The Klein-Gordon-Fock equation in the curved spacetime of the Kerr-Newman (anti) de Sitter black hole

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Motivation

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- Now that scalar particles have been observed in Nature solving exactly the massive KGF equation in curved backgrounds is fundamentally important
- The recent spectacular observation of gravitational waves predicted by the theory of General Relativity from the binary black hole merger GW150914 adds further motivation for investigating the interaction of scalar particles with the curved black hole spacetime.

The Kerr-Newman-de Sitter black hole metric

The metric in Boyer-Lindquist coordinates:

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The metric in Boyer-Lindquist coordinates:

$$\mathrm{d}s^2 = \frac{\Delta_r^{KN}}{\Xi^2\rho^2}(c\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)^2 - \frac{\rho^2}{\Delta_r^{KN}}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 - \frac{\Delta_\theta\sin^2\theta}{\Xi^2\rho^2}(ac\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)^2 \tag{1}$$

$$\Delta_\theta := 1 + \frac{a^2\Lambda}{3}\cos^2\theta, \quad \Xi := 1 + \frac{a^2\Lambda}{3}, \tag{2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{3}$$

$$\Delta_r^{KN} := \left(1 - \frac{\Lambda}{3}r^2\right)(r^2+a^2) - 2\frac{GM}{c^2}r + \frac{Ge^2}{c^4}, \tag{4}$$

This is accompanied by a non-zero electromagnetic field $F = \mathrm{d}A$ with vector potential ($G = c = 1$):

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This is accompanied by a non-zero electromagnetic field $F = \mathrm{d}A$ with vector potential $(G = c = 1)$:

$$A = -\frac{er}{\Xi(r^2 + a^2\cos^2\theta)}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi). \tag{5}$$

The massive KGF equation in the curved KN(a)dS black hole spacetime

The Klein-Gordon-Fock (KGF) equation for a scalar field Φ that describes the dynamics of a massive scalar electrically charged particle of charge q, in a curved spacetime is described by the equation:

$$\Box\Phi + \mu^2\Phi = 0, \tag{6}$$

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Using the the ansatz:

$$\boxed{\Phi = \Phi(\vec{r},t) = R(r)S(\theta)e^{im\varphi}e^{-i\omega t},} \quad (12)$$

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while the angular equation remains unaltered.

Heun's differential equation

$$\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}z^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-a}\right)\frac{\mathrm{d}y}{\mathrm{d}z}+\frac{\alpha\beta z-q}{z(z-1)(z-a)}y=0} \tag{17}$$

In (17), y and z are regarded as complex variables and $\alpha, \beta, \gamma, \delta, \varepsilon, q, a$ are parameters, generally complex and arbitrary, except that $a \in \mathbb{C} \setminus \{0, 1\}$. The first five parameters are linked by the equation

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Heun's equation is thus of Fuchsian type with regular singularities at the points $z=0,1,a,\infty$. The *exponents* at these singularities are computed through the indicial equation to be:
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$\{0, 1-\gamma\}; \{0, 1-\delta\}; \{0, 1-\varepsilon\}; \{\alpha, \beta\}$. The sum of these exponents must take the value 2, according to the general theory of Fuchsian equations. The Heun equation includes an *accessory* or *auxiliary* parameter, namely the quantity $q \in \mathbb{C}$, which in many applications appears as a spectral parameter.

The Confluent Heun Equation (CHE)

This is obtained by merging the singularity at $z = a$ of Heun's equation with that at $z = \infty$, resulting in an equation still having regular singularities at $z = 0$ and $z = 1$, and an *irregular* singularity of rank 1 at $z = \infty$ (RONVEAUX). Indeed, dividing (17) by a we derive:

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in which γ, δ, α are the same parameters as in the original equation (17) while ν, σ are new. CHE

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The angular Fuchsian equation (23) has four regular singularities at the points $\pm 1, \pm \frac{i}{\sqrt{\alpha_\Lambda}}$, which we denote with the tuple $(a_1, a_2, a_3, a_4) = (-1, 1, -\frac{i}{\sqrt{\alpha_\Lambda}}, \frac{i}{\sqrt{\alpha_\Lambda}})$. The automorphism group of the parameter space of Heun's equation has recently been determined, thus we apply first to equation (23) the *homographic transformation of the independent variable* :

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$$z = \frac{a_2 - a_4}{a_2 - a_1}\frac{x - a_1}{x - a_4} = \frac{1 - \frac{i}{\sqrt{\alpha_\lambda}}}{2}\frac{x + 1}{x - \frac{i}{\sqrt{\alpha_\Lambda}}}, \quad \alpha_\Lambda := \frac{a^2\Lambda}{3}, \tag{24}$$

where such a transformation is designed to map the three singularities a_1, a_2, a_4 into $0, 1, \infty$. The fourth singularity $a_3 \overset{(24)}{\rightarrow} z_3 = \frac{a_3 - a_1}{a_3 - a_4}\frac{a_2 - a_4}{a_2 - a_1}$. With this transformation we have:

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$$(1 + \alpha_\Lambda x^2)(1 - x^2) = \frac{\alpha_\Lambda 16 i \Xi^2}{\sqrt{\alpha_\Lambda}} \frac{z(z-1)(z-z_3)}{[2z\sqrt{\alpha_\Lambda} - \sqrt{\alpha_\Lambda} + i]^4}, \tag{25}$$

where

$$z_3 = -\frac{1}{2}\left(-1 + \frac{\alpha_\Lambda - 1}{2i\sqrt{\alpha_\Lambda}}\right). \tag{26}$$

Equation (23) with the aid of (24) becomes:

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

$$\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}$$
$$-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{2}{(z-z_\infty)^2}+$$
$$\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}-\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]$$
$$+\frac{1}{z-z_\infty}\frac{-8i\sqrt{\alpha_\Lambda}}{\Xi}\Bigg\}S(z)=0, \qquad (27)$$

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}\right.$$
$$-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{2}{(z-z_\infty)^2}+$$
$$\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}-\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]$$
$$+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]$$
$$\left.+\frac{1}{z-z_\infty}\frac{-8i\sqrt{\alpha_\Lambda}}{\Xi}\right\}S(z)=0, \tag{27}$$

where $z_\infty=-\frac{-i(1+\sqrt{\alpha_\Lambda}i)}{2\sqrt{\alpha_\Lambda}}$ and $\xi:=a\omega$.

