Holographic renormalisation group flows from a Wilsonian perspective

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with

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Motivation

- Holography (or the AdS/CFT correspondence) allows to rewrite a quantum field theory in terms of new degrees of freedom different from the Lagrangian fields.
- At strong coupling, the new degrees of freedom are fields in a theory including gravity in one more dimension:
$$ds^2 = \frac{L^2}{z^2} dz^2 + h_{\mu\nu}(z, x^\mu) dx^\mu dx^\nu$$
- The new coordinate is identified with the energy scale of the field theory.
- Several formulations of the holographic renormalization group (Wilsonian vs non-Wilsonian).
- Computation of the holographic β-functions.
- Relation between the different formulations?

Table of Contents

Holographic Wilson Renormalization Review

- UV/IR duality in holography $\Longrightarrow$ Holographic cutoff

$$ds^2 = \frac{L^2}{z^2}dz^2 + h_{\mu\nu}(z, x^\mu)dx^\mu dx^\nu$$

$$\int [\mathcal{D}\omega]^{\Lambda=1/l}\, e^{-S(g)+\text{sing. trace def.}} = \int_{\text{Dirich. condition}} [\mathcal{D}\phi]_{z>l}\, e^{-S^G[\phi]}.$$

- Adding $S^B_l(g)[\phi(l)]$ localized at $z = l$ defines a more general boundary condition, dual to a more general deformation (including higher trace deformations):

$$\pi = \frac{\delta S^B}{\delta \phi}$$

Space of Wilsonian actions $\Longleftrightarrow$ Space of boundary conditions

$$e^{-S^W[\mathcal{O}]} = \int \mathcal{D}\varphi e^{-S^B[\varphi]+\int \varphi\mathcal{O}}$$

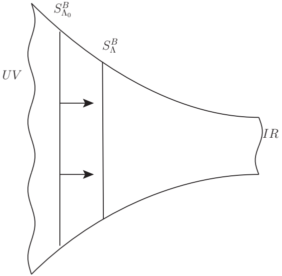
Heemskerk, Polchinski, '10

- Evolution of S^B:

$$e^{-S^B_l} = \int \mathcal{D}\phi^l_{l_0} e^{-S^B_{l_0}-S^G},$$

$$l\partial_l S^B_l[\varphi] = -H\left[\varphi, \frac{\delta S^B_l}{\delta\varphi}\right] - \frac{\delta S^B}{\delta\varphi^i}D\varphi^i \Rightarrow \beta\text{−functions.}$$

$$(D\varphi)(x) = x^\mu \frac{\partial}{\partial x^\mu}\varphi(x)$$



Fixed points

- Analysis of the fixed points in the space of boundary actions for a theory in which we only active the scalar excitations (after some rescalings),

$$0 = -H\left[\varphi, \frac{\delta S_l^B}{\delta \varphi}\right] - \frac{\delta S^B}{\delta \varphi^i} D\varphi^i = -\hat{H}\left[\varphi, \frac{\delta S^B}{\delta \varphi^i}\right],$$

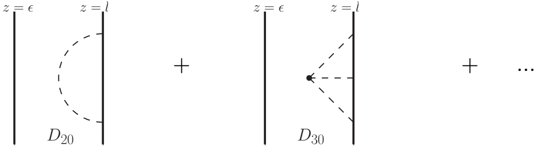
$$H\left[\varphi, \pi\right] = -\frac{1}{2}\pi_i \pi_i + \frac{1}{2}\partial_\mu \varphi^i \partial_\mu \varphi^i + V(\varphi).$$

- Kind of solutions:
    - Non analytic solutions in ϕ around their critical point.
    - Analytic solutions in ϕ in their critical point but non-analytic in the expansion of momenta.
    - Analytic solutions in ϕ in their critical point and analytic in the expansion of momenta.

$$S^{B*}(\varphi) = -\frac{d-1}{L} + \frac{1}{2}\Delta_{(i)}\varphi^{i\;2} + \mathcal{O}\left(\varphi^i\right)^3 + \text{derivatives},$$

$\Delta(\Delta - d) = m^2$

- Witten diagrams:



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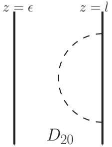

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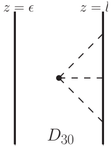

