Generalised Unitarity in d-dimensions and Colour/Kinematics duality

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Based on the collaborations with
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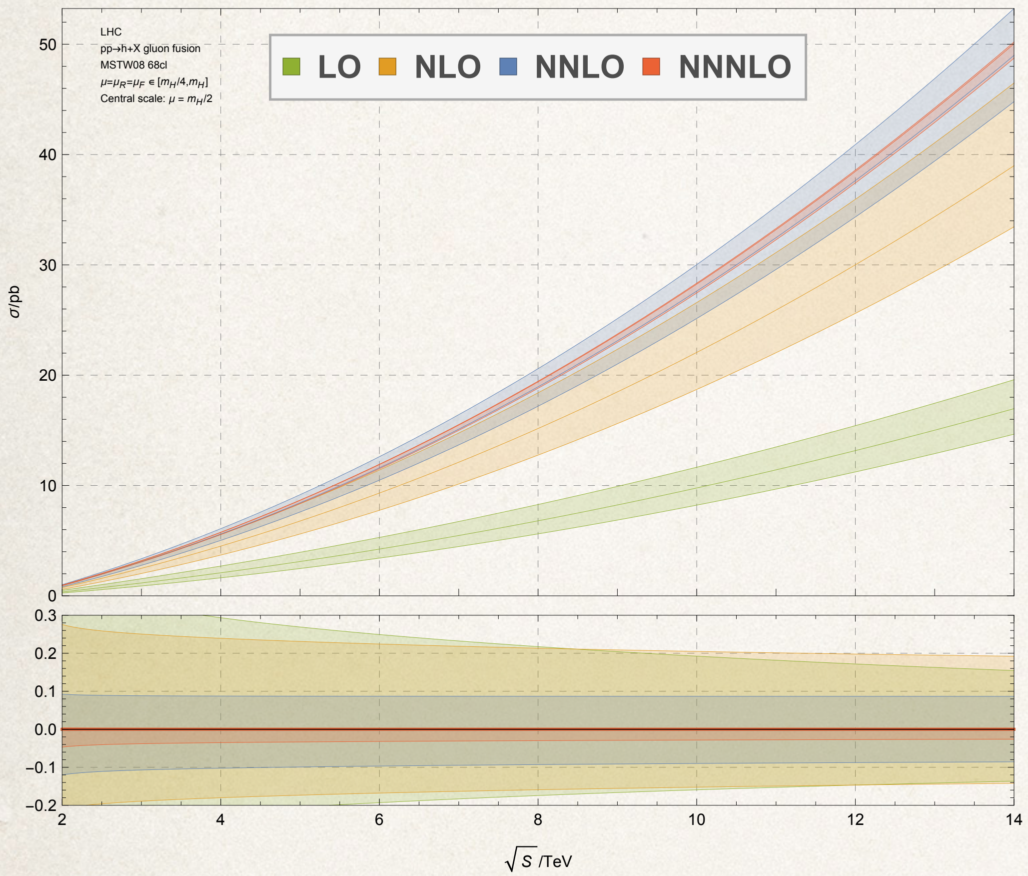
Introduction

- Scattering amplitudes are necessary to test our theoretical models by comparing their predictions against the experiments.

[Anastasiou, Duhr, Dulat, Herzog, Mistlberger (2015)]



- Tree-level (LO) predictions are qualitative due to the poor convergence of the truncated expansion at strong coupling.

$$\alpha_S\,(100\mathrm{GeV}) \sim 0.12$$

- K factors

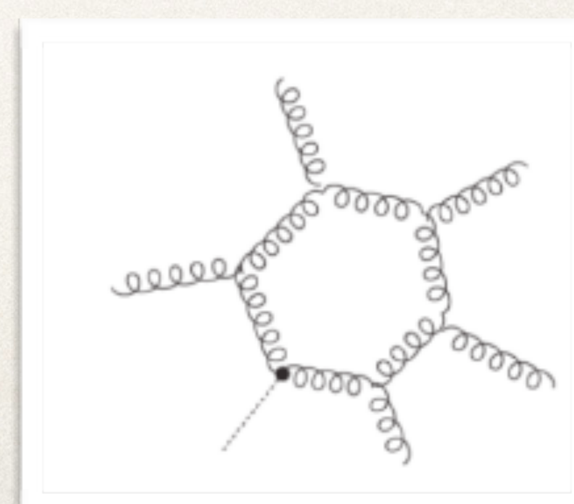
$$K = \frac{\mathrm{NLO}}{\mathrm{LO}} \sim 30\% \div 80\%$$

- Feynman diagrams, based on the Lagrangian, are not optimised for these processes.

- On-shell methods are based on amplitudes and take full advantage of the analyticity of the S-matrix.

Motivation

- Compute the uncomputable.
- Simplify the calculations in High-Energy Physics.
- Discover hidden properties of Quantum Field Theories
- Towards NNLO is the **Next Frontier.**



Introduction

NLO timeline

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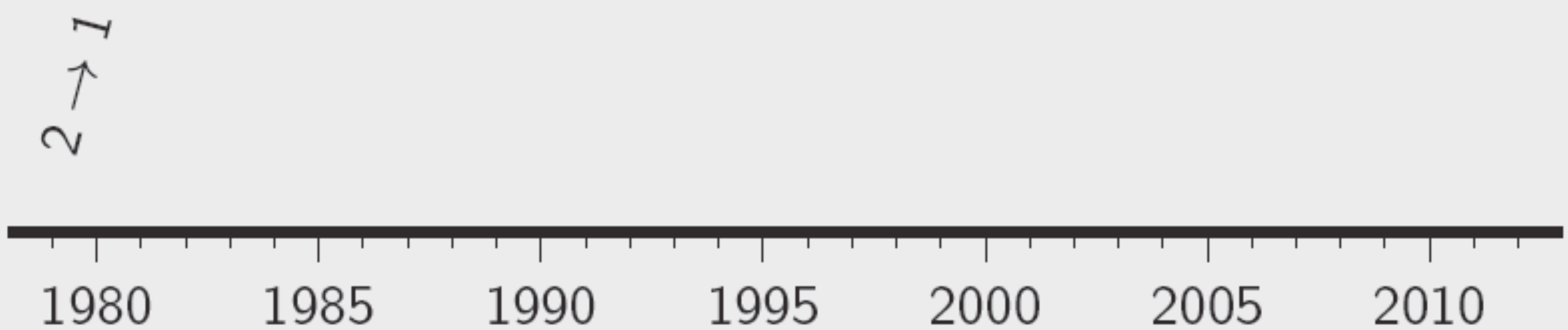
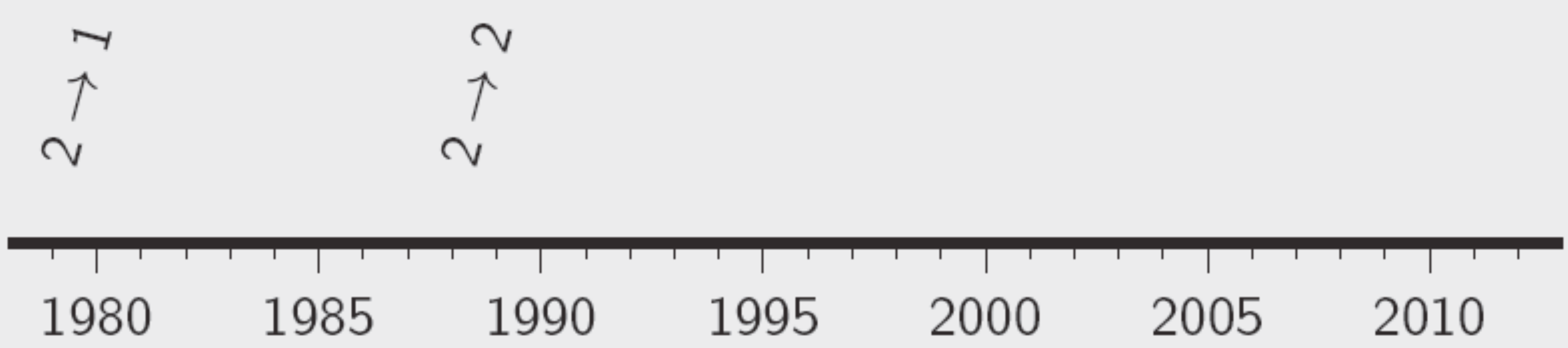
NLO timeline



1979: NLO Drell-Yan [Altarelli, Ellis & Martinelli]
1991: NLO $gg \to$ Higgs [Dawson; Djouadi, Spira & Zerwas]

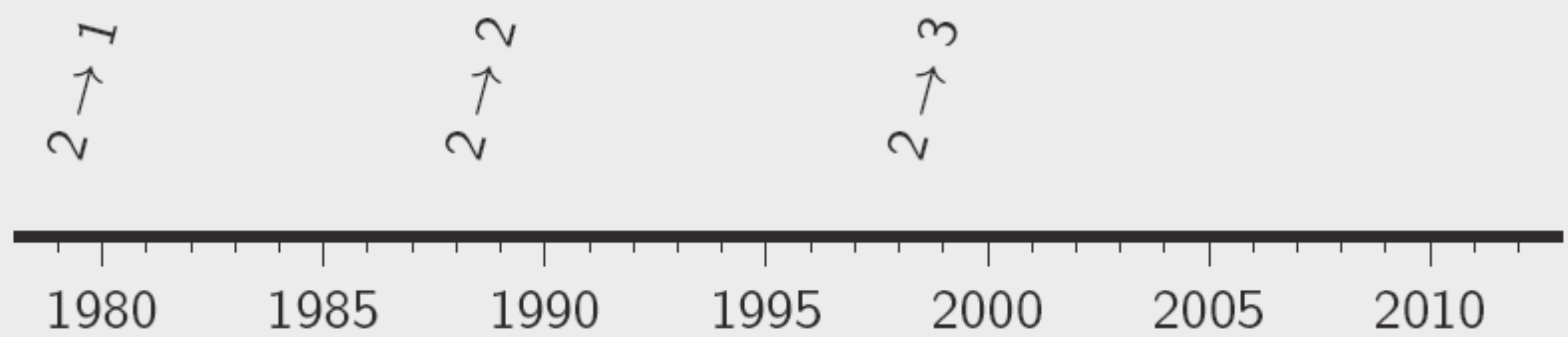
Introduction

NLO timeline



1987: NLO high-p_t photoproduction [Aurenche et al]
1988: NLO $b\bar{b}$, $t\bar{t}$ [Nason et al]
1988: NLO dijets [Aversa et al]
1993: Vj [JETRAD, Giele, Glover & Kosower]

