Relativistic thermal rates: a common tool in cosmology and relativistic heavy-ion collisions

Mirco Cannoni

Universidad de Huelva

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(1)

Reaction rates and rate equations:
non-relativistic and relativistic gas

Motivation and applications

(2)

Relative velocity, flux, cross section, rate
and all that… in Special Relativity

Rates in a non-relativistic classical gas

Reaction rate depends only on the relative motion of colliding pairs

$$R = n_1 n_2 \sigma v_r \qquad v_r = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$$

The distribution of the relative velocity is a Maxwell distribution with the reduced mass μ in place of m and v_r in place of v:

$$P(v_r) = \sqrt{\frac{2}{\pi}} x^{3/2} v_r^2 \, e^{-\frac{x}{4} v_r^2} \qquad \int_0^\infty dv_r P(v_r) = 1 \qquad \text{p.d.f}$$

Knowing the p.d.f. we can calculate average quantities which depend on temperature

Averaged relative velocity $\langle v_r \rangle = \int_0^\infty dv P(v_r) v_r = \sqrt{\frac{8T}{\pi\mu}}$ $\qquad x=m/T$

Averaged cross section $\langle \sigma_{nr} v_r \rangle_{nr} = \int_0^\infty dv_r P(v_r) \sigma_{nr} v_r$

Averaged reaction rate $\langle \mathcal{R} \rangle = \frac{n_1 n_2}{1+\delta_{12}} \int dv_r P(v_r) \sigma v_r = \frac{n_1 n_2}{1+\delta_{12}} \langle \sigma v_r \rangle$

Chemical reactions: rate equations

Consider a gas which is mixture of different species that can react and the volume expand

$$\nu_A A + \nu_B B \rightleftarrows \nu_C C + \nu_D D$$

For a 2-->2 reaction with elementary particles the value of stoichiometric coefficients is 1. If A=B then $\nu_A = \nu_B$, say 2A-->C+D, then $\nu_A = 2$

The number density or concentration of each specie change in time, rate equation:

$$\frac{1}{\nu_A}\frac{1}{V}\frac{d(n_A V)}{dt} = \mathcal{R}_f - \mathcal{R}_b$$

$$\mathcal{R}_f = k_f\, n_A^{|\nu_A|} n_B^{|\nu_B|} \qquad \mathcal{R}_b = k_b\, n_C^{|\nu_C|} n_D^{|\nu_D|} \qquad k = \frac{1}{1+\delta_{ij}}\langle \sigma v_{\rm rel}\rangle_{ij}$$

In chemical equilibrium, detailed balance implies $\mathcal{R}_f = \mathcal{R}_b$

$$K_{\rm eq} = \frac{n_{A,eq}^{|\nu_A|} n_{B,eq}^{|\nu_B|}}{n_{C,eq}^{|\nu_C|} n_{D,eq}^{|\nu_D|}} = \frac{1+\delta_{AB}}{1+\delta_{CD}}\frac{\langle \sigma v_{\rm rel}\rangle_{CD}}{\langle \sigma v_{\rm rel}\rangle_{AB}}$$

The rate equation then is $\frac{1}{\nu_A}\frac{1}{V}\frac{d(n_A V)}{dt} = \frac{\langle \sigma v_{\rm rel}\rangle_{AB}}{1+\delta_{AB}}(n_A^{|\nu_A|} n_B^{|\nu_B|} - K_{\rm eq} n_C^{|\nu_A|} n_D^{|\nu_B|})$

The microscopic foundation reside in kinetic theory, hydrodynamics, the Boltzmann Equation and is valid in the relativistic case as well
The macroscopic version is non-equilibrium thermodynamic **M.C. EPJ C75 (2015) [1407.4108]**

Applications of Rate equations

Early Universe: Big Bang

(how matter emerged from primordial plasma)

Neutron/proton ratio and nucleosynthesis	**Alpher, Follin, Herman (1953) Zeldovich (1965)** **Peebles, A. J. (1966) Wagoner, Fowler, Hoyle (1967)**
Relics (neutrinos, dark matter, WIMPS)	**Zeldovich. Okun, Pikelner (1965) Lee, Weinberg (1977)** **Visotsky, Dolgov, Zeldovich, (1977)**
Baryon asymmetry generation	**Kolb, Wolfram, (1980)**

