Conformal Dilaton Gravity: Classical Noninvariance Begets Quantum Invariance

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EXCELENCIA SEVERO OCHOA

Introduction

- When doing a conformal transformation in the metric

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}(x)$$

the Einstein-Hilbert action transforms as

$$\int d^n x \sqrt{|g|} \left[\Omega^{n-2} R + (n-1)(n-2)\Omega^{n-4} (\nabla\Omega)^2\right]$$

- Conformal Dilaton Gravity (CDG) is defined by promoting the Weyl parameter to a Stückelberg field through

$$\Omega = \frac{1}{M_p} \left(\frac{n-2}{8(n-1)}\right)^{\frac{1}{n-2}} \phi^{\frac{2}{n-2}}$$

Then the action reads

$$S_{CDG} = \int d^n x \sqrt{|g|} \left(-\frac{n-2}{8(n-1)} R\phi^2 - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi\right)$$

- Classically CDG reduces to General Relativity in the gauge

$$\phi = \sqrt{\frac{8(n-1)}{n-2}} M_p^{\frac{n-2}{2}}$$

- and to Unimodular Gravity in the gauge

$$\phi + 2^{\frac{3}{2}} M_p^{\frac{n-2}{2}} \sqrt{\frac{n-1}{n-2}} g^{\frac{2-n}{4n}} = 0$$

We can even extend CDG with a self interaction for the scalar field

$$S_{\mathrm{CDG}} = -\int d^4x \sqrt{|g|} \left(-\frac{1}{12} R\phi^2 - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \lambda \phi^4 \right)$$

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We want to compute loop corrections, so we expand the action using the background field method

$$\phi = \bar{\phi} + \eta$$
$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

There is a four-dimensional Ward identity stemming from conformal invariance

$$\mathscr{D}W[\bar{g},\bar{\phi}] \equiv \left(-2\bar{g}_{\mu\nu}\frac{\delta}{\delta\bar{g}_{\mu\nu}} + \bar{\phi}\frac{\delta}{\delta\bar{\phi}}\right)W[\bar{g},\bar{\phi}] = 0$$

where the free energy $W[\bar{g},\bar{\phi}]$ is defined as usual

$$W[\bar{g},\bar{\phi}] \equiv -\log\, Z[\bar{g},\bar{\phi}]$$

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We say that there is a conformal anomaly in the theory whenever this Ward identity is broken.

When computing the effective action to the one loop level the counterterm reads

$$\Gamma_\infty = \frac{1}{n-4}\frac{1}{16\pi^2}\frac{53}{45}\int \, d(vol)\, E_4$$

which is exactly the same that in GR and vanishes for manifolds with vanishing Euler characteristic. $\Rightarrow$ No conformal anomaly in CDG at one loop level.

In the case of the self-interaction the result is

$$a_4 = \frac{53}{45}W_4 - \frac{4568}{15}\lambda^2\phi^4$$

which is the conformal transformation of the Christensen and Duff result

$$\phi \to \sqrt{12}M_p,\ \Lambda = 6\lambda\phi^2 \Rightarrow a_4^{\mathrm{GR}} = \frac{53}{45}W_4 - \frac{1142}{135}\lambda^2\phi^4$$

S. M. Christensen and M. J. Duff, "Quantizing Gravity with a Cosmological Constant", Nucl. Phys. B **170** (1980) 480.

Based on the previous result we shall assume that the pole part at two loops is given by performing a Weyl transformation from the Goroff and Sagnotti counterterm

$$\Gamma_{\infty}^{(GS)}[\bar{G}_{\mu\nu}] = \frac{1}{n-4}\frac{1}{(4\pi)^4 M_p^2}\frac{209}{2880}\int d^4x\sqrt{|\bar{G}|}\, W_6^{(4)}[\bar{G}_{\mu\nu}],$$

M. H. Goroff and A. Sagnotti, "Quantum Gravity At Two Loops," Phys. Lett. B **160** (1985) 81. doi:10.1016/0370-2693(85)91470-4

where

$$W_6^{(4)} \equiv W^{(4)\,\alpha_1\alpha_2\alpha_3\alpha_4}\;W^{(4)}_{a_3\alpha_4\alpha_5\alpha_6}\;W^{(4)\,a_5\alpha_6}{}_{\alpha_1\alpha_2}.$$

By applying a Weyl transformation

$$\bar{G}_{\mu\nu} = \frac{1}{12\,M_p^2}\,\bar{\phi}^2\,\bar{g}_{\mu\nu}$$

one obtains

$$\Gamma_{\infty}^{(GS)}[\bar{g}_{\mu\nu},\bar{\phi}] = \frac{1}{n-4}\,\frac{12}{(4\pi)^4}\,\frac{209}{2880}\int d^4x\sqrt{|\bar{g}|}\,\frac{1}{\bar{\phi}^2}\,W_6^{(4)}[\bar{g}_{\mu\nu}],$$

and the Ward identity no longer holds

$$\left[2\,\bar{g}^{\mu\nu}\,\frac{\delta}{\delta\bar{g}^{\mu\nu}} + \bar{\phi}\,\frac{\delta}{\delta\bar{\phi}}\right]\Gamma_0[\bar{g}_{\mu\nu},\bar{\phi}] \neq 0$$

where $\Gamma_0[\bar{g}_{\mu\nu},\bar{\phi}]$ is the renormalized action obtained by minimal substraction.

