Vacuum Stability of a General Scalar Potential of a Few Fields

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K. K. [arXiv:1603.02680]

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2 The Importance of Being Bounded From Below

- Scalar potentials must be bounded from below to be physical
- Limit of large fields $\phi \to \infty$
- Ignore dimensionful terms: constraints on the *quartic* part of the potential

3 The Importance of Being Bounded From Below

- Tree-level potential with running couplings (scale of validity)
- Classical scale invariance: violate at *finite* field values

4 Biquadratic Scalar Potential

For a biquadratic scalar potential

$$V(\phi) = \lambda_{ij}\phi_i^2\phi_j^2$$

the vacuum is stable if the λ_{ij} matrix is copositive

K.K., Eur. Phys. J. C72 (2012) 2093 [arXiv:1205.3781]

5 General Scalar Potential

- The general renormalisable scalar quartic potential is

$$V(\phi) = \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l$$

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- Scaling the fields ϕ by constant c gives

$$V(c\phi) = c^4 V(\phi)$$

- Hence we can take unit ϕ:

$$V(\phi) = V(\hat{\phi})\, r^4 \quad \text{with} \quad \hat{\phi}^2 = 1, \quad r \geqslant 0$$

6 General Scalar Potential

We can force unit ϕ with the help of a Lagrangian multiplier,

$$V(\phi, \lambda) = V(\phi) + \lambda \left(1 - \phi^2 \right) ,$$

yielding the minimisation equations

$$\frac{\partial V(\phi)}{\partial \phi_i} = 2\lambda\phi_i \quad \text{and} \quad \phi^2 = 1$$

7 General Scalar Potential

- When we set one or more fields to zero, the resulting potential must still be positive
- With n fields there are $2^n - 1$ different possibilities

8 General Potential of Two Real Scalars

$$
\begin{aligned}
V(\phi_1, \phi_2) &= \lambda_{ij}\phi_1^i\phi_2^j \\
&= \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4
\end{aligned}
$$

8 General Potential of Two Real Scalars

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$$

9 General Potential of Two Real Scalars

$$V_{\phi_2=0} = \lambda_{40}\phi_1^4 > 0 \quad \text{and} \quad V_{\phi_1=0} = \lambda_{04}\phi_2^4 > 0$$

imply

$$\lambda_{40} > 0 \quad \text{and} \quad \lambda_{04} > 0$$

If $\lambda_{31} = \lambda_{13} = 0$, these conditions and

$$\lambda_{22} + 2\sqrt{\lambda_{40}\lambda_{04}} > 0$$

are necessary and sufficient

10 General Potential of Two Real Scalars

Reduce the vacuum stability of the general potential

$$V(\phi_1, \phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$

to the positivity of the general quartic

$$P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

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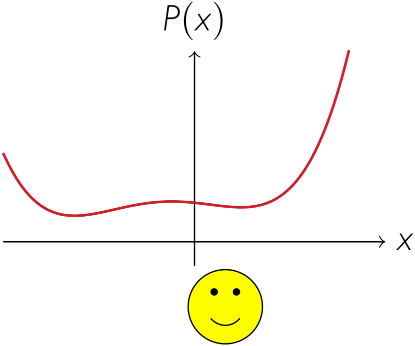
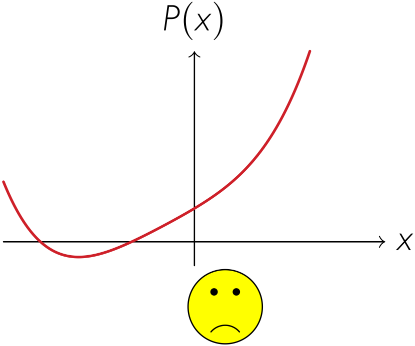
$$P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

- Dehomogenise V by $\phi_2 = 1$
- Equivalently divide V by ϕ_2^4, and choose $x = \dfrac{\phi_1}{\phi_2}$ as the variable

11 General Quartic

$$P = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$





12 General Quartic: Positivity

$$P = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

- P has no real roots if

$$D > 0 \wedge (Q > 0 \vee R > 0)$$

Rees, Amer. Math. Monthly 29 (1922) 2: 51–55;