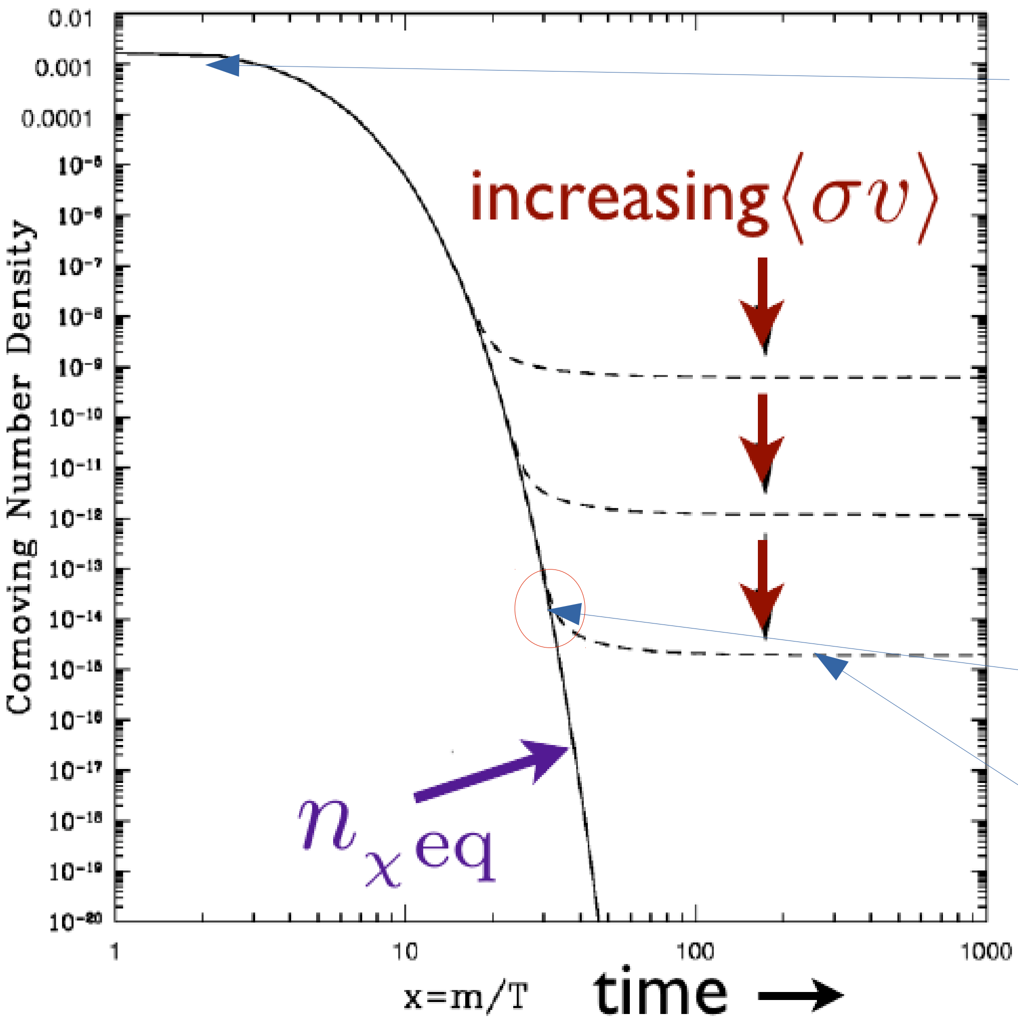
Relativistic heavy-ion collisions: Little Bang

(how matter emerges from the QGPlasma)

Quarkochemistry-hadrochemistry	**Montvay, Zimanyi, Biro, Mishustin (1980)**
Hydrodynamical approach	**Kajantie, McLerran, Bjorken, Kapusta (1983)** **Koch, Muller, Rafelski (1986)**
Thermostatical model	**Braun-Munzinger et al (1996)**

Decoupling of relics in the Early Universe

$$\frac{1}{a^3}\frac{\mathrm{d}(na^3)}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = \langle \sigma_{\mathrm{ann}} v_{\mathrm{r}} \rangle (n_0^2 - n^2) \qquad H = (1/a)\mathrm{d}a/\mathrm{d}t$$



Thermal equilibrium in the early Universe plasma

$$n_{eq} \simeq \left(\frac{m_\chi T}{2\pi}\right)^{3/2} e^{-m_\chi/T}$$

Chemical potentials are zero → chemical eq.