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

$$\Bigg\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}$$
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$$+\frac{1}{z-z_\infty}\frac{-8i\sqrt{\alpha_\Lambda}}{\Xi}\Bigg\}S(z)=0, \quad (27)$$

where $z_\infty=-\frac{-i(1+\sqrt{\alpha_\Lambda}i)}{2\sqrt{\alpha_\Lambda}}$ and $\xi:=a\omega$. The four singularities $z=0,1,z_3,z_\infty$ have exponents:
$\{\frac{|m|}{2},-\frac{|m|}{2}\},\{\frac{|m|}{2},-\frac{|m|}{2}\},\{\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}}-m\sqrt{\alpha_\Lambda}\right),-\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}}-m\sqrt{\alpha_\Lambda}\right)\},\{2,1\}$.

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

$$\begin{aligned}
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\left[\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_3}-\frac{2}{z-z_\infty}\right]\frac{\mathrm{d}}{\mathrm{d}z}\right.\\
&-\frac{m^2}{4}\frac{1}{z^2}-\frac{m^2}{4}\frac{1}{(z-1)^2}+\left(\frac{\Xi a\omega}{2\sqrt{\alpha_\Lambda}}-\frac{m\sqrt{\alpha_\Lambda}}{2}\right)^2\frac{1}{(z-z_3)^2}+\frac{2}{(z-z_\infty)^2}+\\
&\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2}+\frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2}+\frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]\\
&+\frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2}-\frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2}-\frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]\\
&+\frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi}+\frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2}+\frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]\\
&\left.+\frac{1}{z-z_\infty}\frac{-8i\sqrt{\alpha_\Lambda}}{\Xi}\right\}S(z)=0, \qquad (27)
\end{aligned}$$

where $z_\infty=-\frac{-i(1+\sqrt{\alpha_\Lambda}i)}{2\sqrt{\alpha_\Lambda}}$ and $\xi:=a\omega$. The four singularities $z=0,1,z_3,z_\infty$ have exponents: $\{\frac{|m|}{2},-\frac{|m|}{2}\},\{\frac{|m|}{2},-\frac{|m|}{2}\},\{\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}}-m\sqrt{\alpha_\Lambda}\right),-\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}}-m\sqrt{\alpha_\Lambda}\right)\},\{2,1\}$. Thus equation (27) is *not* of a Heun type.

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

The *F-homotopic transformation* or *index transformation* of the dependent variable S:

$$S(z) = z^{\alpha_1}(z-1)^{\alpha_2}(z-z_3)^{\alpha_3}(z-z_\infty)^{\alpha_4}\bar{S}(z) \tag{28}$$

where $\alpha_1 = \alpha_2 = \frac{|m|}{2}, \alpha_3 = \pm\frac{i}{2}\left(\frac{\Xi\xi}{\sqrt{\alpha_\Lambda}} - m\sqrt{\alpha_\Lambda}\right), \alpha_4 = 1$ is designed to reduce one of the exponents of the finite singularities $0, 1, z_3$ to zero and to eliminate the finite z_∞ singularity. In other words transforms (27) into the Heun form (17). Indeed application of (28) into (27) yields:

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

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$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{2\alpha_1+1}{z} + \frac{2\alpha_2+1}{z-1} + \frac{2\alpha_3+1}{z-z_3}\right]\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-z_3)}\right\}\bar{S}(z) = 0, \tag{29}$$

where the *auxiliary parameter q* is calculated in terms of the cosmological constant, spin of the black hole,the parameters m, ω and is given by the expression:

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where the *auxiliary parameter q* is calculated in terms of the cosmological constant, spin of the black hole,the parameters m, ω and is given by the expression:

$$q = \frac{i}{4\sqrt{\alpha_\Lambda}}\Big\{-(1+i\sqrt{\alpha_\Lambda})^2[2\alpha_1\alpha_2+\alpha_2+\alpha_1] - 4\sqrt{\alpha_\Lambda}i[2\alpha_1\alpha_3+\alpha_3+\alpha_1]$$
$$-\frac{m^2}{2}((1+i\sqrt{\alpha_\Lambda})^2 + 4\alpha_\Lambda) + K_{lm} - 2i\sqrt{\alpha_\Lambda} + 2\Xi m\xi\Big\}$$

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

The parameters α, β are given in terms of the physical parameters by the expression:

Value of scalar mass for which the angular Fuchsian equation is solved in terms of Heun functions