Lazard, J. Symbolic Computat. 5 (1988) 1–2: 261–266

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- For P to be positive, also

$$a_0 > 0, \quad a_4 > 0$$

13 General Quartic: Positivity

$$
\begin{aligned}
D &= 256a_0^3a_4^3 - 4a_1^3a_3^3 - 27a_0^2a_3^4 + 16a_0a_2^4a_4 - 6a_0a_1^2a_3^2a_4 \\
&\quad - 27a_1^4a_4^2 - 192a_0^2a_1a_3a_4^2 - 4a_2^3(a_0a_3^2 + a_1^2a_4) \\
&\quad + 18a_2(a_1a_3 + 8a_0a_4)(a_0a_3^2 + a_1^2a_4) \\
&\quad + a_2^2(a_1^2a_3^2 - 80a_0a_1a_3a_4 - 128a_0^2a_4^2), \\
Q &= 8a_2a_4 - 3a_3^2, \\
R &= 64a_0a_4^3 + 16a_2a_3^2a_4 - 16a_4^2(a_2^2 + a_1a_3) - 3a_3^4
\end{aligned}
$$

14 General Potential of Two Real Scalars

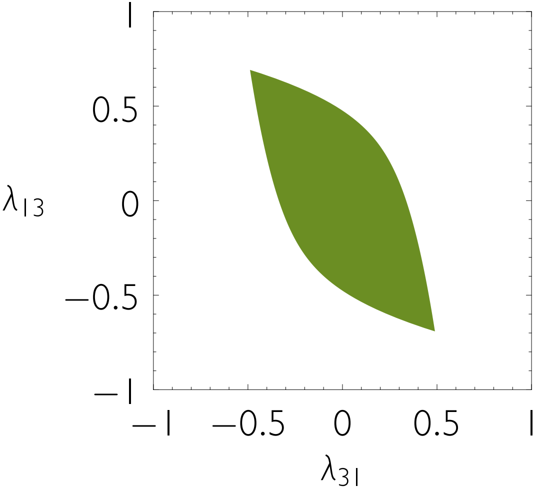
$$\lambda_{40} > 0, \quad \lambda_{04} > 0, \quad D > 0 \wedge (Q > 0 \vee R > 0),$$

where

$$\begin{aligned}
D = {} & 256\lambda_{40}^3\lambda_{04}^3 - 4\lambda_{31}^3\lambda_{13}^3 - 27\lambda_{31}^4\lambda_{04}^2 + 16\lambda_{40}\lambda_{22}^4\lambda_{04} \\
& - 6\lambda_{40}\lambda_{31}^2\lambda_{04}\lambda_{13}^2 - 27\lambda_{40}^2\lambda_{13}^4 - 192\lambda_{40}^2\lambda_{31}\lambda_{04}^2\lambda_{13} \\
& - 4\lambda_{22}^3(\lambda_{31}^2\lambda_{04} + \lambda_{40}\lambda_{13}^2) + 18\lambda_{22}(8\lambda_{40}\lambda_{04} + \lambda_{31}\lambda_{13}) \\
& \times (\lambda_{31}^2\lambda_{04} + \lambda_{40}\lambda_{13}^2) + \lambda_{22}^2(\lambda_{31}^2\lambda_{13}^2 - 80\lambda_{40}\lambda_{31}\lambda_{04}\lambda_{13} \\
& - 128\lambda_{40}^2\lambda_{04}^2), \\
Q = {} & 8\lambda_{40}\lambda_{22} - 3\lambda_{31}^2, \\
R = {} & 64\lambda_{40}^3\lambda_{04} + 16\lambda_{40}\lambda_{22}\lambda_{31}^2 - 16\lambda_{40}^2(\lambda_{22}^2 + \lambda_{31}\lambda_{13}) - 3\lambda_{31}^4
\end{aligned}$$

15 General Potential of Two Real Scalars

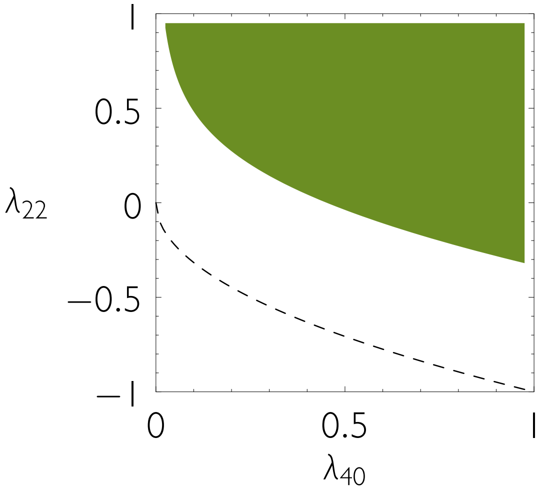
$$V(\phi_1, \phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$