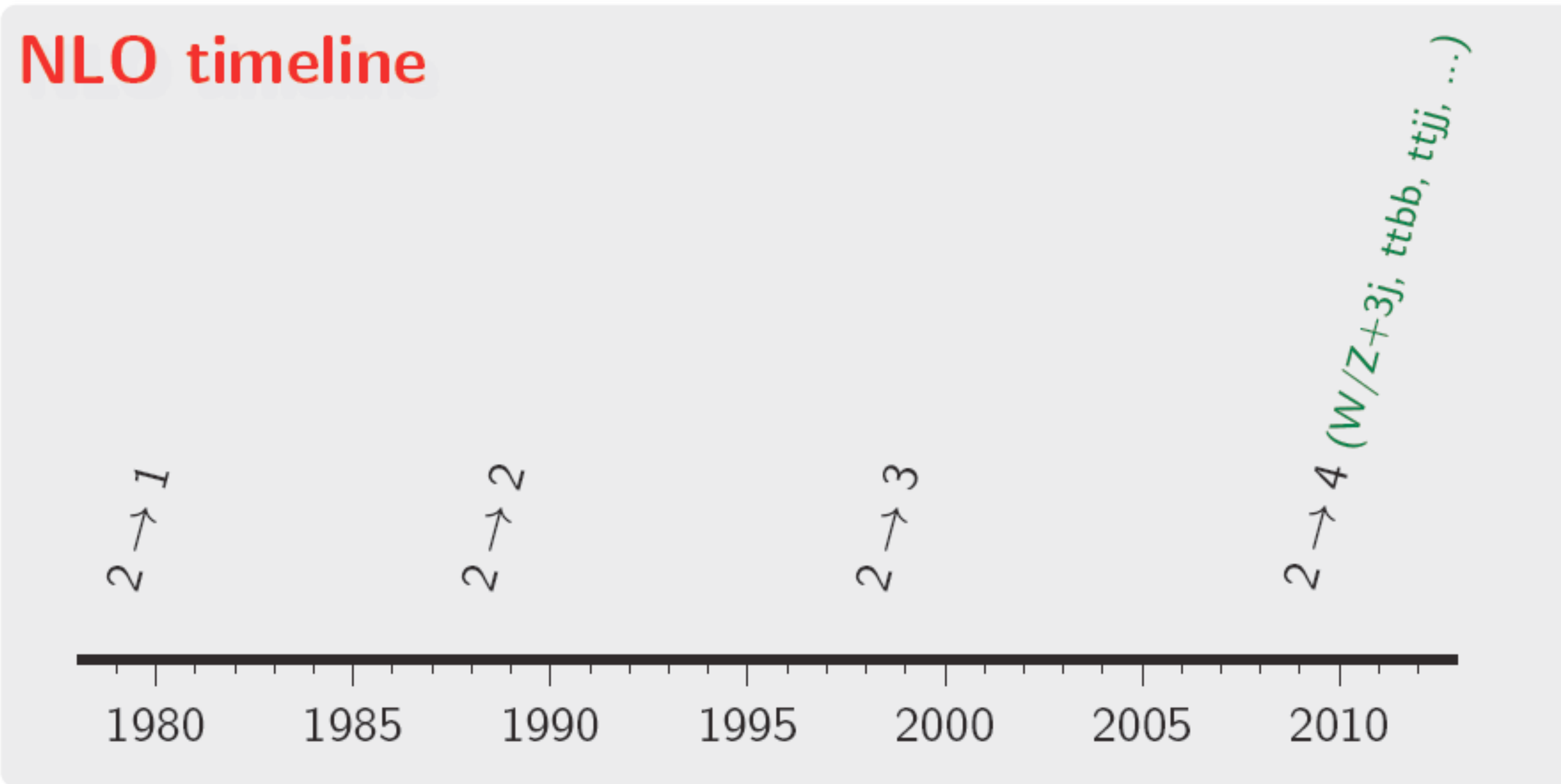
NLO timeline



1998: NLO $Wb\bar{b}$ [MCFM: Ellis & Veseli]
2000: NLO $Zb\bar{b}$ [MCFM: Campbell & Ellis]
2001: NLO $3j$ [NLOJet++: Nagy]
...
2007: NLO $t\bar{t}j$ [Dittmaier, Uwer & Weinzierl '07]
...

Introduction

G. Salam (SILAFAE 2012)



2009: NLO $W+3j$ [Rocket: Ellis, Melnikov & Zanderighi] [unitarity]

2009: NLO $W+3j$ [BlackHat+Sherpa: Berger et al] [unitarity]

2009: NLO $t\bar{t}b\bar{b}$ [Bredenstein et al] [traditional]

2009: NLO $t\bar{t}b\bar{b}$ [HELAC-NLO: Bevilacqua et al] [unitarity]

2009: NLO $q\bar{q} \to b\bar{b}b\bar{b}$ [Golem: Binoth et al] [traditional]

2010: NLO $t\bar{t}jj$ [HELAC-NLO: Bevilacqua et al] [unitarity]

2010: NLO $Z+3j$ [BlackHat+Sherpa: Berger et al] [unitarity]

...

Introduction

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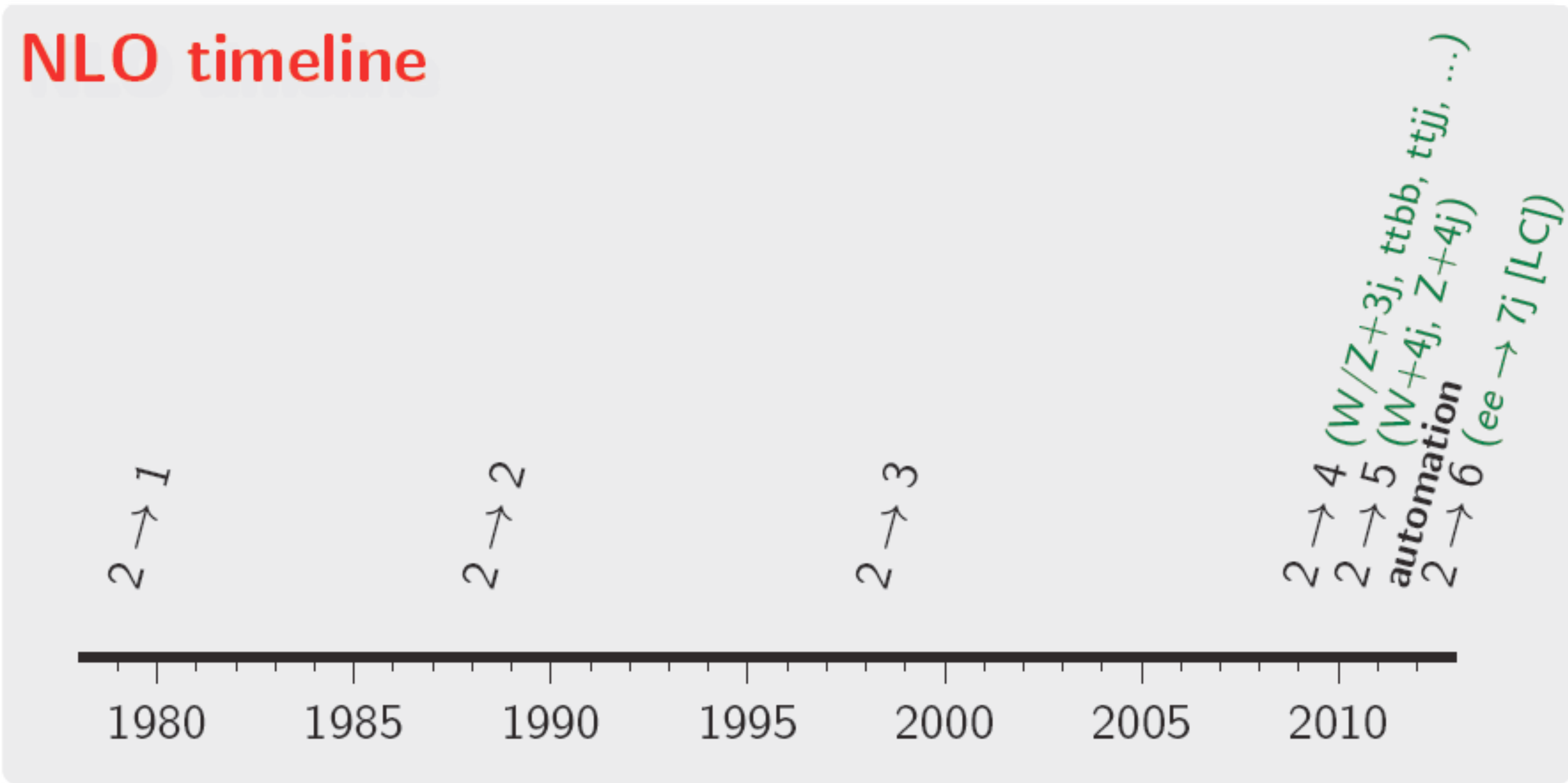

2010: NLO $W+4j$ [BlackHat+Sherpa: Berger et al] [unitarity]

2011/12: NLO $WWjj$ [Rocket: Melia et al; GoSaM+MadX Greiner et al] [unitarity]

2011: NLO $Z+4j$ [BlackHat+Sherpa: Ita et al] [unitarity]

2011/12: NLO $4j$ [BlackHat/NGluons+Sherpa: Bern et al; Badger et al] [unitarity]

2011–: first automation [MadNLO: Hirschi et al] [unitarity + feyn.diags]

2011–: first automation [Helac NLO: Bevilacqua et al] [unitarity]

2011–: first automation [GoSam: Cullen et al] [feyn.diags(+unitarity)]

2011: $e^+e^- \to 7j$ [Becker et al, leading colour] [numerical loops]

Outline

- Analytic one-loop amplitudes
  — d dimensional generalised unitarity
  — Four dimensional formulation of dimensional regularisation

- Automation of analytic one-loop amplitudes
  — Results

- Further simplifications from colour/kinematics duality
  — C/K relations @ tree-level in dimensional regularisation
  — C/K relations @ one-loop

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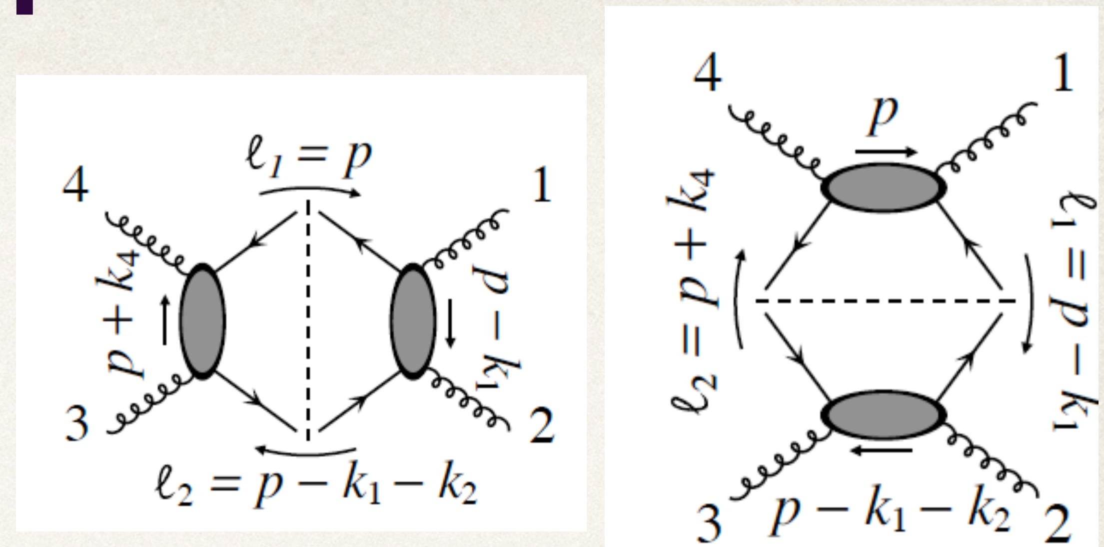
Analytic one-loop scattering amplitudes



[Bern, Dixon, Dunbar, Kosower (1994)]

Standard Unitarity in 4D

Glue together the two amplitudes and uplift the integral with

$$2\pi\delta^{(+)}\left(p^2 - m^2\right) \to \frac{i}{p^2 - m^2 - i\epsilon}$$

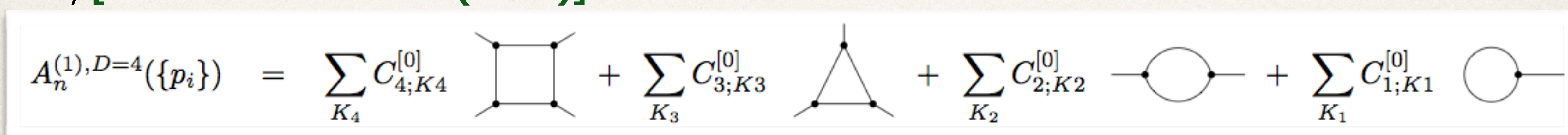
Generalised Unitarity in 4D

[Bern, Dixon, Kosower (1998)]
[Britto, Cachazo, Feng (2004)]

Isolate the leading discontinuity

$$\mathcal{A}^{(L)} = \sum_i c_i \mathcal{I}_i^{(L)}$$ → **Known** basis of L-loop scalar integrals

For L=1, **[Passarino - Veltman (1979)]**



$$A_n^{(1),D=4}(\{p_i\}) = \sum_{K_4} C^{[0]}_{4;K4} + \sum_{K_3} C^{[0]}_{3;K3} + \sum_{K_2} C^{[0]}_{2;K2} + \sum_{K_1} C^{[0]}_{1;K1}$$

Scalar Master Integrals: Made of polylogarithmic functions

- If an amplitude is determined by its branch cuts, it is said to be cut-constructible.
- All one-loop amplitudes are cut-constructible in dimensional regularisation.