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$$\begin{aligned}
\alpha\beta &= q - (z_3 - 1) \times coef.of \frac{1}{z-1} \\
&= \frac{i}{4\sqrt{\alpha_\Lambda}} \Big\{ -(1 + i\sqrt{\alpha_\Lambda})^2 [2\alpha_1\alpha_2 + \alpha_2 + \alpha_1] - 4\sqrt{\alpha_\Lambda} i [2\alpha_1\alpha_3 + \alpha_3 + \alpha_1] \\
&- \frac{m^2}{2}((1 + i\sqrt{\alpha_\Lambda})^2 + 4\alpha_\Lambda) + K_{lm} - 2i\sqrt{\alpha_\Lambda} + 2\Xi m\xi \Big\} \\
&+ \frac{i}{4\sqrt{\alpha_\Lambda}} \Big\{ \frac{m^2}{2} \left((1 - i\sqrt{\alpha_\Lambda})^2 + 4\alpha_\Lambda\right) - 2m\xi\Xi - K_{lm} - 2\sqrt{\alpha_\Lambda} i \\
&+ (1 - i\sqrt{\alpha_\Lambda})^2 [2\alpha_1\alpha_2 + \alpha_2 + \alpha_1] + i4\sqrt{\alpha_\Lambda}(-2\alpha_2\alpha_3 - \alpha_3 - \alpha_2) \Big\}
\end{aligned} \tag{30}$$

Closed form solution of the radial equation for a massive charged particle in the KNdS black hole spacetime in terms of Heun functions for specific values of the scalar mass

The massive radial Fuchsian equation:

Closed form solution of the radial equation for a massive charged particle in the KNdS black hole spacetime in terms of Heun functions for specific values of the scalar mass

The massive radial Fuchsian equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta_r^{KN}\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\frac{R(r)}{\Delta_r^{KN}}\left[\Xi^2\left(K-\frac{eqr}{\Xi}\right)^2-r^2\mu^2\Delta_r^{KN}-K_{lm}\Delta_r^{KN}\right]=0} \tag{31}$$

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We write the quantity Δ_r^{KN} in terms of the radii of the event and Cauchy horizons r_+, r_- and the cosmological horizon r_Λ^+ for positive cosmological constant:

$$\Delta_r^{KN}=-\frac{\Lambda}{3}(r-r_+)(r-r_-)(r-r_\Lambda^+)(r-r_\Lambda^-) \tag{32}$$

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There are five regular singularities in (31), at the points $r_\pm, r_\Lambda^\pm, \infty$. Applying the homographic substitution

$$\boxed{z=\left(\frac{r_+-r_\Lambda^-}{r_+-r_-}\right)\left(\frac{r-r_-}{r-r_\Lambda^-}\right)} \quad (33)$$

Exact solution for the massive-charged radial KGF Fuchsian equation

Equation (32) in terms of the new variable is written:

$$\boxed{\Delta_r^{KN} = -\frac{\Lambda}{3}\frac{Hz_\infty^3 z(z-1)(z-z_r)}{(z_\infty - z)^4},} \tag{34}$$

where $H := \frac{(r_- - r_\Lambda^-)^2(r_+ - r_-)(r_\Lambda^+ - r_-)}{z_r}$. Also we have the following relations:

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$$r = \frac{r_- z_\infty - r_\Lambda^- z}{z_\infty - z}, \tag{35}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{z_\infty(r_- - r_\Lambda^-)}{(r - r_\Lambda^-)^2} = \frac{1}{z_\infty}\frac{1}{r_- - r_\Lambda^-}(z_\infty - z)^2 = \frac{r_+ - r_-}{r_+ - r_\Lambda^-}\frac{1}{r_- - r_\Lambda^-}(z_\infty - z)^2 \tag{36}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = \frac{-2z_\infty(r_- - r_\Lambda^-)}{(r - r_\Lambda^-)^3}, \quad \frac{\frac{\mathrm{d}^2 z}{\mathrm{d}r^2}}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2} = \frac{-2}{z_\infty - z}. \tag{37}$$

The quantities z_∞, z_r are defined as follows:

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The quantities z_∞, z_r are defined as follows:

$$z_\infty := \frac{r_+ - r_\Lambda^-}{r_+ - r_-}, \quad z_r := z_\infty\left(\frac{r_\Lambda^+ - r_-}{r_\Lambda^+ - r_\Lambda^-}\right). \tag{38}$$

Applying the homographic transformation (33) in the radial equation for a massive charged particle (16) we obtain:

Applying the homographic transformation (33) in the radial equation for a massive charged particle (16) we obtain:

$$\frac{1}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2}\frac{1}{\Delta_r^{KN}}\frac{\mathrm{d}\Delta_r^{KN}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}r}=\left\{\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-z_r}-\frac{4}{z-z_\infty}\right\}\frac{\mathrm{d}R}{\mathrm{d}z}. \quad (39)$$

However the term proportional to $\frac{\mathrm{d}R}{\mathrm{d}z}$, taking into account a contribution from the second derivative, will eventually be:

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We also have for the term $\frac{-r^2\mu^2 R}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Delta_r^{KN}}$, the expansion:

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We also have for the term $\frac{-r^2\mu^2 R}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Delta_r^{KN}}$, the expansion:

$$\frac{-r^2\mu^2 R}{\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\Delta_r^{KN}}=\frac{A}{(z_\infty-z)^2}+\frac{B}{z_\infty-z}+\frac{C}{z}+\frac{D}{z-1}+\frac{F}{z-z_r}, \quad (41)$$