Cooling and expansion of the Universe:
the number density decreases

When the annihilation rate per particle falls below the expansion rate, WIMP “freeze out”

$$\Gamma = n\langle \sigma v \rangle \leq H$$

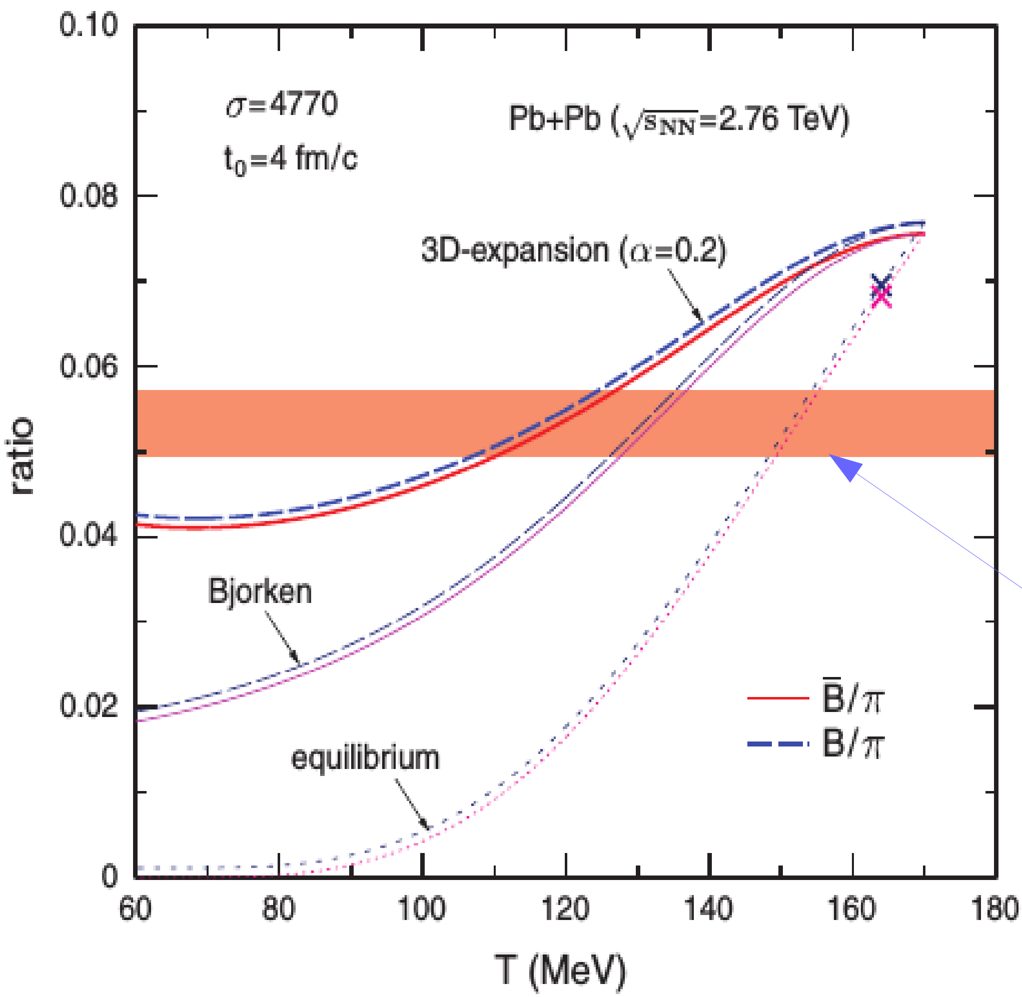
leaving a constant relic density

Baryon yield relativistic heavy-ion collisions

$$\frac{1}{V}\frac{d(n_{\overline{B}}V)}{dt} = \frac{1}{V}\frac{d(n_B V)}{dt} = \langle\sigma_{\rm ann}v_{\rm rel}\rangle\left(n_B^{\rm (eq)}n_{\overline{B}}^{\rm (eq)} - n_B n_{\overline{B}}\right)$$

Satarov, Mishutin, Greiner 2014

$$\frac{B}{\pi} = \frac{N}{\pi} + \frac{\Lambda+\Sigma}{\pi} + \frac{\Xi}{\pi} + \frac{\Omega^-}{\pi}$$



Iso-entropic expansion of the fireball

$$s(T,\mu)V(t) = s(T_0,\mu_0)V(t_0)$$

+

Volume parametrization from hydrodynamic simulations

$$\frac{V}{V_0} = \frac{t}{t_0}\left(\frac{1+\alpha\, t/t_0}{1+\alpha}\right)^2$$

Gives the temperature-time relation

ALICE @LHC experimental yield

Lead+Lead collisions @2.76 TeV

(2)

Relative velocity and rates in Special Relativity

(Why textbooks are conceptually wrong)

Relative velocity and cross section

Quantum field theory and particle physics textbook:

$$\sigma = \frac{\mathcal{R}}{F} = \frac{1}{F}\int \overline{|\mathcal{M}|^2} \mathrm{d}\Phi(f) \qquad d\Phi(f) = (2\pi)^4 \delta(\sum p_i - \sum p_f) \prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f}$$

Define the flux as in the nonrelativistic case $F = n_1 n_2 |\mathbf{v}_1 - \mathbf{v}_2|$ Lab or cm frame