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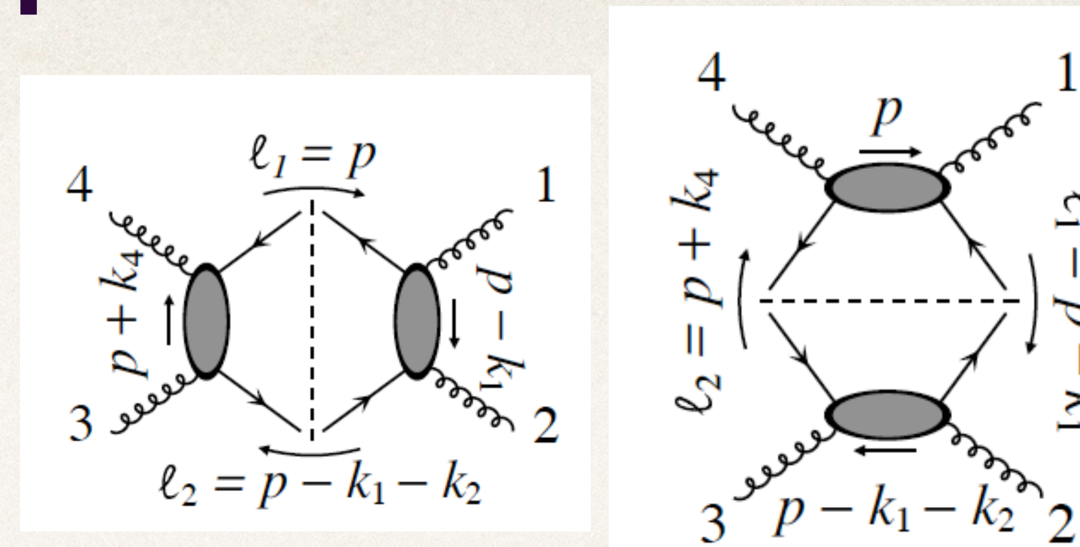
Analytic one-loop scattering amplitudes



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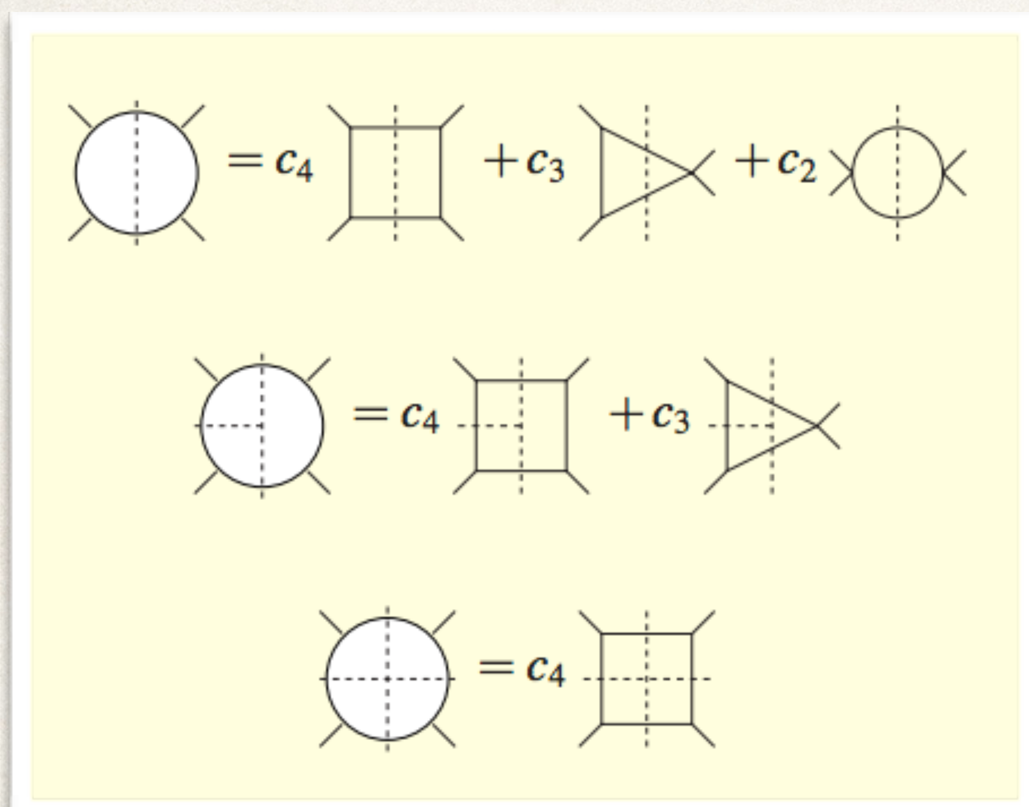


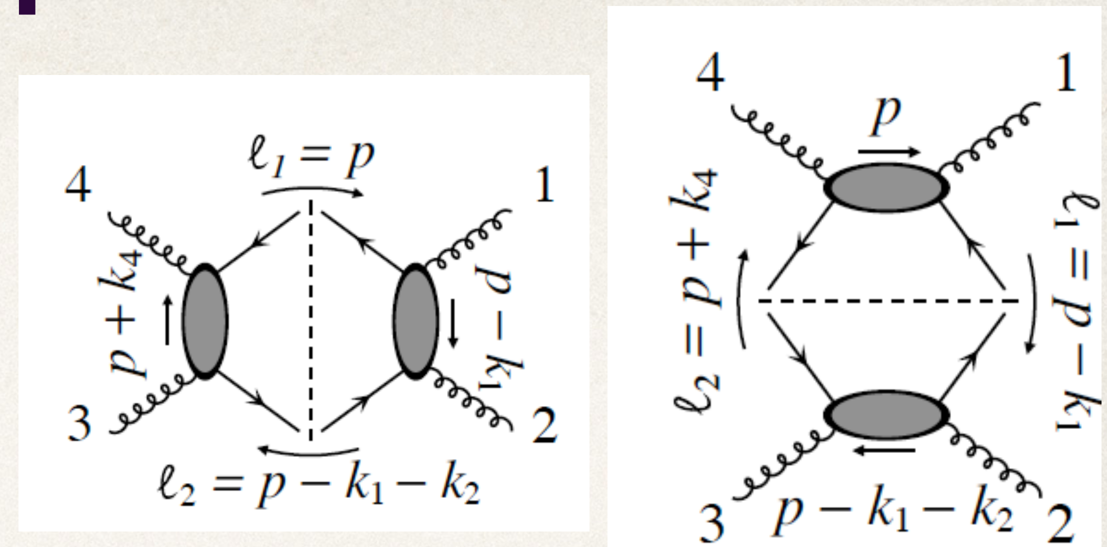
cut-4 :: Britto Cachazo Feng

cut-3 :: Forde
Bjerrum-Bohr, Dunbar, Ita, Perkins
Mastrolia

cut-2 :: Bern, Dixon, Dunbar, Kosower.
Britto, Buchbinder, Cachazo, Feng.
Britto, Feng, Mastrolia.

Analytic one-loop scattering amplitudes

Standard Unitarity in 4D



Glue together the two amplitudes and uplift the integral with

$$2\pi\delta^{(+)}\left(p^2 - m^2\right) \to \frac{i}{p^2 - m^2 - i\epsilon}$$

[Bern, Dixon, Dunbar, Kosower (1994)]

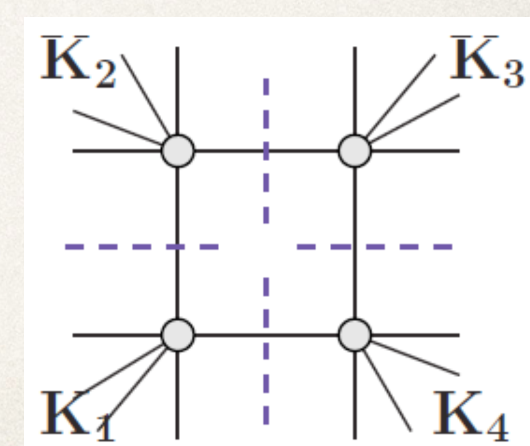
Generalised Unitarity in 4D

Isolate the leading discontinuity

[Britto, Cachazo, Feng (2004)]

Quadruple-cut: read our single box coefficient

$$\Delta_4 A^{1-\text{loop}} = \int d^4\ell\ \delta(\ell_1^2)\ \delta(\ell_2^2)\ \delta(\ell_3^2)\ \delta(\ell_4^2)\ A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$



The cut expansion collapses to a single term

$$\Delta_4 A^{1-\text{loop}} = c_4(K_1, K_2, K_3, K_4)\Delta_4 I_4(K_1, K_2, K_3, K_4)$$

Analytic one-loop scattering amplitudes

In D=4-2ε we can do the decomposition $\bar{\ell}^\nu = \ell^\nu + \tilde{\ell}^\nu$

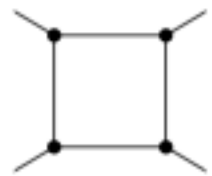
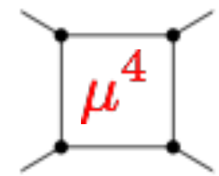
ℓ^ν → D=4 $\qquad$ $\tilde{\ell}^\nu$ → D=-2ε

The on-shell condition

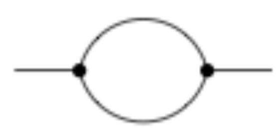
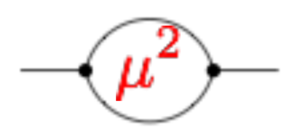
$$\bar{\ell}^2 = \ell^2 - \mu^2 = 0 \;\longrightarrow\; \ell^2 = \mu^2$$

μ^2 → Mass term

Any massless one-loop becomes

$$A_n^{(1),D=4-2\epsilon}(\{p_i\}) = \sum_{K_4} C^{[0]}_{4;K4}$$  $$+ \sum_{K_4} C^{[4]}_{4;K4}$$ 

$$+ \sum_{K_3} C^{[0]}_{3;K3}$$  $$+ \sum_{K_3} C^{[2]}_{3;K3}$$ 

$$+ \sum_{K_2} C^{[0]}_{2;K2}$$  $$+ \sum_{K_2} C^{[2]}_{2;K2}$$ 

$$+ \sum_{K_1} C^{[0]}_{1;K1}$$ 

[Ossola,Papadopoulos,Pittau (2006)]
[Giele,Kunszt,Melnikov (2008)]
[Badger (2008)]
[Mastrolia, Mirabella, Peraro (2012)]

How to compute those coefficients?