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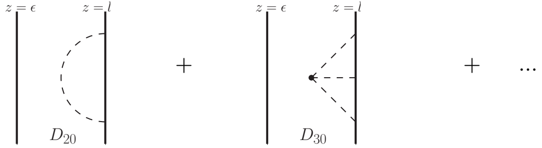

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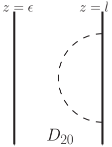
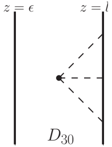
- Witten diagrams: 

$+$ 

$+ \quad \dots$

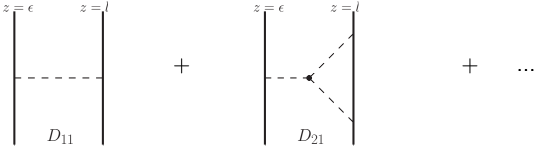
- At the linearized level $S^B = S^{B*} + \int dx g(x) \mathcal{S}'^B(x)$,

$$l\partial_l \mathcal{S}'^B[\varphi] = \hat{\Psi}\mathcal{S}'^B - D\mathcal{S}'^B.$$

- Requiring analyticity in fields and momenta, the *basic* eigenperturbations ($\hat{\Psi}\mathcal{S}'^B = -\Delta\mathcal{S}'^B$), associated to primary single trace operators can be found (order by order in the number of fields expansion),

$$\mathcal{T}^i(\varphi) = \varphi^i + O(\partial\varphi) + O(\varphi^2),$$

- And products of $\mathcal{T}^i(\varphi)$ and derivatives of them.
- Dimensions of the eigenperturbation given by $\Delta^{\pm}_{(i)}$ and standard rules for descendants and products.
- Witten diagrams:



Perturbations

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- Requiring analyticity in fields and momenta, the *basic* eigenperturbations ($\hat{\Psi}\mathcal{S}'^B = -\Delta\mathcal{S}'^B$), associated to primary single trace operators can be found (order by order in the number of fields expansion),

$$\mathcal{T}^i(\varphi) = \frac{(-\partial^2)^{\pm\frac{\nu_{(j)}}{2}}}{\Gamma(1 \pm \nu_{(j)}) 2^{\nu_{(j)}} I_{\pm\nu_{(j)}}(\sqrt{-\partial^2})} \varphi^i + O(\varphi^2),$$

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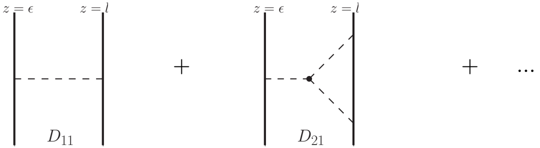
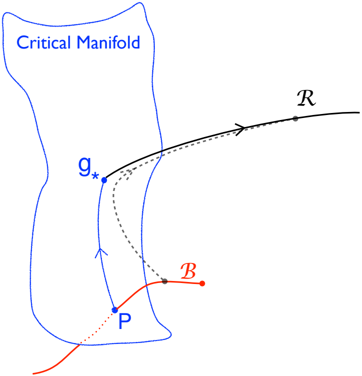


General Picture

- Space of possible theories (boundary actions):



- Gell-Mann-Low β-func. vs Wilson β-func.
- "Scheme of renorm." = "Parametrization",
- Chain rule,

$$\beta_W^{\alpha}(g(g_R)) = \frac{\partial g^{\alpha}(g_R)}{\partial g_R^a}\beta^a(g_R)$$

- Method of Renormalization:
  - Choosing a bare manifold $\mathcal{B}$
  - Find the bare couplings rate to absorb the divergences (get a finite answer after an infinite evolution of the Wilsonian RG flow).
- Holographic renormalization.

Holographic Schemes

- Projection over the Wilsonian couplings.
- UV scheme. General case and exceptional case ($\Delta_i^- = \Delta_j^- + \Delta_k^-$),

$$g_W^\alpha(x) = g_*^\alpha + (\Lambda/\mu)^{\Delta_{(a)}-d} t_a^\alpha g_R^{\mathrm{UV}\,a}(x\mu/\Lambda), \quad \text{as} \quad \Lambda/\mu \to \infty, \quad \text{(linearised)}$$
$$\beta^i(x) = \Delta_{(i)}^- g_R^i(x) - D g_R^i(x).$$

- Is it possible to parametrize $\mathcal{R}$ with solutions of the EoM in the bulk?