Rewrite the relative velocity using v=p/E $F = n_1 n_2 |\boldsymbol{v}_1 - \boldsymbol{v}_2| = n_1 n_2 \left| \frac{E_2 \boldsymbol{p}_1 - E_1 \boldsymbol{p}_2}{E_1 E_2} \right|$

After rearranging, $F = n_1 n_2 \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}, \ \text{if} \ \boldsymbol{v}_1 \parallel \boldsymbol{v}_2$

Normalize fields and one particle states such that number density per unit volume is

$$n = 2E$$

Finally we get the invariant expression $F = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$

1^{st} problem: The collinearity paradox

Many authors insist on the fact $F = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$ (1)

is valid only for collinear velocities, because the cross section must be invariant for boosts along collision axis

Bjorken and Drell, Schweber, Brown, Kaku, Peskin and Shroeder, Zee, Tully, Halzen and Martin.......

But (1) is Lorentz invariant, this a sort of “negation of the evidence”

Actually, using $(\boldsymbol{a} \times \boldsymbol{b})^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$ it is easy to show that

$$n_1 n_2 \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} = n_1 n_2 \sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}$$

Møller (1945)

Pauli 1933?
Landau Lifschitz Vol2

Thus formula (1) is valid in any frame and for any direction of the velocities

At the quantum level is meaningless to think the cross section as an area transverse area to the collision direction

2^{nd} problem: Who is the relative velocity?

- For many authors the relative velocity in Special Relativity is

$$\frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2} = \sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2} \equiv \bar{v}$$

- For other authors the non-relativistic expression remain valid also SR

$$v_r = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$$

Two problems:

1) Both expressions are not Lorentz invariant

2) For example in the CMF their value is $2v_*$, thus for $v_* > 0.5$ c, they are superluminal

These expressions are not physical in Special Relativity,
they cannot be the relative velocity

3rd problem: Is the flux formulated correctly?

In the rest frame of particle 1, the relative velocity is the velocity of particle 2 as seen in that frame

$$v_{\rm rel} = |\boldsymbol{v}_2| = \frac{|\boldsymbol{p}_2|}{E_2} = \frac{\sqrt{E_2^2 - m_2^2}}{E_2}$$

Use $E_2 = p_1 \cdot p_2 / m_1$ to find

$$v_{\rm rel} = |\boldsymbol{v}_2| = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2}$$

This expression is invariant and never superluminal, much better candidate.

But $n_1 n_2 = 4E_1 E_2$ is not invariant, multiplying with $v_{\rm rel}$ we get a non invariant flux!

Densities are just the time component of a 4-vector,
their product cannot be a Lorentz scalar

Relativistic relative velocity (1)

In non-relativistic physics the relative velocity: $v_r = |\boldsymbol{v}_1 - \boldsymbol{v}_2|$

1) Is invariant under Galileo transformations $\boldsymbol{v}_i' = \boldsymbol{v}_i \pm \boldsymbol{V}$

2) The velocities $\boldsymbol{v} = (v_1, v_2, v_3)$ are points of the velocity space $\mathcal{V}_{NR}$

The relative velocity is the Euclidean distance in this space $v_r = d_{\mathcal{V}_{NR}}$

In Special relativity

1) The relative velocity must be invariant under Lorentz transformations

2) The points of the velocity space $\boldsymbol{v} \in \mathcal{V}_{SR}$ are such that $v_1^2 + v_2^2 + v_3^2 \leq c^2$

This space is not Euclidean! It is the Lobachesky- Bolyai hyperbolic space with negative constant curvature equal to the velocity of light.

V. Fock, The theory of Space Time & Gravitation 1955

The distance is $d_{\mathcal{V}_{SR}} = \frac{1}{2} \ln \left(\frac{1 + v_{\rm rel}}{1 - v_{\rm rel}} \right) = \tanh^{-1}(v_{\rm rel})$ or $v_{\rm rel} = \tanh(d_{\mathcal{V}_{SR}})$

3) Must be equal to c for two massless particles or one massive and a massless particle

4) Recover the known expression in the non-relativistic limit c-->∞

Relativistic relative velocity (2)

V. Fock, The theory of Space Time & Gravitation

The expression that satisfy this property is

$$v_{\rm rel} = \frac{\sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - \frac{(\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}{c^2}}}{1 - \frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{c^2}}$$

Basically, it is the Einstein-Poincare' rule for the "composition'' of velocties

The Lorentz factor $\gamma_{\rm r} = \frac{1}{\sqrt{1 - v_{\rm rel}^2}}$ reads $\gamma_{\rm r} = \gamma_1 \gamma_2 (1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2)$