A: Separated computation of cut-constructible and rational terms

A1: Computing the rational term separately (using non gauge invariant terms)

— R1 and R2 separation [Ossola, Papadopoulos, Pittau(2008); Pittau, Draggiotis, Garzelli (2009)]
— Supersymmetric decomposition [Bern, Dixon, Kosower]

B: D-dimensional unitarity offers the determination of all pieces together

B1: 6-dimensional spinor-helicity formalism [Cheung and O'Connell(2009); Davies (2012)]

— New rules for spinor products
— No automatic generator exists

B2: Gamma algebra in extended dimension [Ellis,Giele,Kunszt,Melnikov (2008)]

— The explicit representation of the polarisation states is avoid
— Gamma algebra has to be extended everywhere.
— Automatic generator has to be modified

B3: Don't leave 4 dimensions! [Fazio, Mastrolia, Mirabella, W.J.T (2014)]

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B: D-dimensional unitarity offers the determination of all pieces together

Four Dimensional Formulation of Dimensional Regularisation (FDF)

B3: Don't leave 4 dimensions! [Fazio, Mastrolia, Mirabella, W.J.T (2014)]

— Explicit 4D representation of polarisation and. spinors
— 4D representation of D-reg loop propagators
— 4D Feynman rules + (-2ε)-Selection Rules
— Easy to implement in existing generators

FDH: 4D helicity scheme

[Bern and Kosower (1992)]

The d-dimensional metric tensor can be split as

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$$

d-dimensional, 4-dimensional, -2ε-dimensional

Where

$$\tilde{g}^{\mu\nu} g_{\mu\nu} = 0, \qquad \tilde{g}^{\mu}_{\mu} = -2\epsilon \underset{d\to 4}{\longrightarrow} 0, \qquad g^{\mu}_{\mu} = 4 \qquad \tilde{q}^2 = \tilde{g}^{\mu\nu}\bar{q}_\mu \bar{q}_\nu = -\mu^2$$

Projections of the vectors q and $\tilde{q}$.

$$\tilde{q}^{\mu} g_{\mu\nu} = \tilde{g}^{\mu\sigma}\bar{q}_\sigma g_{\mu\nu} = 0$$

As well for the gamma matrices

$$[\tilde{\gamma}^\alpha, \gamma^5] = 0, \qquad \{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2\,\tilde{g}^{\alpha\beta}, \qquad \{\tilde{\gamma}^\alpha, \gamma^\mu\} = 0.$$

Can we implement it with 4D-object only?

In 4-dimensions, one can infer: $\tilde{\gamma} \sim \gamma^5$

And the Clifford algebra $\tilde{\gamma}^\mu \tilde{\gamma}_\mu \underset{d\to 4}{\longrightarrow} 0$ while $\gamma^5\gamma^5 = 1$

Excludes any four-dimensional representation of the —2ε-subspace

—2ε-subspace → **—2ε-Selection Rules (—2ε)-SRs**

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

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-2ε-Selection Rules

The Clifford algebra conditions are satisfied by imposing

$$\tilde{g}^{\alpha\beta} \to G^{AB}, \qquad \tilde{\ell}^\alpha \to i\, \mu\, Q^A\,, \qquad \tilde{\gamma}^\alpha \to \gamma^5\, \Gamma^A\,.$$

A,B := —2ε-dimensional vectorial indices traded for (—2ε)-SRs

$$\begin{aligned} G^{AB} G^{BC} &= G^{AC}, & G^{AA} &= 0, & G^{AB} &= G^{BA}, \\ \Gamma^A G^{AB} &= \Gamma^B, & \Gamma^A \Gamma^A &= 0, & Q^A G^A &= 1, \\ Q^A G^{AB} &= Q^B, & Q^A Q^A &= 1. \end{aligned}$$

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

Completeness relations within FDF

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

Gluon propagator

The helicity sum of the transverse polarisation vector is

$$\sum_{i=1}^{d-2} \varepsilon^{\mu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right)\varepsilon^{*\nu}_{i\,(d)}\left(\bar{\ell},\bar{\eta}\right) = \left(-g^{\mu\nu} + \frac{\ell^{\mu}\ell^{\nu}}{\mu^2}\right) - \left(\tilde{g}^{\mu\nu} + \frac{\tilde{\ell}^{\mu}\tilde{\ell}^{\nu}}{\mu^2}\right).$$

d = 4 **d = -2ε**

massive gluon

$$\left(-g^{\mu\nu} + \frac{l^{\mu}l^{\nu}}{\mu^2}\right) = \sum_{\lambda=\pm,0} \varepsilon^{\mu}_{\lambda}\left(l\right)\varepsilon^{*\nu}_{\lambda}\left(l\right)$$

$$\left(\tilde{g}^{\mu\nu} + \frac{\tilde{l}^{\mu}\tilde{l}^{\nu}}{\mu^2}\right) \longrightarrow \hat{G}^{AB} = G^{AB} - Q^A Q^B$$

Fermion propagator

$$\sum_{\lambda=\pm} u_{\lambda}\left(l\right)\bar{u}_{\lambda}\left(l\right) = \not{l} + i\mu\gamma^5 + m$$

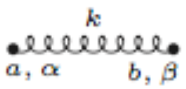
$$\sum_{\lambda=\pm} v_{\lambda}\left(l\right)\bar{v}_{\lambda}\left(l\right) = \not{l} + i\mu\gamma^5 - m$$

Allows to generalise the Dirac Equation

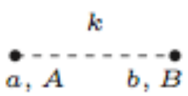
$$\left(\not{\ell} + i\mu\gamma^5 + m\right) u_{\lambda}\left(\ell\right) = 0\,, \quad \ell^2 = m^2 + \mu^2\,, \quad \ell = \ell^{\flat} + \frac{m^2+\mu^2}{2\,\ell\cdot q_{\ell}} q_{\ell}\,, \quad (\ell^{\flat})^2 = (q_{\ell})^2 = 0\,.$$

Feynman Rules in FDF

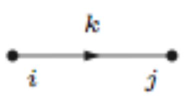
[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]



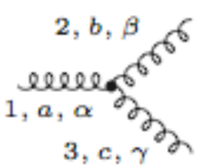
$$= -i\,\delta^{ab}\,\frac{1}{k^2-\mu^2+i0}\left[g^{\alpha\beta}-\frac{k^\alpha k^\beta}{\mu^2}\right] \quad \text{(gluon)},$$



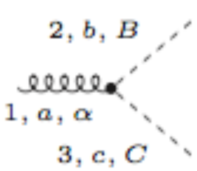
$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k^2-\mu^2+i0}, \quad \text{(scalar)},$$



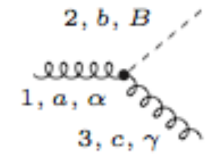
$$= i\,\delta^{ij}\,\frac{\not{k}+i\mu\gamma^5+m}{k^2-m^2-\mu^2+i0}, \quad \text{(fermion)},$$



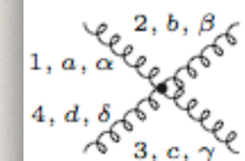
$$= -g\,f^{abc}\left[(k_1-k_2)^\gamma g^{\alpha\beta} + (k_2-k_3)^\alpha g^{\beta\gamma} + (k_3-k_1)^\beta g^{\gamma\alpha}\right],$$



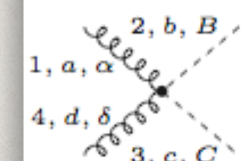
$$= -g\,f^{abc}\,(k_2-k_3)^\alpha\,G^{BC},$$



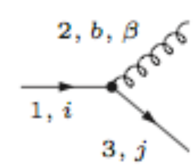
$$= \mp g\,f^{abc}\,(i\mu)\,g^{\gamma\alpha}\,Q^B \qquad (\tilde{k}_1=0, \quad \tilde{k}_3=\pm\tilde{\ell}),$$



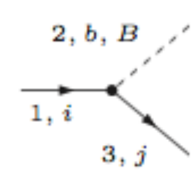
$$= -ig^2\Big[f^{xad}\,f^{xbc}\,\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\gamma}g^{\beta\delta}\right) + f^{xac}\,f^{xbd}\,\left(g^{\alpha\beta}g^{\delta\gamma}-g^{\alpha\delta}g^{\beta\gamma}\right) + f^{xab}\,f^{xdc}\,\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}\right)\Big],$$



$$= 2ig^2\,g^{\alpha\delta}\,\left(f^{xab}\,f^{xcd}+f^{xac}\,f^{xbd}\right)G^{BC},$$



$$= -ig\,\left(t^b\right)_{ji}\,\gamma^\beta,$$

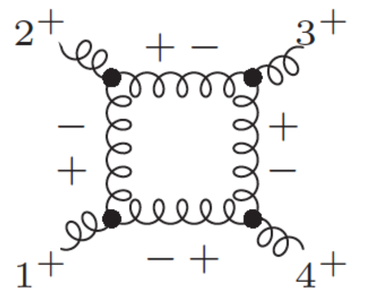


$$= -ig\,\left(t^b\right)_{ji}\,\gamma^5\,\Gamma^B.$$

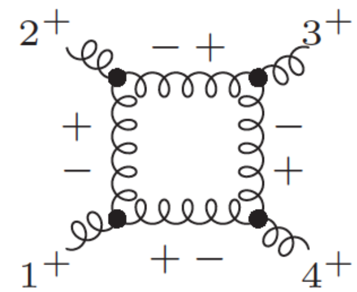
The simplest example: the gggg amplitude

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

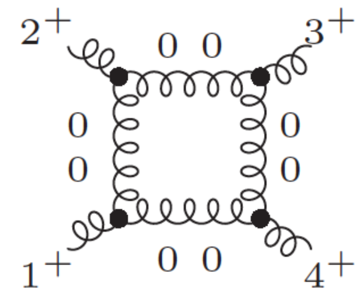
$$C^{[0]}_{1|2|3|4} =$$



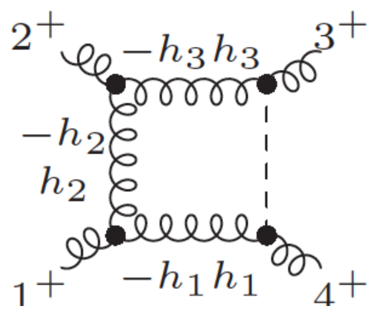
$$+$$



$$+$$

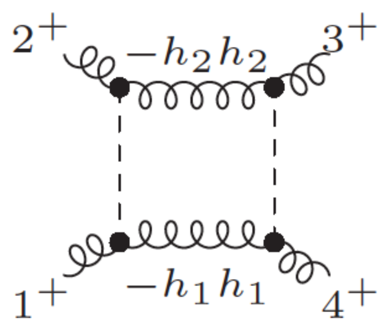


$$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$$

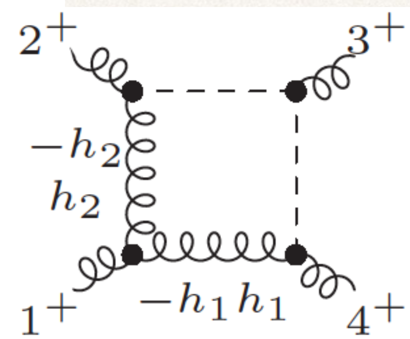


$$+\ \text{c.p.}\,,$$

$$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$$

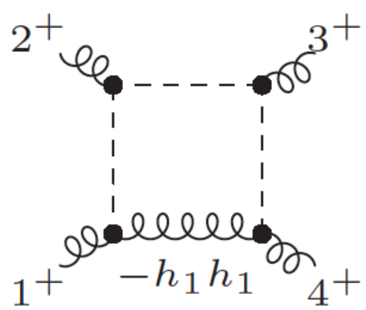


$$+\ \mathcal{T}_2$$



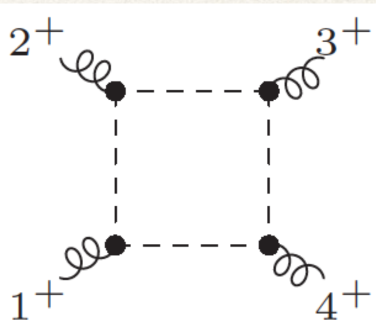
$$+\ \text{c.p.}$$

$$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$$



$$+\ \text{c.p.}$$

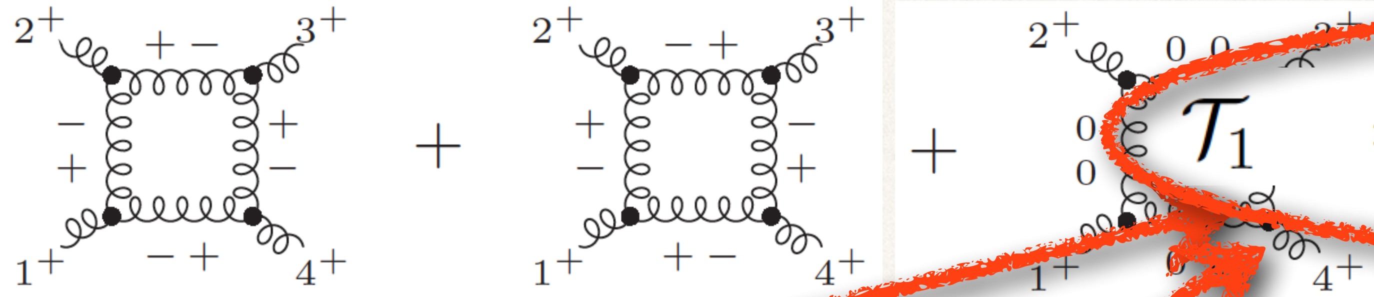
$$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$$