Exact solution for the massive-charged radial KGF Fuchsian equation

where we compute the coefficients of the expansion as follows:

$$A = \frac{3\mu^2}{\Lambda}, \tag{42}$$

$$B = \frac{3\mu^2}{\Lambda}\frac{1}{r_- - r_\Lambda^-}\left[\frac{(r_\Lambda^- + r_-)z_r - 2r_- z_\infty - 2r_- z_r z_\infty - (r_\Lambda^- - 3r_-)z_\infty^2}{(1-z_\infty)(z_r - z_\infty)z_\infty}\right], \tag{43}$$

$$C = \frac{3\mu^2}{\Lambda}\frac{1}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{r_-^2}{z_\infty}, \tag{44}$$

$$D = -\frac{3\mu^2}{\Lambda}\frac{z_r}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{1}{z_\infty}\frac{[r_\Lambda^- - r_- z_\infty]^2}{(z_r - 1)(z_\infty - 1)}, \tag{45}$$

$$F = \frac{3\mu^2}{\Lambda}\frac{1}{r_+ - r_-}\frac{1}{r_\Lambda^+ - r_-}\frac{1}{z_\infty}\frac{(r_\Lambda^- z_r - r_- z_\infty)^2}{(z_r - 1)(z_r - z_\infty)^2} \tag{46}$$

Let us calculate the exponents of the singularity at z_∞. The indicial equation takes the form:

$$F(r) = r(r-1) - 2r + \frac{3\mu^2}{\Lambda} = 0, \tag{47}$$

and the exponents are computed to be:

$$r_{\mu z_\infty}^{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{9 - \frac{12\mu^2}{\Lambda}}. \tag{48}$$

Exact solution for the massive-charged radial KGF Fuchsian equation

Subsequently we compute the exponents for the regular singularities $z = 0, z = 1, z = z_r$. Indeed the indicial equation for the $z = 1$ singularity takes the form:

$$F(r) = r(r-1) + r + \frac{a^4}{\alpha_\Lambda^2} \frac{[\Xi K(r_+) - eqr_+]^2}{(r_+ - r_\Lambda^-)^2 (r_+ - r_\Lambda^+)^2 (r_+ - r_-)^2} = 0 \qquad (49)$$

Thus the roots are calculated to be:

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Thus the roots are calculated to be:

$$\boxed{r_{z=1}^{1,2} \equiv \mu_2 = \pm \frac{ia^2}{\alpha_\Lambda} \frac{\Xi K(r_+) - eqr_+}{(r_\Lambda^- - r_+)(r_- - r_+)(r_\Lambda^+ - r_+)}} \tag{50}$$

Likewise we compute the exponents of the other two singularities:

$$r_{z=0}^{1,2} \equiv \mu_1 = \pm \frac{ia^2}{\alpha_\Lambda} \frac{\Xi K(r_-) - eqr_-}{(r_- - r_\Lambda^-)(r_+ - r_-)(r_\Lambda^+ - r_-)}, \tag{51a}$$

$$r_{z=z_r}^{1,2} \equiv \mu_3 = \pm \frac{ia^2}{\alpha_\Lambda} \frac{[\Xi K(r_\Lambda^+) - eqr_\Lambda^+]}{(r_\Lambda^- - r_\Lambda^+)(r_+ - r_\Lambda^+)(r_- - r_\Lambda^+)}. \tag{51b}$$

Exact solution for the massive-charged radial KGF Fuchsian equation

Subsequently we compute the exponents for the regular singularities $z = 0, z = 1, z = z_r$. Indeed the indicial equation for the $z = 1$ singularity takes the form:

$$F(r) = r(r-1) + r + \frac{a^4}{\alpha_\Lambda^2} \frac{[\Xi K(r_+) - eqr_+]^2}{(r_+ - r_\Lambda^-)^2 (r_+ - r_\Lambda^+)^2 (r_+ - r_-)^2} = 0 \quad (49)$$

Thus the roots are calculated to be:

$$\boxed{r_{z=1}^{1,2} \equiv \mu_2 = \pm \frac{ia^2}{\alpha_\Lambda} \frac{\Xi K(r_+) - eqr_+}{(r_\Lambda^- - r_+)(r_- - r_+)(r_\Lambda^+ - r_+)}} \quad (50)$$

Likewise we compute the exponents of the other two singularities:

$$r_{z=0}^{1,2} \equiv \mu_1 = \pm \frac{ia^2}{\alpha_\Lambda} \frac{\Xi K(r_-) - eqr_-}{(r_- - r_\Lambda^-)(r_+ - r_-)(r_\Lambda^+ - r_-)}, \quad (51a)$$

$$r_{z=z_r}^{1,2} \equiv \mu_3 = \pm \frac{ia^2}{\alpha_\Lambda} \frac{[\Xi K(r_\Lambda^+) - eqr_\Lambda^+]}{(r_\Lambda^- - r_\Lambda^+)(r_+ - r_\Lambda^+)(r_- - r_\Lambda^+)}. \quad (51b)$$

Thus we see that in general the massive radial Fuchsian KGF equation for a charged particle in the curved spacetime of a cosmological rotating charged black hole possess five singularities including the infinity.

Choosing a value of the scalar mass in terms of Λ as $\mu = \sqrt{\frac{2}{3}\Lambda}$ the exponents of the z_∞ singularity become $r^{1,2}_{z_\infty,\mu^2=\frac{2}{3}\Lambda} = 2, 1$. Thus applying the F-homotopic transformation of the dependent variable R

$$\boxed{R(z) = z^{\mu_1}(z-1)^{\mu_2}(z-z_r)^{\mu_3}(z-z_\infty)^{r^2_{z_\infty}}\bar{R}(z)} \tag{52}$$

we *eliminate the z_∞ singularity* and reduce one of the exponents of the three finite singularities $z = 0, 1, z_r$ to zero. Consequently for this value for the scalar mass the radial part of the KGF Fuchsian equation in the curved spacetime of the KNdS black hole becomes a Heun differential equation:

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$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{2\mu_1+1}{z} + \frac{2\mu_2+1}{z-1} + \frac{2\mu_3+1}{z-z_r}\right]\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-z_r)}\right\}\bar{R}(z) = 0. \tag{53}$$