-Using scalar product of 4-vectors we find various equivalent forms

$$u_i = \gamma(1, \boldsymbol{v}_i) \quad p_i = (E_i, \boldsymbol{p}_i) \quad J = (n, n\boldsymbol{v}_i) = n^0(\gamma, \gamma\boldsymbol{v}) = n^0 u$$

$$v_{\rm rel} = \frac{\sqrt{(u_1 \cdot u_2)^2 - 1}}{u_1 \cdot u_2} = \frac{\sqrt{(J_1 \cdot J_2)^2 - (n_1^0)^2 (n_2^0)^2}}{J_1 \cdot J_2} = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{s - (m_1^2 + m_2^2)}$$

$$= \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2}$$

$$\gamma_{\rm r} = u_1 \cdot u_2 = \frac{p_1 \cdot p_2}{m_1 m_2} = \frac{J_1 \cdot J_2}{n_1^0 n_2^0} = \frac{s - (m_1^2 + m_2^2)}{2 m_1 m_2}$$

The forgotten formula: cannot find in any QFT or Particle Physics book (or article)

The correct formulation of flux (1)

Non relativistic: density in the rest frame, the proper density, is the same as in the moving frame, there is no volume contraction

$$F_{nr} = n_1 n_2 v_r = n_1^0 n_2^0 v_r$$

Relativistic: density is time component of the 4-current, let's take the scalar product of two currents

$$J_1 \cdot J_2 = n_1 n_2 (1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2) = n_1^0 n_2^0 \gamma_1 \gamma_2 (1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2) = n_1^0 n_2^0 \gamma_\mathrm{r}$$

The natural invariant definition of the flux is $\boxed{F = (J_1 \cdot J_2) v_{\mathrm{rel}}}$

$$F = n_1 n_2 \cancel{(1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2)} \frac{\sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}}{\cancel{1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2}}$$

The cancellation explains why textbook get the correct result with wrong reasoning!

M.C. [1506.07475]

The correct formulation of flux (2)

Write in another way: $F = n_1 n_2 k_{\rm r} v_{\rm rel}$ where I define

$$k_{\rm r} = \frac{\gamma_{\rm r}}{\gamma_1\gamma_2} = 1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \frac{p_1 \cdot p_2}{E_1 E_2}$$

Flux correlation factor

The flux correlation factor is not Lorentz invariant and takes values between 0 and 2

This explains why the Moller velocity is superluminal and not invariant!

$$k_{\rm r} v_{\rm rel} = \sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}$$

Much better to not consider it a velocity at all!

It is a factor that when multiplied by the density n_1 and n_2 gives the invariant flux

$$F = n_1 n_2 \bar{F}$$

Møller flux factor

Flux and luminosity (2)

Accelerator physicist prefer to talk about luminosity instead of flux $N_f = \sigma \mathcal{L}_{\rm int}$

Integrated luminosity $\mathcal{L}_{\rm int} = \int dt d^3\boldsymbol{x} n_1(\boldsymbol{x}_1, t) n_2(\boldsymbol{x}_2, t) \bar{F}$

Bunched beams of ultra-relativistic particles colliding head-on with Gaussian shaped spatial distributions colliding along the z axis

$$n_i(\boldsymbol{x}, t) = N_i \rho(\boldsymbol{x}, t) \quad \rho(\boldsymbol{x}, t) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z \pm vt}{2\sigma_z^2}\right)}$$

Performing the Gaussian integrations results in the well known formula, see PDG

$$\mathcal{L}_{\rm int} = 2\frac{N_1 N_2}{8\pi\sigma_x\sigma_y}$$

$$\bar{F} = k_{\rm r} v_{\rm rel} = 2$$

$$k_{\rm r} = 2. \qquad v_{\rm rel} \sim 1$$

The relative velocity is 1
But the flux correlation factor is 2
The Møller factor is 2

Misuse of the Moller "velocity"

Total cross section for $e^+ e^- \rightarrow \gamma\gamma$ $\quad y = \frac{s}{4m^2}$

For example L&L Vol4

$$\sigma(s) = \frac{\pi r_e^2}{2y^2(y-1)}\left[(y^2+y-\frac{1}{2})\ln\frac{\sqrt{y}+\sqrt{y-1}}{\sqrt{y}-\sqrt{y-1}} - (y+1)\sqrt{y}\sqrt{y-1}\right]$$

Use the relation between v_{rel} and the Mandelstam variable: $\quad s = 2m^2(1+\gamma_r)$

The cross section then reads

$$\sigma(\gamma_r) = \frac{\pi r_e^2}{1+\gamma_r}\left[\frac{\gamma_r^2+4\gamma_r+1}{\gamma_r^2-1}\ln(\gamma_r+\sqrt{\gamma_r^2-1}) - \frac{\gamma_r+3}{\sqrt{\gamma_r^2-1}}\right]$$

(2) is valid in any frame as well!