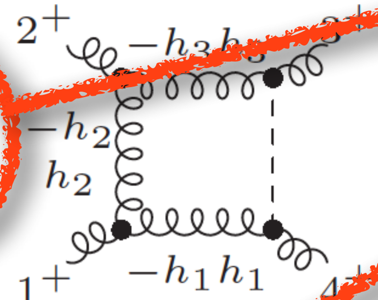
The simplest example: the gggg amplitude

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

$C^{[0]}_{1|2|3|4} =$

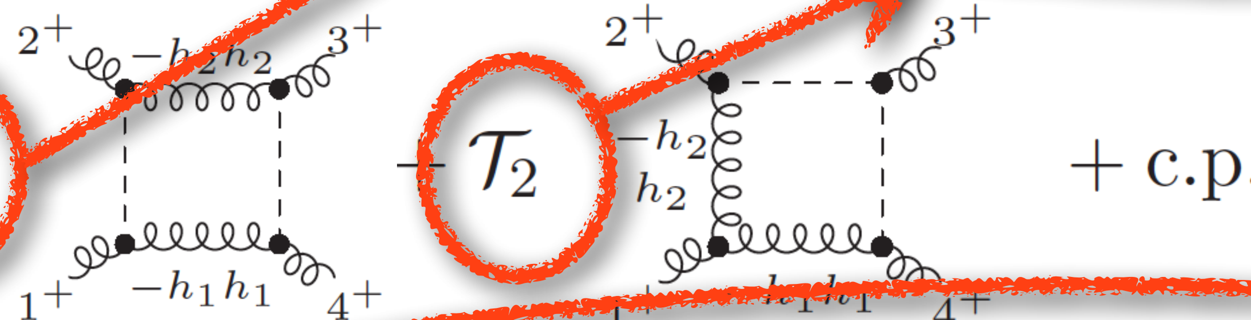


$C^{[1]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1$



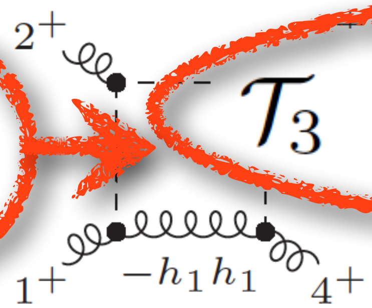
$+ \text{c.p.}$

$C^{[2]}_{1|2|3|4} = \sum_{h_i=\pm,0} \mathcal{T}_1^2$

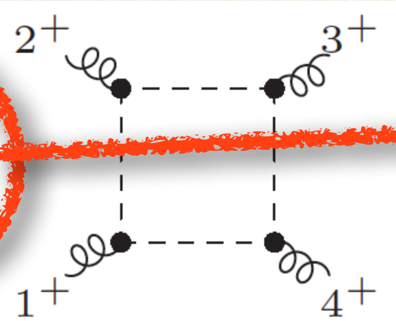


$+ \text{c.p.}$

$C^{[3]}_{1|2|3|4} = \sum_{h_1=\pm,0} \mathcal{T}_3$



$C^{[4]}_{1|2|3|4} = \mathcal{T}_4$



$$\mathcal{T}_1 = Q^A \hat{G}^{AB} Q^B = 0$$

$$\mathcal{T}_2 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} Q^D = 0$$

$$\mathcal{T}_3 = Q^A \hat{G}^{AB} G^{BC} \hat{G}^{CD} G^{DE} \hat{G}^{EF} Q^F = 0$$

$$\mathcal{T}_4 = \text{tr}\left(G \hat{G} G \hat{G} G \hat{G} G \hat{G}\right) = -1$$

The simplest example: the gggg amplitude

[Fazio, Mastrolia, Mirabella, W.J.T. (2014)]

$$A_4^{1-\text{loop}}\left(1_g^+, 2_g^+, 3_g^+, 4_g^+\right)$$

Contributions come only from the coefficients:

(Gluon loop)
$$c^{[0]}_{1|2|3|4;\,0} = 0\,, \qquad c^{[0]}_{1|2|3|4;\,4} = 3i\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$$

(-2ε-Scalar loop)
$$c^{[4]}_{1|2|3|4;\,0} = 0\,, \qquad c^{[4]}_{1|2|3|4;\,4} = -i\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$$

which amounts

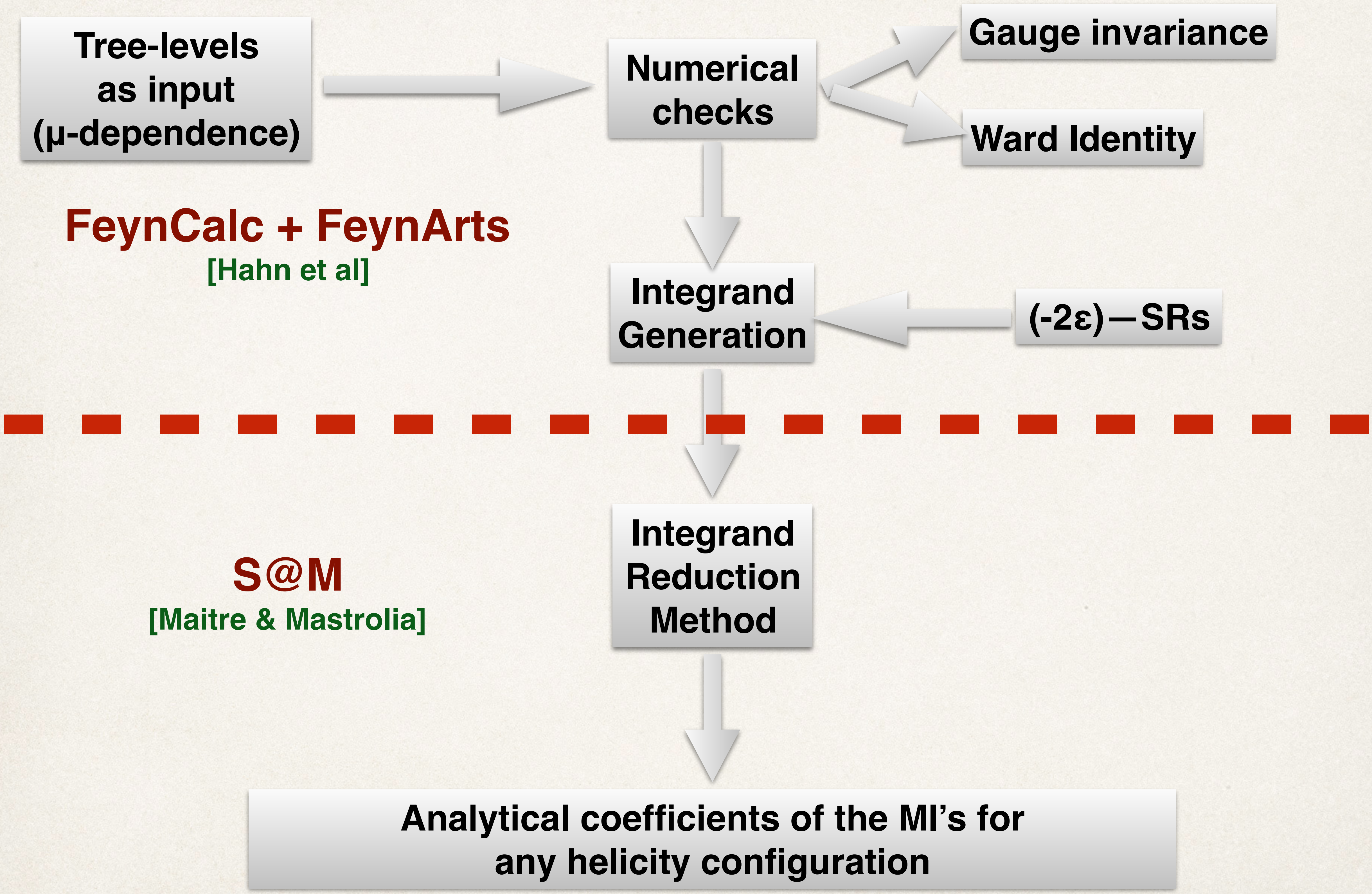
$$c_{1|2|3|4;\,4} = c^{[0]}_{1|2|3|4;\,4} + c^{[4]}_{1|2|3|4;\,4} = 2i\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}$$

Being the full one-loop amplitude

$$\begin{aligned} A_4\left(1_g^+, 2_g^+, 3_g^+, 4_g^+\right) &= c_{1|2|3|4;\,4}\, I_{1|2|3|4}[\mu^4] \\ &= -\frac{i}{48\,\pi^2}\,\frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\,, \end{aligned}$$

[Bern & Kosower (1992)]