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$$\left.+\frac{\alpha\beta z-q}{z(z-1)(z-z_r)}\right\}\bar{R}(z)=0. \tag{53}$$

The $F-$ *homotopic transformation* (52) *factors out* the z_∞ singularity because it eliminates both terms $\propto \frac{1}{(z-z_\infty)^2}$ and $\propto \frac{1}{z-z_\infty}$ respectively. Indeed the last term vanishes:

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$$\frac{1}{z-z_\infty}\left(\frac{1}{z_\infty}-\frac{1}{1-z_\infty}-\frac{1}{z_r-z_\infty}\right)-\frac{B}{z-z_\infty}$$
$$=\frac{1}{z-z_\infty}\frac{(r_--r_+)(r_\Lambda^-+r_\Lambda^++r_-+r_+)}{(r_\Lambda^--r_-)(r_\Lambda^--r_+)}=0, \tag{54}$$

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due to Vieta's relations, i.e. $r_\Lambda^- + r_\Lambda^+ + r_- + r_+ = 0$.

Theorem

For the value: $\mu = \sqrt{\frac{2\Lambda}{3}}$ *both radial and angular Fuchsian dif.equations are solved in closed analytic form in terms of general Heun functions. Thus both radial* $\bar{R}(z)$ *and angular parts* $\bar{S}(z)$ *are expressed locally in terms of Heun functions:* $Hl(a_i, q_i; \alpha_i, \beta_i, \gamma_i, \delta_i; z), \quad i = \bar{R}, \bar{S}$.

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determine the Weierstraß invariants (g_2, g_3) with the result:

$$g_2 = \frac{1}{12}\left(-\frac{3}{\Lambda}+a^2\right)^2 - \frac{3}{\Lambda}(a^2+e^2), \tag{61}$$

$$g_3 = -\frac{1}{216}\left(-\frac{3}{\Lambda}+a^2\right)^3 - \frac{3}{\Lambda}\frac{1}{6}(a^2+e^2)\left(-\frac{3}{\Lambda}+a^2\right) - \frac{9}{4\Lambda^2}. \tag{62}$$

False singular points and exact solution of the angular KGF equation

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- In this case too, the angular Fuchsian equation can in principle be solved exactly in terms of Heun functions.
- There is a deep connection between a Fuchsian equation with false singular points and *finite-gap* elliptic Schrödinger equation. It is worth exploring further generalisations of this connection from closed form solutions of massive KGF equation in curved BH backgrounds with false singular point(s).

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- We proceed to introduce the concept of *false* or *apparent* singularity.

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Definition

We call a singular point a_i *false* if both exponents α_i and β_i are non-negative integers and there are no logarithmic terms in the local expansion near the singular point.

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We discuss briefly these restrictions on the coefficients of eqn.(63) so that the singular point a_j is false.

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Considering the simplest false point with the exponents equal to 0 and 2, then, from the general theory of Fuchsian equations, the exponents follow from a characteristic equation, and local to a_j:

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$$f(z) = \frac{1}{z - a_j} + f_0 + O(z - a_j), \; g(z) = \frac{g_{-1}}{z - a_j} + g_0 + O(z - a_j), \quad (64)$$

for some constants f_0, g_{-1}, g_0. The solution corresponding to the exponent zero can be written in the form:

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$$Y(z) = \sum_{m=0}^{\infty} c_m (z - a_j)^m = c_0 + c_1(z - a_j) + c_2(z - a_j)^2 + O((z - a_j)^3), \quad (65)$$

for some constants c_0, c_1, c_2. Substituting (65) into (63) we obtain recursive equations for the coefficients at different orders of $(z - a_j)$ which yields the following condition that guarantees the absence of logarithmic terms local to a_j:

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$$\boxed{g_{-1} f_0 - g_0 + (g_{-1})^2 = 0.} \quad (66)$$

fs

Conditions on the coefficients of the massive Fuchsian angular KGF equation such that the fifth singular point is a false singular point

Equation (22) with the aid of (24) becomes:

$$\Bigg\{ \frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-z_3} - \frac{2}{z-z_\infty} \right] \frac{\mathrm{d}}{\mathrm{d}z}$$

$$- \frac{m^2}{4}\frac{1}{z^2} - \frac{m^2}{4}\frac{1}{(z-1)^2} + \left(\frac{\Xi a \omega}{2\sqrt{\alpha_\Lambda}} - \frac{m\sqrt{\alpha_\Lambda}}{2} \right)^2 \frac{1}{(z-z_3)^2} + \frac{a^2\mu^2}{\alpha_\Lambda (z-z_\infty)^2} +$$

$$\frac{1}{z}\left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2} + \frac{2m\Xi\xi}{(1+i\sqrt{\alpha_\Lambda})^2} + \frac{a^2\mu^2}{(-i+\sqrt{\alpha_\Lambda})^2} + \frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2} \right]$$

$$+ \frac{1}{z-1}\left[\frac{-m^2(1-2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(i+\sqrt{\alpha_\Lambda})^2} - \frac{-2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2} - \frac{a^2\mu^2}{(i+\sqrt{\alpha_\Lambda})^2} - \frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2} \right]$$

$$+ \frac{1}{z-z_3}\left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2} + \frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi} + \frac{4ia^2\mu^2}{\sqrt{\alpha_\Lambda}\Xi^2} + \frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2} \right]$$

$$+ \frac{1}{z-z_\infty}\frac{-4ia^2\mu^2}{\sqrt{\alpha_\Lambda}\Xi} \Bigg\} S(z) = 0, \tag{67}$$