- Take the electron at rest, LAB frame: $v_{\rm rel} = v_+ \quad \gamma_r = \gamma_+ \quad$ σ has the same form as (2)

-In the CMF instead: $v_{\rm rel} = \frac{2v_*}{1+v_*^2} \quad \gamma_r = \frac{1+v_*}{1-v_*} \quad \sigma(v_*) = \frac{1-v_*^2}{4v_*}\left[\frac{3-v_*^4}{v_*}\ln\frac{1+v_*}{1-v_*} - 2(2-v_*^2)\right]$

But if you use $\bar{v}_* = 2v_*$ you get a wrong formula with a wrong non-relativistic expansion

-This misunderstanding (and wrong formulas) is found in many papers in dark matter literature where the relation $s_* = \frac{4m^2}{1-\frac{\bar{v}_*^2}{4}}$ is used instead of $s = 2m^2(1+\gamma_r)$

M.C. [1506.07475]

From beams to the Relativistic gas (1)

One particle phase space distribution: $N = \int d^3\boldsymbol{x} d^3\boldsymbol{p} f(\boldsymbol{x},\boldsymbol{p})$

Maxwell-Boltzmann-Juttner distribution: $f = e^{\frac{\mu}{T}} e^{-\frac{p\cdot u}{T}}$

Co-moving or local rest frame: $u = (1, \mathbf{0})$

$$f = \lambda f_0 \quad f_0 = e^{-\frac{E}{T}} \quad E = \sqrt{\boldsymbol{p}^2 + m^2}$$

The average number density at zero chemical potential μ=0:

$$n_0 = \frac{g}{(2\pi)^3} \int d^3\boldsymbol{p} f_0 = \frac{g}{(2\pi)^3} 4\pi m^2 T K_2(x)$$

Momentum distribution: $\int d^3\boldsymbol{p} f_{0,p}(\boldsymbol{p}) = 1$

$$f_{0,p}(\boldsymbol{p}) = \frac{1}{4\pi m^2 T K_2(x)} e^{-\sqrt{\boldsymbol{p}^2+m^2}/T}$$

The 4-current J is replaced by the 4-number flow:

$$N_0^\mu = \frac{g}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{E} p^\mu f_0 = n_0 \int \frac{d^3\boldsymbol{p}}{E} p^\mu f_{0,p}$$

From beams to the Relativistic gas (2)

Ideal classical gas: no shear or bulk viscosity, no heat transport, only reactions

Take the scalar product of two 4-flows

$$\hat{\mathcal{C}} = \mathcal{N}_{0,1} \cdot \mathcal{N}_{0,2} = n_{0,1} n_{0,2} \int \frac{d^3\boldsymbol{p}_1}{E_1} \frac{d^3\boldsymbol{p}_2}{E_2} p_1 \cdot p_2 f_{0,p}(\boldsymbol{p}_1) f_{0,p}(\boldsymbol{p}_2)$$

$$= n_{0,1} n_{0,2} \langle \frac{p_1 \cdot p_2}{E_1 E_2} \rangle = n_{0,1} n_{0,2} \langle k_{\mathrm{r}} \rangle = \langle J_1 \cdot J_2 \rangle = n_{0,1} n_{0,2}$$

-The flux correlation factor appears automatically
-Its average value is 1
-This average corresponds to integral of the p.d.f. of v_{rel}

$$\langle k_{\mathrm{r}} \rangle = \int_0^1 dv_{\mathrm{rel}} \mathcal{P}(v_{\mathrm{rel}}) = 1$$