Automation

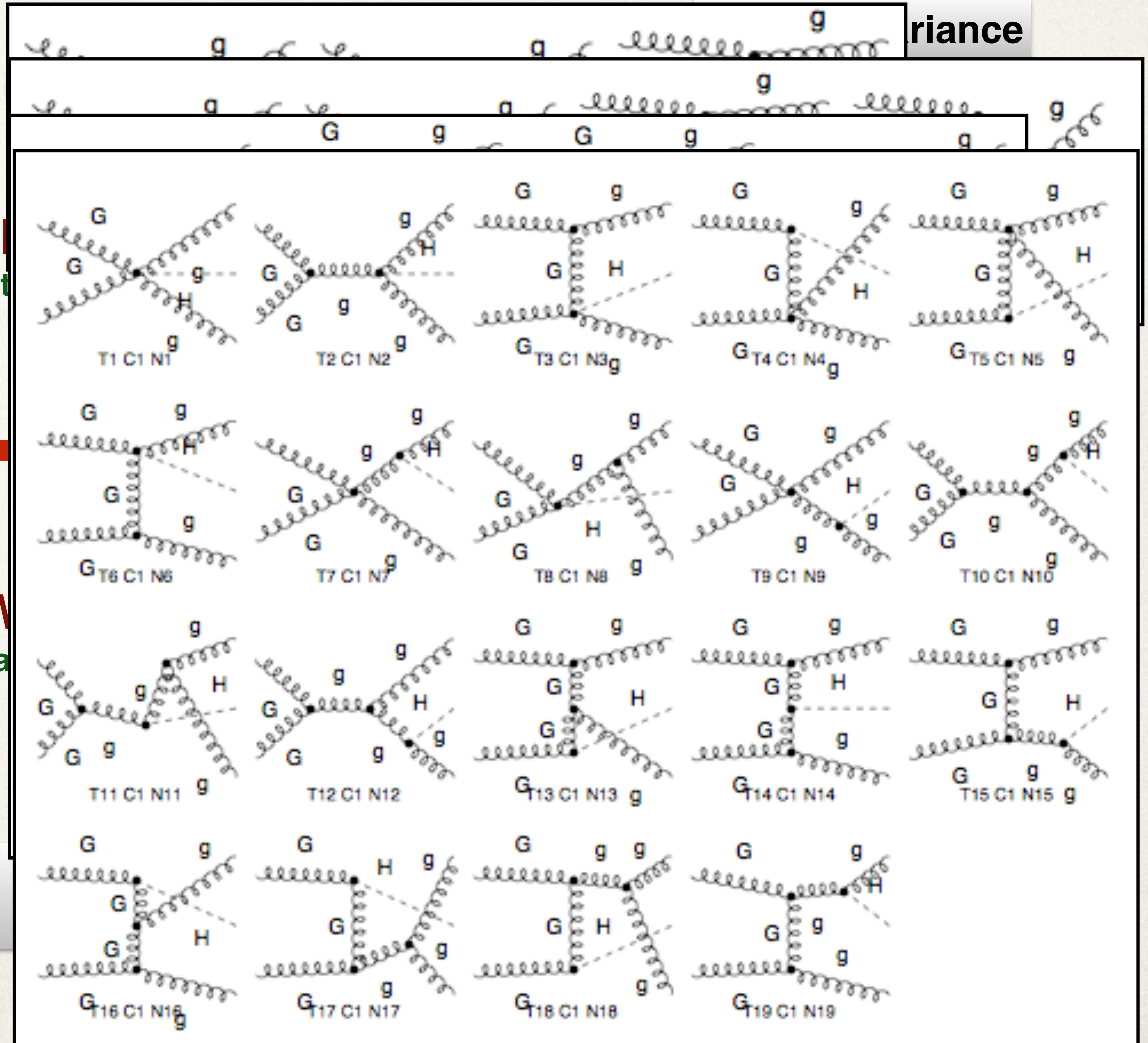


Tree-levels as input (μ-dependence)
Numerical checks
Gauge invariance
Ward Identity
FeynCalc + FeynArts
[Hahn et al]
Integrand Generation
(-2ε)—SRs
S@M
[Maitre & Mastrolia]
Integrand Reduction Method
Analytical coefficients of the MI's for any helicity configuration

Automation

Tree-levels as input (μ-dependence)

riance



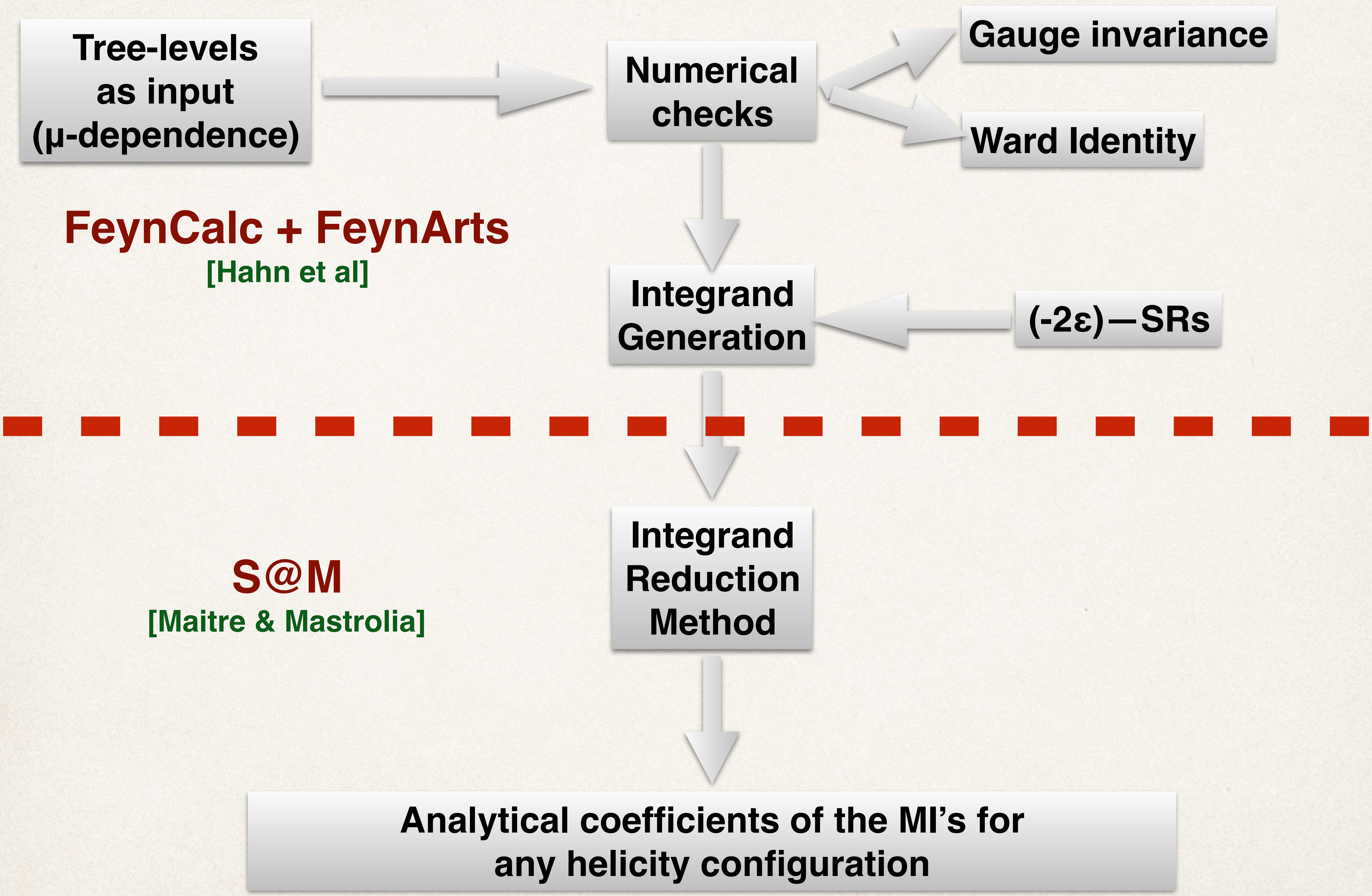
FeynCalc +

[Hahn et

S@M

[Maitre & Ma

Automation



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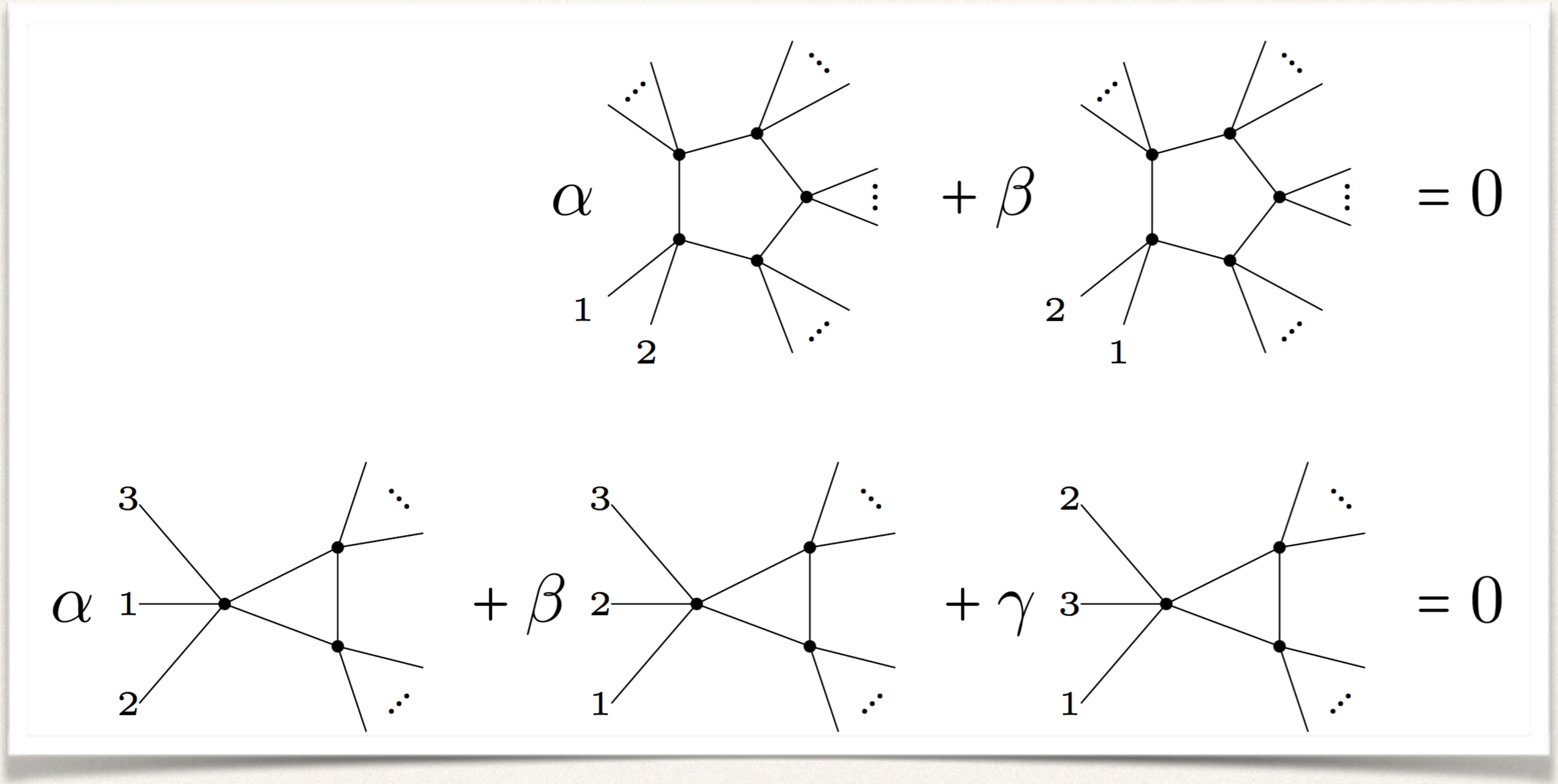
Results

- 4-gluons amplitudes **[Bern and Kosower (1992)]**
- Annihilation of quark & antiquark in two gluons **[Kunszt, Signer and Trocsanyi (1993)]**
- Higgs + 3-gluon amplitudes **[Schmidt (1997)]**
- 5-gluon amplitudes **[Njet]**
- 6-gluon amplitudes **[Njet]**
- Higgs + 4-gluon amplitudes **[Badger, Glover, Mastrolia, Williams (2009)]**
- Higgs + 5-gluon amplitudes (preliminary results) **[GoSam]**

Further simplifications from colour/kinematics duality

At integrand level,



Generalised Unitarity and C/K duality **dance** together.