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If we generalized the GS counterterm to n dimensions

$$\Gamma_\infty[\bar{g}_{\mu\nu},\bar{\phi}] = \frac{1}{n-4}\,\widetilde{W}[\bar{g}_{\mu\nu},\bar{\phi}],$$

where

$$\widetilde{W}[\bar{g}_{\mu\nu},\bar{\phi}] = \frac{12}{(4\pi)^4}\frac{209}{2880}\int d^n x\sqrt{|\bar{g}|}\,\frac{1}{\bar{\phi}^2}\;W_6^{(n)}[\bar{g}_{\mu\nu}].$$

In the previous equation

$$W_6^{(n)}[\bar{g}_{\mu\nu}] \equiv W_{(n)}^{\alpha_1\alpha_2\alpha_3\alpha_4}\;W^{(n)}_{\alpha_3\alpha_4\alpha_5\alpha_6}\;W_{(n)}^{\alpha_5\alpha_6}{}_{\alpha_1\alpha_2},$$

and this $\widetilde{W}[\bar{g}_{\mu\nu},\bar{\phi}]$ generates an evanescent operator

$$\left[2\,\bar{g}^{\mu\nu}\frac{\delta}{\delta\bar{g}^{\mu\nu}} + \frac{n-2}{2}\bar{\phi}\,\frac{\delta}{\delta\bar{\phi}}\right]\widetilde{W}[\bar{g}_{\mu\nu},\bar{\phi}] = -2(n-4)\,\widetilde{W}[\bar{g}_{\mu\nu},\bar{\phi}]$$

$\Gamma_0[\bar{g}_{\mu\nu}, \bar{\phi}]$ satisfies the following broken Ward identity

$$\left[2\,\bar{g}^{\mu\nu}\,\frac{\delta}{\delta\bar{g}^{\mu\nu}} + \bar{\phi}\,\frac{\delta}{\delta\bar{\phi}}\right]\Gamma_0[\bar{g}_{\mu\nu}, \bar{\phi}] = 2\,\widetilde{W}[\bar{g}_{\mu\nu}, \bar{\phi}]. \tag{1}$$

If one wants to restore Weyl invariance in CDG, counterterms with logarithmic dependence on the field must be accepted

$$W_0[\bar{g}_{\mu\nu}, \bar{\phi}] = \frac{12}{(4\pi)^4}\frac{209}{2880}\int d^4x\sqrt{|\bar{g}|}\,\frac{1}{\bar{\phi}^2}\ \log(\bar{\phi}^2)\ W_6^{(4)}[\bar{g}_{\mu\nu}],$$

where now

$$\left[2\,\bar{g}^{\mu\nu}\,\frac{\delta}{\delta\bar{g}^{\mu\nu}} + \bar{\phi}\,\frac{\delta}{\delta\bar{\phi}}\right]W_0[\bar{g}_{\mu\nu}, \bar{\phi}] = 2\,\widetilde{W}[\bar{g}_{\mu\nu}, \bar{\phi}] \tag{2}$$

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So we can add this finite counterterm in order to make the effective energy Weyl invariant again

$$\Gamma_0[\bar{g}_{\mu\nu}, \bar{\phi}] \to \Gamma_0^{(new)}[\bar{g}_{\mu\nu}, \bar{\phi}] = \Gamma_0[\bar{g}_{\mu\nu}, \bar{\phi}] - W_0[\bar{g}_{\mu\nu}, \bar{\phi}] \tag{3}$$

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1. The general form of the finite counterterm necessary to cancel the anomaly do not have any inverse power of any mass scale in front of it (as a consequence of conformal invariance).