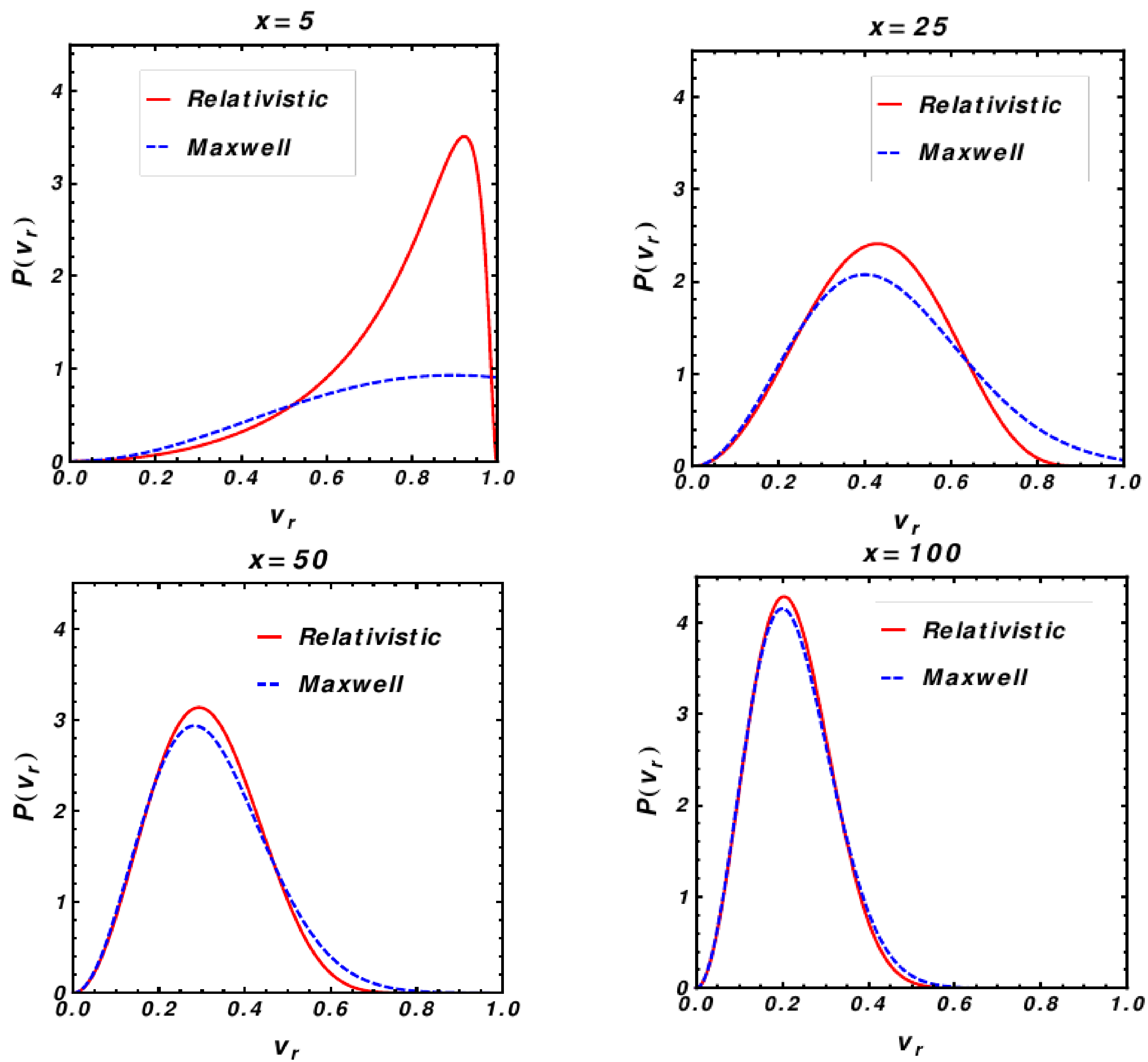
$$\mathcal{P}(v_{\mathrm{rel}}) = \frac{X}{\sqrt{2} \prod_i K_2(x_i)} \frac{\gamma_{\mathrm{r}}^3 (\gamma_{\mathrm{r}}^2 - 1)}{\sqrt{\gamma_{\mathrm{r}} + \varrho}} K_1(\sqrt{2} X \sqrt{\gamma_{\mathrm{r}} + \varrho}) \qquad X = \sqrt{x_1 x_2} \qquad \varrho = \frac{x_1^2 + x_2^2}{2 x_1 x_2}$$

M.C. Phys. Rev. D89(2014) [1311.4494] [1311.4508]

Allows the calculate average values in the same way as the non-relativistic gas

$$\hat{\mathcal{C}} v_{\mathrm{rel}} = n_{0,1} n_{0,2} \langle v_{\mathrm{rel}} \rangle = \langle F \rangle \qquad \hat{\mathcal{C}} \sigma v_{\mathrm{rel}} = n_{0,1} n_{0,2} \langle \sigma v_{\mathrm{rel}} \rangle = \langle \mathcal{R} \rangle$$

Relativistic VS Maxwell: m1 = m2



Average relative velocity

$$\langle v_{\rm rel}\rangle = \int_0^1 dv_{\rm rel}\mathcal{P}(v_{\rm rel})\ v_{\rm rel}$$

$$\langle v_{\rm rel}\rangle = \frac{2}{\alpha}\frac{(1+\varrho)^2 K_3(\alpha) - (\varrho^2-1)K_1(\alpha)}{K_2(x_1)K_2(x_2)}$$