Which "gauge" theories obey C-K duality

- Pure $\mathcal{N}$=0,1,2,4 super-Yang-Mills (any dimension) — Bern, Carrasco, HJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Stieberger; Feng et al. Mafra, Schlotterer, etc ('08-'11)
- Self-dual Yang-Mills theory O'Connell, Monteiro ('11)
- Heterotic string theory Stieberger, Taylor ('14)
- Yang-Mills + F^3 theory Broedel, Dixon ('12)
- QCD, super-QCD, higher-dim QCD HJ, Ochirov ('15)
- Generic matter coupled to $\mathcal{N}$= 0,1,2,4 super-Yang-Mills — Chiodaroli, Gunaydin, Roiban; HJ, Ochirov ('14)
- Spontaneously broken $\mathcal{N}$= 0,2,4 SYM Chiodaroli, Gunaydin, HJ, Roiban ('15)
- Yang-Mills + scalar ϕ^3 theory Chiodaroli, Gunaydin, HJ, Roiban ('14)
- Bi-adjoint scalar ϕ^3 theory — Bern, de Freitas, Wong ('99), Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell
- NLSM/Chiral Lagrangian Chen, Du ('13)
- *D*=3 Bagger-Lambert-Gustavsson theory (Chern-Simons-matter) Bargheer, He, McLoughlin; Huang, HJ, Lee ('12 -'13)

Colour-kinematics duality

- Both string theory and field theory frameworks suggest an intimate connection between Yang-Mills theory and gravity

  $\cdot\ \text{Gravity} \sim (\text{Yang-Mills})^2$

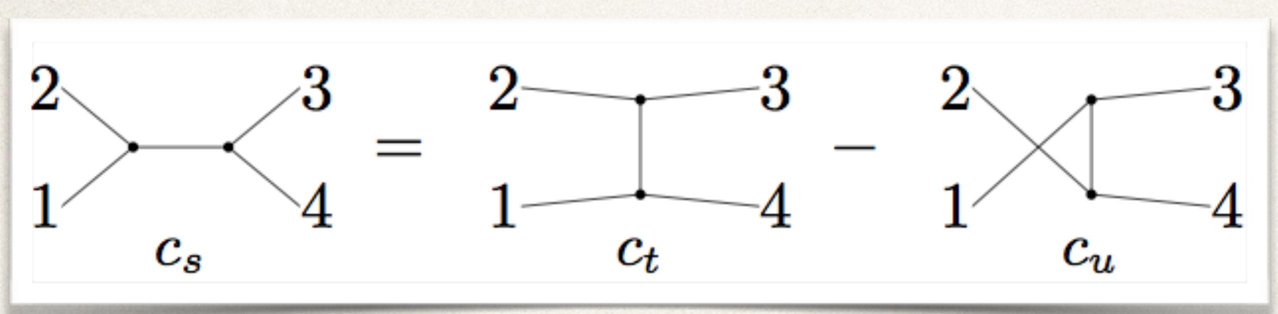
- Colour-kinematics duality provides a construction of gravity amplitudes from knowledge of Yang-Mills amplitudes **[Bern, Carrasco, Johansson (2008),(2010)]**

- In general, Yang-Mills amplitudes can be written as a sum over trivalent graphs

$$\mathcal{A}_n = g^{n-2} \sum \frac{n_i c_i}{D_i}$$

  - Colour factors $\quad c_i \sim f^{abc} f^{ced}$
  - Kinematic factors $\quad n_i \sim (\varepsilon_1 \cdot k_2)(\varepsilon_2 \cdot k_3)(\varepsilon_3 \cdot \varepsilon_4) + \ldots$

Jacobi Relation (colour)



$$c_s = c_t - c_u$$

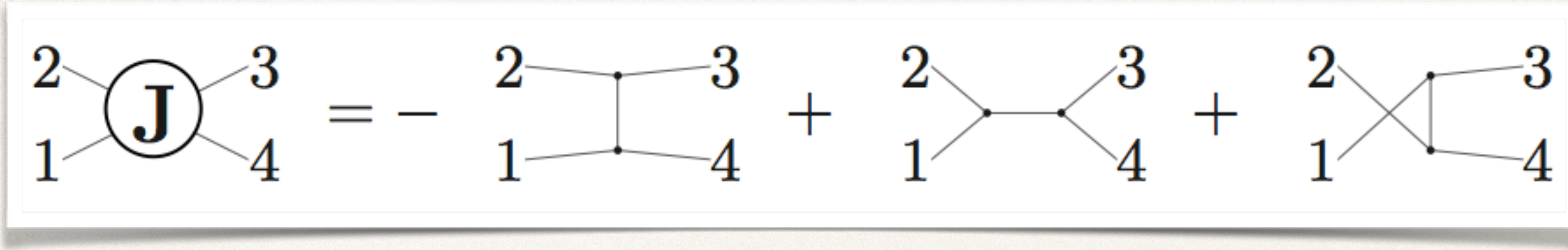
$$f^{a_1 a_2 b} f^{a_3 a_4 b} = f^{a_4 a_1 b} f^{a_2 a_3 b} - f^{a_1 a_3 b} f^{a_2 a_4 b}$$

$$f^{a_1 a_2 b} T^b = T^{a_1} T^{a_2} - T^{a_2} T^{a_1}$$

— Satisfied automatically for 4-point tree amplitudes $\quad n_s = n_t - n_u$

Off-shell Colour-kinematics duality

Consider a tensor as the Jacobi identity of numerators



Four-gluon identity $N_g^{\text{tree}} = J^{\mu_1 \cdots \mu_4} \varepsilon_{\mu_1}(p_1) \varepsilon_{\mu_2}(p_2) \varepsilon_{\mu_3}(p_3) \varepsilon_{\mu_4}(p_4),$

$$
\begin{aligned}
N_g^{\text{tree}} = \varepsilon\left(p_1\right) \cdot p_1 [& \left(\varepsilon\left(p_2\right) \cdot p_1 + 2\varepsilon\left(p_2\right) \cdot p_4\right) \varepsilon\left(p_3\right) \cdot \varepsilon\left(p_4\right) \\
& - \varepsilon\left(p_2\right) \cdot \varepsilon\left(p_4\right)\left(\varepsilon\left(p_3\right) \cdot p_1 + 2\varepsilon\left(p_3\right) \cdot p_4\right) \\
& + \varepsilon\left(p_2\right) \cdot \varepsilon\left(p_3\right)\left(\varepsilon\left(p_4\right) \cdot p_1 + 2\varepsilon\left(p_4\right) \cdot p_3\right)] \\
& + \text{cyclic permutations.}
\end{aligned}
$$

[Zhu (1980)]

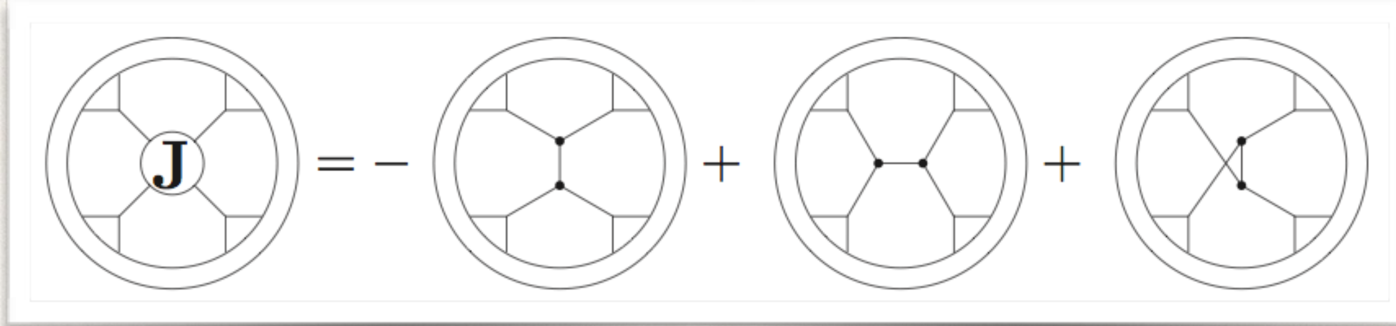
$$N_g^{\text{tree}} = 0$$

by imposing Momentum Conservation and Transversality condition.

Off-shell Colour-kinematics duality

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

At loop level



Outgoing particles in the J-block are now considered as internal:

$$u\left(p_i\right), v\left(p_i\right) \to \not{p}_i$$

$$\varepsilon^{\mu_i}\left(p_i\,;q_i\right) \to \Pi^{\mu_i\nu_i}\left(p_i\,;q_i\right)$$

with

$$\Pi^{\mu_i\nu_i}\left(p_i\,;q_i\right) = -g^{\mu_i\nu_i} + \frac{p_i^{\mu_i}q_i^{\nu_i} + p_i^{\nu_i}q_i^{\mu_i}}{p_i\cdot q_i}$$

$$p_{i,\,\mu_i}\Pi^{\mu_i\nu_i}\left(p_i\,;q_i\right) = p_i^2\frac{q_i^{\nu}}{p_i\cdot q_i}$$

$$\not{p}_i\not{p}_i = p_i^2$$

Numerator built from the J-block decomposed in terms of squared momenta

$$\left(N_{\rm g}^{\rm loop}\right)_{\alpha_1\ldots\alpha_4} = J^{\mu_1..\mu_4}\Pi_{\mu_1\alpha_1}(p_1,q_1)\,\Pi_{\mu_2\alpha_2}(p_2,q_2)\,\Pi_{\mu_3\alpha_3}(p_3,q_3)\,\Pi_{\mu_4\alpha_4}(p_4,q_4)\,,$$

$$\left(N_{\rm g}^{\rm loop}\right)_{\alpha_1\ldots\alpha_4} = \sum_{i=1}^{4} p_i^2(A_g^i)_{\alpha_1\ldots\alpha_4} + \sum_{\substack{i,j=1\\ i\neq j}}^{4} p_i^2p_j^2(C_g^{ij})_{\alpha_1\ldots\alpha_4}.$$

with

$$A_g = A_g(\{p_i\})$$

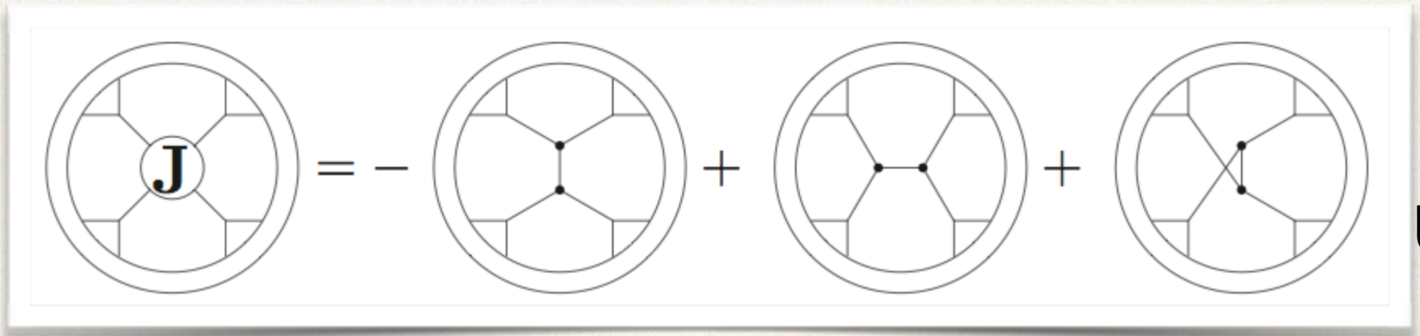
$$C_g = C_g(\{p_i\})$$

We studied the identities for QCD, in a **pure diagrammatic approach**

Off-shell Colour-kinematics duality

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

At loop level



Use of Axial gauge is **new**

Outgoing particles in the J-block are now considered as internal:

$$u(p_i), v(p_i) \to \not{p}_i$$
$$\varepsilon^{\mu_i}(p_i\,;q_i) \to \Pi^{\mu_i\nu_i}(p_i\,;q_i)$$

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$$\Pi^{\mu_i\nu_i}(p_i\,;q_i) = -g^{\mu_i\nu_i} + \frac{p_i^{\mu_i}q_i^{\nu_i} + p_i^{\nu_i}q_i^{\mu_i}}{p_i\cdot q_i}$$
$$p_{i,\mu_i}\Pi^{\mu_i\nu_i}(p_i\,;q_i) = p_i^2\frac{q_i^{\nu}}{p_i\cdot q_i}$$
$$\not{p}_i\not{p}_i = p_i^2$$