We have five singular points. The exponentials at the singular point z_∞ are obtained by solving the indicial equation:

$$F(r) = r(r-1) + p_0 r + q_0 = 0 \tag{68}$$

where $p_0 = \lim_{z \to z_\infty}(z - z_\infty)\frac{-2}{z - z_\infty} = -2$ and

$q_0 = \lim_{z \to z_\infty}(z - z_\infty)^2 Q(z) = \frac{a^2\mu^2}{\alpha_\Lambda}$. Thus we obtain $r_{1,2}(\mu) = \frac{3 \pm \sqrt{9 - 4\frac{a^2\mu^2}{\alpha_\Lambda}}}{2}$.
Now choosing

$$\frac{5}{4} = \frac{a^2\mu^2}{\alpha_\Lambda}, \tag{69}$$

and performing the homotopy transformation for the dependent variable

$$S(z) = z^{\alpha_1}(z-1)^{\alpha_2}(z-z_3)^{\alpha_3}(z-z_\infty)^{\alpha_4}\bar{S}(z) \tag{70}$$

now with $\alpha_4 = \frac{1}{2}$ one transforms (67) into an equation with the same singularities however the exponents of the singular point z_∞ will be now $\{0, 2\}$, i.e. non-negative integers. Thus, for this choice of scalar mass we can arrange matters so that the singularity z_∞ becomes false. However in order for this to be true, also the condition (66), that guarantees the absence of logarithmic terms needs to be satisfied. The terms appearing in (66) are calculated to be:

Conditions on the coefficients of the massive Fuchsian angular KGF equation such that the fifth singular point is a false singular point

For the choice of scalar mass $\mu = \sqrt{\frac{5}{12}\Lambda}$ the coefficients in (66) are:

$$g_{-1} = \frac{-i\sqrt{\alpha_\Lambda}}{\Xi}, \tag{71}$$

$$f_0 = \frac{2\alpha_1+1}{z_\infty} + \frac{2\alpha_2+1}{z_\infty - 1} + \frac{2\alpha_3+1}{z_\infty - z_3}, \tag{72}$$

$$g_0 = \left[\frac{m^2(1+2i\sqrt{\alpha_\Lambda}+3\alpha_\Lambda)}{2(-i+\sqrt{\alpha_\Lambda})^2} + \frac{2m\xi\Xi}{(1+i\sqrt{\alpha_\Lambda})^2} + \frac{-2\alpha_\Lambda}{(1+i\sqrt{\alpha_\Lambda})^2} + \frac{K_{lm}}{(1+i\sqrt{\alpha_\Lambda})^2}\right]\frac{1}{z_\infty}$$
$$+ \left[\frac{m^2}{2}\left(1 + \frac{4\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2} - \right)\frac{2m\xi\Xi}{(1-i\sqrt{\alpha_\Lambda})^2} + \frac{2\alpha_\Lambda}{(1-i\sqrt{\alpha_\Lambda})^2} - \frac{K_{lm}}{(1-i\sqrt{\alpha_\Lambda})^2}\right]\frac{1}{z_\infty - 1}$$
$$+ \left[\frac{-8im^2\alpha_\Lambda\sqrt{\alpha_\Lambda}}{\Xi^2} + \frac{8im\sqrt{\alpha_\Lambda}\xi}{\Xi} + \frac{8i\sqrt{\alpha_\Lambda}}{\Xi^2} + \frac{4i\sqrt{\alpha_\Lambda}K_{lm}}{\Xi^2}\right]\frac{1}{z_\infty - z_3} \tag{73}$$

Exact solution of Heun's differential equation with a false singular point

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$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\zeta^2}+\left(\frac{\gamma}{\zeta}+\frac{\delta}{\zeta-1}+\frac{-1}{\zeta-a}\right)\frac{\mathrm{d}Y}{\mathrm{d}\zeta}+\frac{(\alpha\beta\zeta-q)Y)}{\zeta(\zeta-1)(\zeta-a)}=0, \tag{74}$$

the point $\zeta=a$ is the false singularity.

Exact solution of Heun's differential equation with a false singular point

Consider the Fuchsian Heun equation with a false singular point:

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\zeta^2} + \left(\frac{\gamma}{\zeta} + \frac{\delta}{\zeta - 1} + \frac{-1}{\zeta - a}\right)\frac{\mathrm{d}Y}{\mathrm{d}\zeta} + \frac{(\alpha\beta\zeta - q)Y)}{\zeta(\zeta-1)(\zeta-a)} = 0, \tag{74}$$

the point $\zeta = a$ is the false singularity. The exponents at this point are equal to 0 and 2 and thus $\varepsilon = -1$. Using the Fuchs relation that the sum of all exponents depend only on the number of singular points we now have that $\delta = 2 - \gamma + \beta + \alpha$.

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We have verified this analytically using properties and recurrence relations of Gauß hypergeometric function-Kraniotis 2016.

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We have verified this analytically using properties and recurrence relations of Gauß hypergeometric function-Kraniotis 2016. This leads us to the conjecture:

Conjecture

We expect that in the case of Fuchsian equation with 5 singularities as it is the case for the radial and angular differential equations for a massive charged scalar particle in the KNdS black hole spacetime for most of the parameter space, that if one of the singularities is false, the solution will be expressed in terms of Heun functions.