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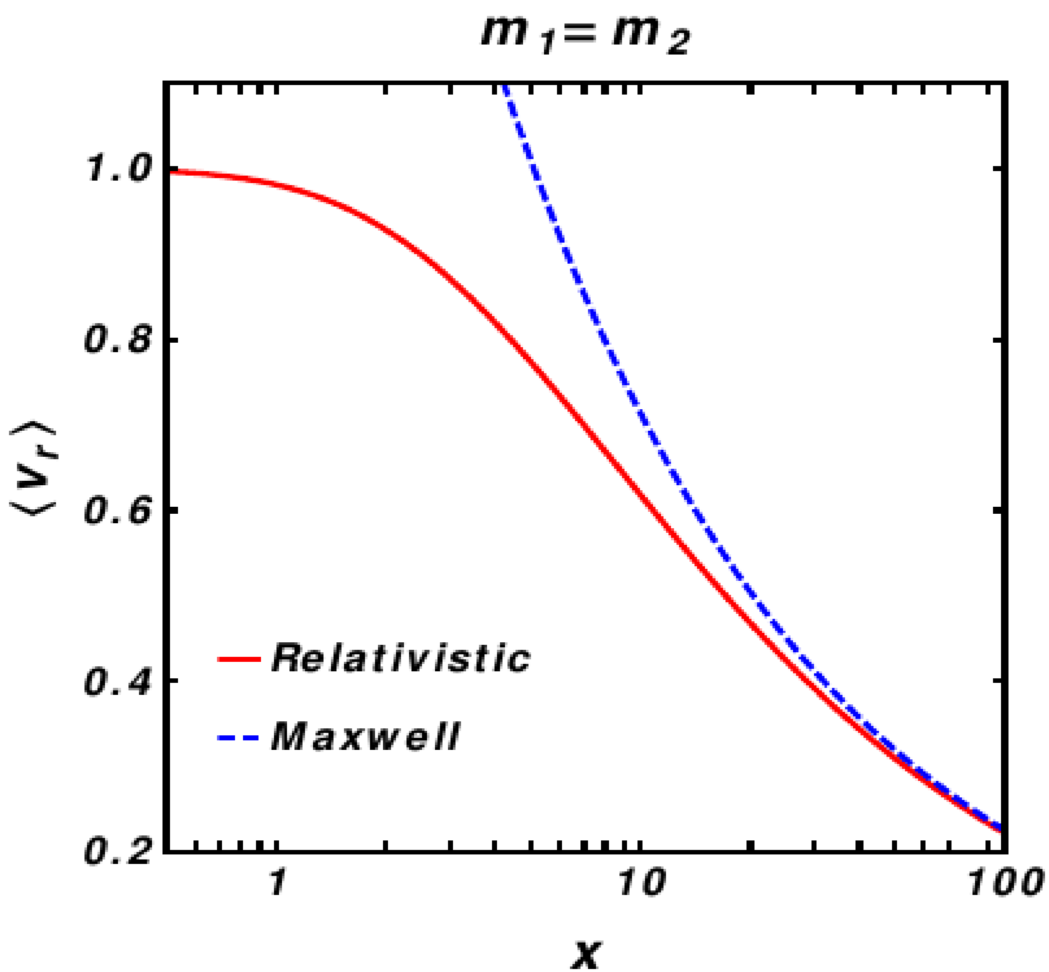
$$\alpha = x_1 + x_2$$

$$\langle v_{\rm rel}\rangle = \frac{4}{x}\frac{K_3(2x)}{K_2^2(x)}$$



Relativistic $\langle \sigma v_{rel} \rangle$

$$\langle \sigma v_{\rm rel}\rangle = \int_0^1 dv_{\rm rel}\mathcal{P}(v_{\rm rel})\sigma v_{\rm rel}$$

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M.C. [1506.07475]

$$\langle \sigma v_{\rm rel}\rangle = \frac{X}{\sqrt{2}\prod_i K_2(x_i)}\int_1^\infty d\gamma_{\rm r}\frac{\gamma_{\rm r}^2-1}{\sqrt{\gamma_{\rm r}+\varrho}}K_1(\sqrt{2}X\sqrt{\gamma_{\rm r}+\varrho})\sigma$$

With T1=T2=T, M=m1+m2 and using the relation between s and γ_r becomes

$$\langle \sigma v_{\rm rel}\rangle = \frac{1}{8T\prod_i m_i^2 K_2(x_i)}\int_{M^2}^\infty ds\frac{\lambda(s,m_1^2,m_2^2)}{\sqrt{s}}K_1(\frac{\sqrt{s}}{T})\sigma$$

Discovered many times …..

Weaver (1976)   Atrophysical plasmas

Zimanyi, Biro (1980) Relativistic heavy-ion collisions

Claudson, Hall, Hintchincliff (1982) Baryogenesis

Gondolo, Gelmini (1991) Dark matter-->A lot of confusion with Moller velocity, lab frame...

M.C. (2014) using the correct formulation of relative velocity