2. Therefore, there are non-negligible effects in the low energy limit.

3. In the present work we derive the corrected scalar field equation of motion and extract some physical consequences.

We can go further; it is natural to speculate that the four dimensional L-loop divergence in CDG has to be of the form

$$W^L_\infty[\bar\phi,\bar g]=\frac{1}{n-4}\int d^nx\sqrt{|g|}\;\bar\phi^{-2L+2}\;\sum_j g_j P^j_{L+1}[\bar g]$$

where $P^j_{(L+1)}\, j=1\ldots N$ stands for the set of purely gravitational terms with conformal weight $\lambda=2L+2$. Like the trace of the product of $L+1$ Weyl tensors.

$$\mathscr{D}\left(d^nx\sqrt{|g|}\;\bar\phi^{-2L+2}\;\sum_j g_j P^j_{L+1}\right)=-(n-4)\left(d^nx\sqrt{|g|}\;\bar\phi^{-2L+2}\;\sum_j g_j P^j_{L+1}[\bar\phi]\right)$$

This means that the integrand of $W^L_\infty[\bar\phi,\bar g]$ is an evanescent operator which yields a putative anomaly

$$\mathscr{A}^L[\bar\phi,\bar g]\equiv-\int d^nx\sqrt{|g|}\;\bar\phi^{-2L+2}\sum g_j P^j_{L+1}[\bar g]$$

This anomaly-to-be can actually be cancelled by a *finite* counterterm

$$\Delta W^L[\bar\phi,\bar g]=\int d^nx\sqrt{|g|}\;\bar\phi^{-2L+2}\;\log\,\bar\phi\sum g_j P^j_{L+1}[\bar g]$$

The full modified action of CDG will be of the form

$$S_{CDG}[\bar{\phi},\bar{g}] = S_{CDG}[\bar{\phi},\bar{g}] + \sum_{L=1}^{\infty}\left(W_R^L[\bar{\phi},\bar{g}] + \Delta W^L[\bar{\phi},\bar{g}]\right)$$

The counterterms needed to cancel the anomalies have no scale in front of them; there is no small coupling constant (i.e. no M_P^{-2} because there is no M_P in the original action). It is justified then to consider those terms already at the classical level.

$$W_{CDG}^{\text{class}}[\bar{\phi},\bar{g}] \equiv \int d^n x\sqrt{g}\left[\,\frac{1}{12}\bar{\phi}^2\bar{R} + \frac{1}{2}\nabla_\mu\bar{\phi}\nabla^\mu\bar{\phi} - \sum_{L=1}^{\infty}\phi^{-(2L-2)}\,\log\,\bar{\phi}\sum_j g_j P^j_{L+1}\right]$$

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The modified EM for the scalar field now reads

$$\Box\phi - \frac{1}{6}\,R\,\phi - \sum_{L=1}^{\infty}(1-(2L-2)\log\phi)\,\phi^{-2L+1}\,\sum_j g_j P_{L+1}^j[\bar{g}_{\mu\nu}] = 0$$

The cancellation of the conformal anomaly at the quantum level implies that we no longer have a conformal theory at the classical level.

Symmetric phase

All the previous analysis is done in the broken phase, i.e $\bar{\phi} \neq 0$. It is not obvious how the symmetric phase of CDG should be understood.

The first problem is that there is no propagator for the gravitational fluctuations (this propagator goes as $\dfrac{1}{\bar{\phi}^2}$ in the broken phase).
Nevertheless, there are some observations that can be made on general grounds. The full partition function can be written

$$Z \equiv \int \mathscr{D}\phi \mathscr{D}h_{\mu\nu} \exp\left\{-\int d^n x \sqrt{|g|}\left[\phi\left(\bar{\nabla}^2 - \frac{1}{6}\overline{R}\right)\phi + O(\phi^2 h, h^3) + O(\phi^2 h^2)\right]\right\}$$

The one loop contribution only involves the quantum fluctuations of the dilaton and its divergent part is easily computed

$$Z_\infty^{(L=1)} = \exp\left\{-\frac{1}{16\pi^2}\frac{1}{n-4}\int d(vol)\left(\frac{1}{180}\left(\overline{R}^2_{\mu\nu\rho\sigma} - \overline{R}^2_{\mu\nu} + \overline{\Box}\,\overline{R}\right)\right)\right\} =$$
$$= \exp\left\{-\frac{1}{16\pi^2}\frac{1}{n-4}\int d(vol)\frac{1}{180}\left(\frac{3}{2}\overline{W}_4 - \frac{1}{2}\overline{E}_4 + \overline{\Box}\,\overline{R}\right)\right\}$$

We are not able to perform loop integrals over the gravitational fluctuations because of the lack of propagator, but assuming that this integral makes sense (for example by discretizing the system) conformal invariance forces it to be

$$Z_\infty = \sum_{L=1}^{\infty} Z_\infty^L \equiv e^{-g\int d(vol)\overline{W}_4}$$

As each divergent piece Z_∞^L has to be conformally invariant and there is only one of those terms in $n=4$.

Conclusions

1. The general form of the finite counterterms necessary to ensure cancellation of the Weyl anomaly to every order has been determined using only conformal invariance. It can be done as long as one is willing to accept logarithmic counterterms.
2. In spite of being loop effects, those finite counterterms does not have any mass scale in front of them so they are not negligible in the low energy limit. Furthermore, there is no small coupling to even ensure that the starting action dominates.
3. The general form of the -gravitational- counterterm in the symmetric phase has been deduced just demanding Weyl invariance.

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Thank you