Exact solution of the radial equation for a massive neutral scalar particle in the Kerr-Newman spacetime

Assuming $\Lambda = 0$, the radial equation for a massive neutral particle ($q = 0$) is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(x+2d)\frac{\mathrm{d}R}{\mathrm{d}x}\right] + \left[\frac{\omega^2}{M^2 x(x+2d)}\{M^2[(x+d+1)^2-(d^2-1)]-e^2\}^2 + \frac{2e^2 a\omega m}{M^2 x(x+2d)} - \frac{4a\omega m(x+d+1)}{x(x+2d)} - \mu^2 M^2 (x+d+1)^2 + \frac{m^2 a^2}{M^2 x(x+2d)} - (\omega^2 a^2 + K_{lm})\right] R = 0, \quad (76)$$

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Using the change of variables:

$$R(x) = e^{2idM\sqrt{\omega^2-\mu^2}z} z^{\pm\frac{i}{2M}\sqrt{4A-M^2}}(z-1)^{\pm\frac{i}{2M}\sqrt{4C-M^2}}Y(z)z^{1/2}(z-1)^{1/2}(x(x+2d))^{-1/2}, \quad (78)$$

yields the *confluent Heun equation*:

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yields the *confluent Heun equation*:

$$\boxed{Y''(z) + \left(\alpha + \frac{\gamma}{z} + \frac{\delta}{z-1}\right)Y'(z) + \frac{wz-\sigma}{z(z-1)}Y(z) = 0} \quad (79)$$

(recall (21))

Exact solution of the radial equation for a massive neutral scalar particle in the Kerr-Newman spacetime

where the coefficients are calculated to be:

$$A = \frac{d^2M^2 + (am + (-2(1+d)M^2 + e^2)\omega)^2}{4d^2}, \tag{80}$$

$$B = \frac{1}{4d^3}(-a^2m^2 + d^2M^2(-1 - 2K_{lm} - 2(1+d)^2M^2\mu^2) + 2am(2M^2 - e^2)\omega \\ - (2a^2d^2M^2 - 4(1+d)^2(-1+2d)M^4 + 4(-1+d^2)M^2e^2 + e^4)\omega^2) \tag{81}$$

$$C = \frac{d^2M^2 + (am + (2(-1+d)M^2 + e^2)\omega)^2}{4d^2} \tag{82}$$

$$D = \frac{1}{4d^3}(d^2M^2(1 + 2K_{lm} + 2(-1+d)^2M^2\mu^2) + 2am(-2M^2 + e^2)\omega \\ + (4(-1+d)^2(1+2d)M^4 + 4(-1+d^2)M^2e^2 + e^4)\omega^2 + a^2(m^2 + 2d^2M^2\omega^2)) \tag{83}$$

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also we made use of a change in the independent variable:

$$z = -\frac{x}{2d}, \tag{84}$$

An exact solution of the radial KGF eqn in the KN spacetime is:

$$R(z) = \frac{M}{\sqrt{\Delta^{KN}}} e^{2idM\sqrt{\omega^2-\mu^2}z} z^{\frac{1}{2}\pm\frac{i}{2M}\sqrt{4A-M^2}} (z-1)^{\frac{1}{2}\pm\frac{i}{2M}\sqrt{4C-M^2}} H_c(\alpha, w, \gamma, \delta, \sigma, z). \tag{85}$$

Exact solution of the radial equation for a massive neutral scalar particle in the Kerr-Newman spacetime

The parameters of the *confluent Heun function* $H_c(\alpha, w, \gamma, \delta, \sigma, z)$ are computed to be:

$$\alpha = 4idM\sqrt{\omega^2-\mu^2},\ \ \gamma = 1 \pm \frac{i}{M}\sqrt{4A-M^2},\ \ \delta = 1 \pm \frac{i}{M}\sqrt{4C-M^2},$$

$$\sigma = \left(\frac{-2dB}{M^2}-\frac{1}{2}\right)+\frac{1}{2}+\frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left(1+\frac{i}{M}\sqrt{4A-M^2}\right)-\left(\frac{1}{2}+\frac{i}{2M}\sqrt{4A-M^2}\right)\left(1+\frac{i}{M}\sqrt{4C-M^2}\right),$$

$$w = \frac{-2d}{M^2}(B+D)+4idM\sqrt{\omega^2-\mu^2}+\frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left[\frac{i}{M}\sqrt{4A-M^2}+\frac{i}{M}\sqrt{4C-M^2}\right]$$

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$$Y = \sum_{\mu} a_{\mu} Y_{\mu},\ Y_{\mu} = F(\alpha_{\mu}, \gamma_{\mu}, s_0 z) \tag{86}$$

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$$\delta = -N \text{ or } \frac{w}{\alpha} = -N \tag{87}$$

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Also if $\alpha_0 = \gamma$ the series is right hand terminated if

$$\gamma+\delta+\left(-\frac{w}{\alpha}\right) = -N \tag{88}$$

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Exact solution of the radial equation for a massive charged scalar particle in the Kerr-Newman spacetime

Following similar steps as in the previous pages the exact solution of the radial part of the KGF differential equation for a massive charged particle in the KN black hole spacetime will involve the confluent Heun function:

$$
Hc(\alpha', w', \gamma', \delta', \sigma', z) \equiv \mathrm{Heun}C\left(4idM\sqrt{\omega^2-\mu^2}, \pm\frac{i}{M}\sqrt{4A'-M^2}, \pm\frac{i}{M}\sqrt{4C'-M^2}, -\frac{2d}{M^2}(B'+D'), \frac{1}{2}+\frac{2dB'}{M^2}, z\right) \tag{89}
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where

$$\alpha' = 4idM\sqrt{\omega^2-\mu^2}, \tag{90}$$

$$\gamma' = 1 \pm \frac{i}{M}\sqrt{4A'-M^2}, \tag{91}$$

$$\delta' = 1 \pm \frac{i}{M}\sqrt{4C'-M^2}, \tag{92}$$

$$\sigma' = \left(\frac{-2dB'}{M^2}-\frac{1}{2}\right)+\frac{1}{2}+\frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left(1+\frac{i}{M}\sqrt{4A'-M^2}\right) - \frac{1}{2}\left(1+\frac{i}{M}\sqrt{4A'-M^2}\right)\left(1+\frac{i}{M}\sqrt{4C'-M^2}\right) \tag{93}$$

$$w' = \frac{-2d}{M^2}(B'+D')+4idM\sqrt{\omega^2-\mu^2}+\frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left[\frac{i}{M}\sqrt{4A'-M^2}+\frac{i}{M}\sqrt{4C'-M^2}\right], \tag{94}$$

the variable z is given in (84) and in (89) we wrote the exact solution also in terms of the confluent Heun function: $\mathrm{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$, defined in Maple.

