^7}\log^{\ell_2} n\right)\right]\end{aligned}$$

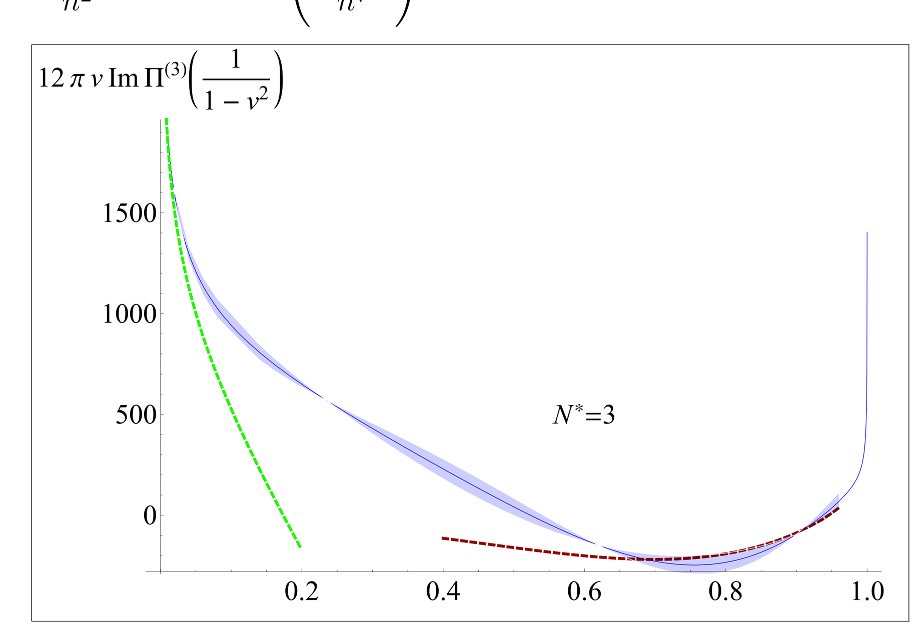
with

$$\begin{cases} \alpha_{-1,0} \simeq 10.5456 \\ \alpha_{0,1} \simeq 31.0063 \\ \alpha_{0,0} \simeq -11.0769 \\ \alpha_{1,0} \simeq 36.3318 \\ \alpha_{1,1} \simeq 37.1514 \\ \alpha_{1,2} \simeq 10.125 \end{cases}, \quad \begin{cases} \beta_{1,0} \simeq -0.181866 \\ \beta_{1,1} \simeq -2.4852 \\ \beta_{1,2} \simeq -0.879515 \\ \beta_{3,0} \simeq -10.4385 \\ \beta_{3,2} \simeq 3.82702 \end{cases}, \quad \begin{cases} \beta_{5,0} \simeq -70.9277 \\ \beta_{5,1} \simeq 56.3093 \\ \beta_{5,2} \simeq 20.9951 \\ \beta_{5,3} \simeq -7.55063 \end{cases}.$$

and we estimate the error,

$$\left[\Omega(n) - \Omega^{AS}(n)\right]_{n>N^*(=30)} \cong \pm\, 15\,\frac{\log^3 n}{n^2} \pm (-1)^n\mathcal{O}\left(\frac{\log^{\ell_2} n}{n^7}\right)$$





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D. Greynat and S. Peris, "Resummation of Threshold, Low- and High-Energy Expansions for Heavy-Quark Correlators", Phys. Rev. D **82** (2010) 034030