Numerator built from the J-block decomposed in terms of squared momenta

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$$\left(N_{\rm g}^{\rm loop}\right)_{\alpha_1\ldots\alpha_4} = \sum_{i=1}^{4} p_i^2 (A_g^i)_{\alpha_1\ldots\alpha_4} + \sum_{\substack{i,j=1\\ i\neq j}}^{4} p_i^2 p_j^2 (C_g^{ij})_{\alpha_1\ldots\alpha_4}.$$

with

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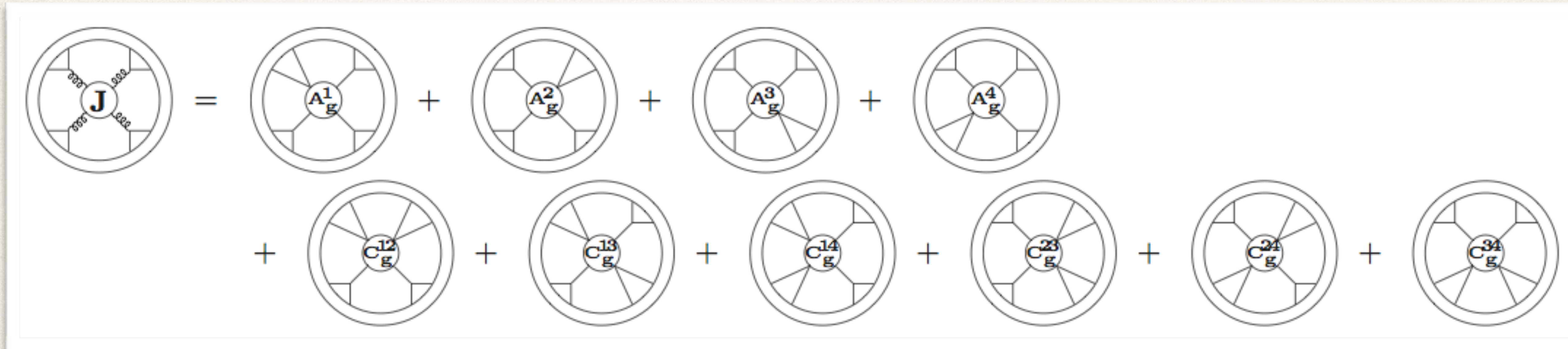
We studied the identities for QCD, in a **pure diagrammatic approach**

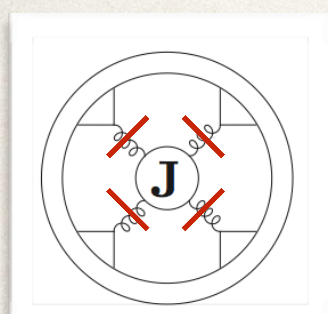
Off-shell Colour-kinematics duality

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

From the J-block we have



- Any loop diagram built from the J-block can be written as the sum of diagrams with one or two propagators less.



$= 0$

- By imposing on-shellness of the four particles

- Colour-kinematics duality is also manifest for d-dimensional regulated amplitudes —> **Novel approach within FDF**

C/K relations @ tree-level in DimReg w/in FDF

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

Consider the 4-point amplitude

$$\mathcal{A}_4(p_1,p_2,p_3,p_4) = c_1 \frac{n_1}{P_{23}^2-\mu^2} + c_2 \frac{n_2}{P_{12}^2} + c_3 \frac{n_3}{P_{24}^2-\mu^2}$$

$$-c_1+c_2+c_3=0$$

$$P_{ij}^2 = (p_i+p_j)^2$$

Solving for c2

$$\mathcal{A}_4(p_1,p_2,p_3,p_4) = c_1 K_1 + c_3 K_3$$

$$K_1 = \frac{n_1}{P_{23}^2-\mu^2} + \frac{n_2}{P_{12}^2},$$

$$K_3 = \frac{n_3}{P_{24}^2-\mu^2} - \frac{n_2}{P_{12}^2}.$$

Kinematic numerators obey Jacobi identity

$$-n_1+n_2+n_3=0.$$

with $K_1 = A(1,2,3,4)$ and $K_3 = A(2,1,3,4)$

$$A(2,1,3,4) = \frac{P_{23}^2-\mu^2}{P_{24}^2-\mu^2} A(1,2,3,4).$$

C/K relations @ tree-level in DimReg w/in FDF

[Mastrolia, Primo, Schubert, W.J.T. (2015)]

As well, for the 5-point

$$A_5(1,3,4,2,5) = \frac{-P_{12}^2 P_{45}^2 A_5(1,2,3,4,5) + (P_{14}^2 - \mu^2)(P_{24}^2 + P_{25}^2 - 2\mu^2) A_5(1,4,3,2,5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)},$$

$$A_5(1,2,4,3,5) = \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1,4,3,2,5) + P_{45}^2 (P_{12}^2 + P_{24}^2 - \mu^2) A_5(1,2,3,4,5)}{P_{35}^2 (P_{24}^2 - \mu^2)},$$

$$A_5(1,4,2,3,5) = \frac{-P_{12}^2 P_{45}^2 A_5(1,2,3,4,5) + (P_{25}^2 - \mu^2)(P_{14}^2 + P_{25}^2 - 2\mu^2) A_5(1,4,3,2,5)}{P_{35}^2 (P_{24}^2 - \mu^2)},$$

$$A_5(1,3,2,4,5) = \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1,4,3,2,5) + P_{12}^2 (P_{24}^2 + P_{45}^2 - \mu^2) A_5(1,2,3,4,5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)}.$$

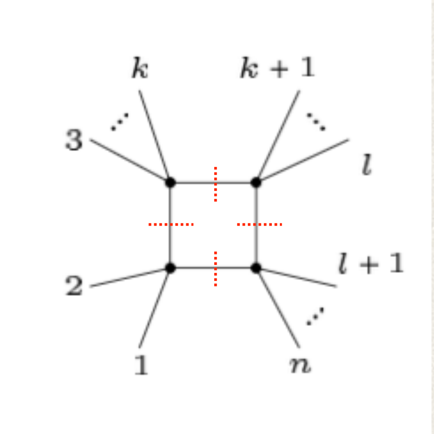
Making use of the photon decoupling identity

$$A_5(1,2,4,3,5) = \frac{(P_{14}^2 + P_{45}^2 - \mu^2) A_5(1,2,3,4,5) + (P_{14}^2 - \mu^2) A_5(1,2,3,5,4)}{(P_{24}^2 - \mu^2)}$$

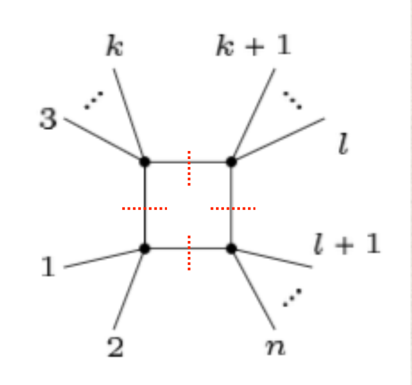
C/K relations @ 1-loop

Inspired by the generalised unitarity

$$C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = A_4^{\text{tree}}\left(-l_1^{\pm}, 1, 2, l_3^{\pm}\right) A_k^{\text{tree}}\left(-l_3^{\pm}, P_{3\cdots k}, l_{k+1}^{\pm}\right) \times A_{l-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm}, P_{k+1\ldots,l}, l_{l+1}^{\pm}\right) A_{n-l+2}^{\text{tree}}\left(-l_{l+1}^{\pm}, P_{l+1\ldots,n}, l_1^{\pm}\right)$$



$$C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n}.$$



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C/K relations @ 1-loop

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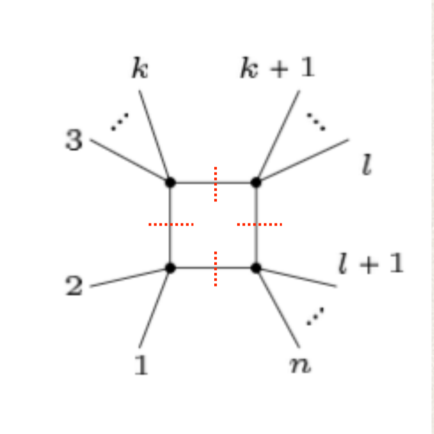
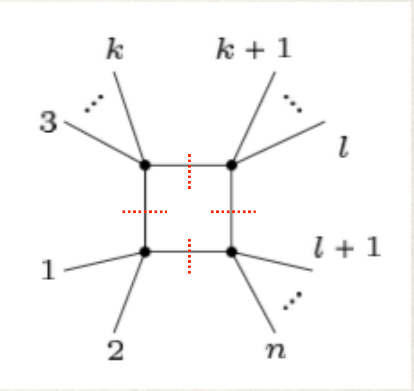
$$C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = A_4^{\text{tree}}\left(-l_1^{\pm}, 1, 2, l_3^{\pm}\right) A_k^{\text{tree}}\left(-l_3^{\pm}, P_{3\ldots k}, l_{k+1}^{\pm}\right) \times A_{l-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm}, P_{k+1\ldots,l}, l_{l+1}^{\pm}\right) A_{n-l+2}^{\text{tree}}\left(-l_{l+1}^{\pm}, P_{l+1\ldots,n}, l_1^{\pm}\right)$$



$$C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n}.$$



C/K relation

$$A(2,1,3,4) = \frac{P^2_{23} - \mu^2}{P^2_{24} - \mu^2} A(1,2,3,4).$$

— One-loop amplitudes in N=4 sYM

[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010)]

— Cut constructible part of One-loop QCD amplitudes **[Chester (2016)]**

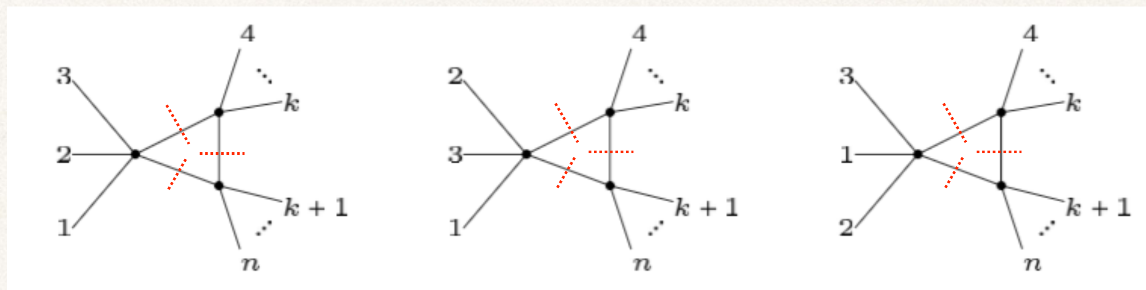
— One-loop QCD amplitudes **[Primo, W.J.T. (2016)]**

C/K relations @ 1-loop

[Primo, W.J.T. (2016)]

Same behaviour for lower topologies

$$C^{\pm}_{123|4\ldots k|(k+1)\ldots n} = A_5^{\text{tree}}\left(-l_1^{\pm},1,2,3,l_4^{\pm}\right) A_{k-1}^{\text{tree}}\left(-l_4^{\pm},P_{4\cdots k},l_{k+1}^{\pm}\right) A_{n-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm},P_{k+1\ldots,n},l_1^{\pm}\right)$$



$$C^{\pm}_{213|4\ldots k|(k+1)\ldots n} = \frac{\left(P^2_{l_4^{\pm}2}+P^2_{23}-\mu^2\right)C^{\pm}_{123|4\ldots k|(k+1)\ldots n}+\left(P^2_{l_4^{\pm}2}-\mu^2\right)C^{\pm}_{132|4\ldots k|(k+1)\ldots n}}{\left(P^2_{-l_1^{\pm}2}-\mu^2\right)}$$

due to

$$A_5(1,2,4,3,5) = \frac{(P^2_{14}+P^2_{45}-\mu^2)A_5(1,2,3,4,5)+(P^2_{14}-\mu^2)A_5(1,2,3,5,4)}{(P^2_{24}-\mu^2)}$$

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Unitarity, On-shellness & Integrand Decomposition

— Dramatic developments for **One-Loop** Amplitudes
— NLO: automating **analytic** one-loop calculations
— NN...LO
    — many legs
    — massive particles in the loops

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— An open problem: C/K duality of higher loop amplitudes.
— Scattering Amplitudes in Gauge theories still reserve a lot of surprises.

New ideas

— Exploiting the **L**oop-**T**ree **D**uality Theorem within FDF.

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Thanks