Field solutions as renormalised couplings

- Generating solutions with the Hamilton-Jacobi method (fake superpotential method).

$$\hat{H}[\varphi, -\frac{\delta W}{\delta \phi}] = 0,$$
$$z\partial_z \hat{\phi}_z^i = \left.\frac{\delta \hat{H}[\hat{\phi}_z, \hat{\Pi}_z]}{\delta \hat{\Pi}_z^i}\right|_{\hat{\Pi}_z = -\delta W[\hat{\phi}_z]/\delta\hat{\phi}_z} .$$

- Looking for a quasilocal reparametrization, the only candidate is $-S_-^{B*}[\varphi]$ when it exists.
- A proper mapping can be done using holographic renormalization,

$$\begin{array}{c}\text{bare couplings}\\ \text{(Dirichlet condition)}\end{array} \left\{ \begin{array}{ll} h_{t_0}^b & = \hat{\phi}^b_{1/(t_0\mu)}, \\ h_{t_0}^0 & = \mathcal{W}(\hat{\phi}_{1/(t_0\mu)}, \partial\hat{\phi}_{1/(t_0\mu)}, \ldots) + g_R^0 t_0^{-d}. \end{array}\right.$$

Interacting scalars in a fixed AdS space

- Let us consider the theory of M real scalar fields in fixed AdS_{d+1} space, with the potential,

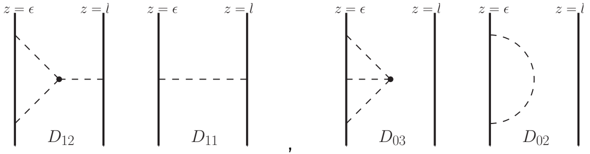
$$V(\phi) = v_0 + \frac{1}{2} m^2_{(i)} \delta_{ij} \phi^i \phi^j + v_{ijk} \phi^i \phi^j \phi^k .$$

- The description of the renormalized manifold up to some order (in our case $O(v_{ijk})$) can be done with holographic renormalization,

$$z\partial_z \phi^i = \Delta_- \phi^i + \frac{\phi^j \phi^k}{\Delta^-_{(i)} + \Delta^-_{(j)} + \Delta^-_{(k)} - d} + \text{derivatives} + O(\phi^3),$$

$$\Rightarrow \phi^i(x) = (\epsilon\mu)^{\Delta^-_{(i)}} g^i_R(\mu x)$$
$$+ (\epsilon\mu)^{\Delta^-_{(j)} + \Delta^-_{(k)}} \frac{v_{ijk} g^j_R(\mu x) g^k_R(\mu x)}{(\Delta^-_{(j)} + \Delta^-_{(k)} - \Delta^-_{(i)})(\Delta^-_{(j)} + \Delta^-_{(k)} - \Delta^+_{(i)})} + \text{derivatives} + O(v^2)$$

- Non local, semilocal and local divergences:



β-functions in different schemes

- This previous method give us the parametrization of the renormalised manifold as a function of the renormalised parameters g_R.
- Computing the β-functions from the asymptotic behaviour of the bare couplings, we find,

$$\beta^i(x) = \Delta^-_{(i)} g^i_R(x) - D g^i_R(x)$$
$$\beta^0(x) = -d - D g^0_R(x)$$

- Wilsonian β-functions computed from the Hamilton-Jacobi equation.
- We can check the consistency of the Gell-Mann-Low and Wilson β-functions.
- We can define other schemes redefining the renormalised couplings (e.g. the field solution scheme),

$$\bar{g}^i_R(x) = g^i_R(x) + v_{ijk} \frac{g^j_R(x) g^k_R(x)}{(\Delta^-_{(j)} + \Delta^-_{(k)} - \Delta^-_{(i)})(\Delta^-_{(j)} + \Delta^-_{(k)} - \Delta^+_{(i)})} + \text{derivatives} + O(v^2).$$

- We studied the connection between Wilsonian and non-Wilsonian approaches to the holographic renormalisation group.
- Different schemes can be understood from the Wilsonian point of view as a reparametrization of the renormalised manifold.
- From this point of view, we can interpret the different schemes that appear naturally in holography.
- In some cases, a schemes can be defined in which the running couplings satisfying the bulk equation of motion.
- Finally, we have studied how these ideas can be implemented in a specific toy model system (scalar theory with a cubic interaction).

Summary

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Thank you!