$$A' = A - \frac{1}{4d^2}\left(-e^2q^2M^2(1+d)^2 + 4eM^3q\omega(1+d)^2 - 2e^3qM\omega(d+1)\right)$$
$$B' = B - \frac{1}{4d^3}\left(2aemMq + e^2M^2q^2(1-d^2) + 2d^2M^4\mu^2(1+d)^2 + 4eM^3q\omega(d^3+2d^2-1) + 2e^3qM\omega\right)$$
$$C' = C - \frac{1}{4d^2}\left(2aeqmM(d-1) - e^2q^2M^2(1-d)^2 + 4eqM^3\omega(d-1)^2 + 2e^3qM\omega(d-1)\right)$$
$$D' = D - \frac{1}{4d^3}(-2aeqmM - e^2q^2M^2(1-d^2) - 2d^2M^4\mu^2(d-1)^2 + 4eq\omega M^3(1-2d^2+d^3) - 2e^3qM\omega)$$

Constraining the parameters of the theory so that the solution when expanded in terms of the confluent Kummer hypergeometric functions is right hand terminated we derive the conditions:

$$A' = A - \frac{1}{4d^2}\left(-e^2q^2M^2(1+d)^2 + 4eM^3q\omega(1+d)^2 - 2e^3qM\omega(d+1)\right)$$

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$$\delta' = 1 \pm \frac{i}{M}\sqrt{4C' - M^2} = -N \text{ or} \tag{95}$$

$$\frac{w'}{\alpha'} = \frac{\frac{-2d}{M^2}(B'+D') + 4idM\sqrt{\omega^2-\mu^2} + \frac{4idM\sqrt{\omega^2-\mu^2}}{2}\left[\frac{i}{M}\sqrt{4A'-M^2} + \frac{i}{M}\sqrt{4C'-M^2}\right]}{4idM\sqrt{\omega^2-\mu^2}} = -N \tag{96}$$

$$
\begin{aligned}
A' &= A - \frac{1}{4d^2}\left(-e^2q^2M^2(1+d)^2 + 4eM^3q\omega(1+d)^2 - 2e^3qM\omega(d+1)\right) \\
B' &= B - \frac{1}{4d^3}\left(2aemMq + e^2M^2q^2(1-d^2) + 2d^2M^4\mu^2(1+d)^2 + 4eM^3q\omega(d^3+2d^2-1) + 2e^3qM\omega\right) \\
C' &= C - \frac{1}{4d^2}\left(2aeqmM(d-1) - e^2q^2M^2(1-d)^2 + 4eqM^3\omega(d-1)^2 + 2e^3qM\omega(d-1)\right) \\
D' &= D - \frac{1}{4d^3}(-2aeqmM - e^2q^2M^2(1-d^2) - 2d^2M^4\mu^2(d-1)^2 + 4eq\omega M^3(1-2d^2+d^3) \\
&\quad - 2e^3qM\omega)
\end{aligned}
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Also if $\alpha_0 = \gamma'$ the series is right hand terminated if:

$$
\gamma' + \delta' + \left(-\frac{w'}{\alpha'}\right) = -N \tag{97}
$$

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- One of the moduli is the QCD axion. There are also other scalar moduli, the so called *string axions* with ultralight masses: e.g. from $10^{-10}\mathrm{eV} - 10^{-33}\mathrm{eV}$, and further below.

Superradiance-constraining the mass of ultralight axionic degrees of freedom

- An axion field of mass $m_A = 10^{-10}\mathrm{eV}$ has a *Compton wavelength* $\frac{h}{m_A c} = 12417\mathrm{m}$ which corresponds to the size of a black hole with a mass $m_{\mathrm{BH}} \sim 10 M_\odot$ while for an axion mass $m_A = 10^{-16}\mathrm{eV}$ its length is comparable to the length $\frac{GM_{BH}}{c^2}$ of the galactic centre *supermassive black hole* $M_{\mathrm{BH}} = 4.04 \times 10^6 M_\odot$ SgrA*.

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- The superradiant instability (SI) has been investigated by theoretical motivations (e.g. Brito *et al* 2015-LNP 906)-the *axiverse scenario* (Arvanitaki *et al* 2010, Yoshino & Kodama 2014) has provoked renewed interest in the topic because it suggests that the SI may happen in the Universe and signals from an axion field may be observed by gravitational wave detectors.

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- Thus an interesting application of our exact analytic solutions of the KGF equation in the curved spacetime of a KNdS black hole will be the investigation of superradiant instabilities in such gravitational backgrounds that can be used to constrain the mass of ultralight axionic degrees of freedon and perhaps vindicating such a scenario- especially when combined with precision measurements of the relativistic effects for the galactic centre SgrA* black hole which will determine its fundamental parameters M, a, e, Λ.

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- The closed form analytic solutions for the radial and angular equations for a massive charged scalar particle in the KN spacetime are expressed in terms of *confluent Heun functions*. The latter, under certain conditions on the parameters they reduce to a sum-with finite number of terms-of *confluent Kummer hypergeometric functions.*