- $\lambda_{40} = 0.125, \quad \lambda_{04} = 0.25, \quad \lambda_{22} = 0.125$

16 General Potential of Two Real Scalars

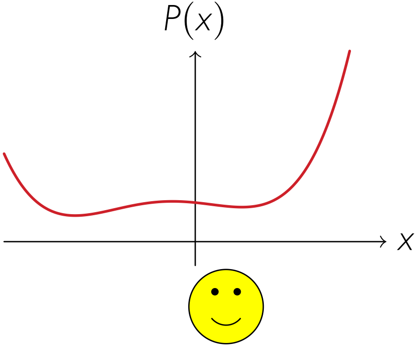
$$V(\phi_1, \phi_2) = \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$



- $\lambda_{04} = 0.25, \quad \lambda_{31} = -0.75, \quad \lambda_{13} = 0$

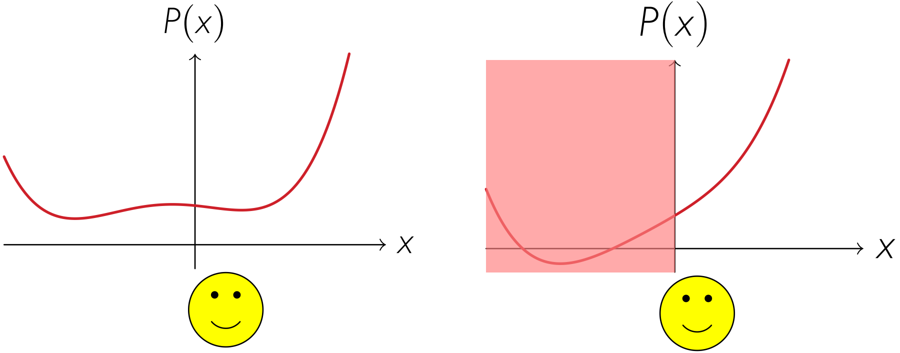
17 General Quartic: Positivity on $\mathbb{R}_+$

$$P = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$



17 General Quartic: Positivity on $\mathbb{R}_+$

$$P = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \quad x \geqslant 0$$



18 General Quartic: Positivity on $\mathbb{R}_+$

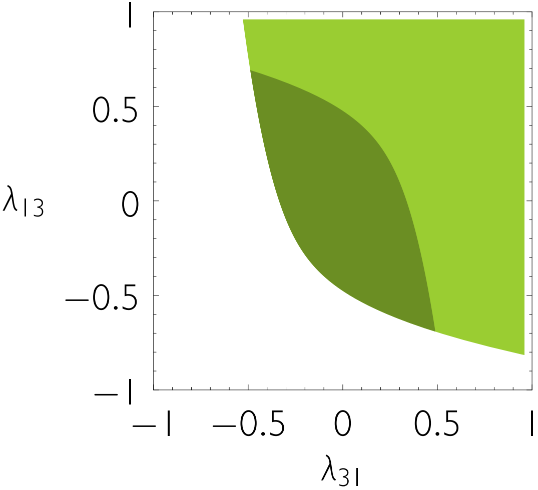
$$
\begin{aligned}
&(D \leqslant 0 \wedge a_3\sqrt{a_0} + a_1\sqrt{a_4} > 0) \\
\vee\, &(-2\sqrt{a_0 a_4} < a_2 < 6\sqrt{a_0 a_4} \wedge D \geqslant 0 \wedge \Lambda_1 \leqslant 0) \\
\vee\, &(6\sqrt{a_0 a_4} < a_2 \wedge [(a_1 > 0 \wedge a_3 > 0) \vee (D \geqslant 0 \wedge \Lambda_2 \leqslant 0)])
\end{aligned}
$$

Ulrich & Watson, TR 90-57 (1990)

19 General Quartic: Positivity on $\mathbb{R}_+$

$$\begin{aligned}
\Lambda_1 &= \left(\sqrt{a_0}a_3 - a_1\sqrt{a_4}\right)^2 - 32\,(a_0a_4)^{\frac{3}{2}} \\
&\quad - 16\left(a_0a_2a_4 + a_0^{\frac{5}{4}}a_3a_4^{\frac{3}{4}} + a_0^{\frac{3}{4}}a_1a_4^{\frac{5}{4}}\right), \\
\Lambda_2 &= \left(\sqrt{a_0}a_3 - a_1\sqrt{a_4}\right)^2 - 4\sqrt{a_0a_4}\frac{a_2 + 2\sqrt{a_0a_4}}{\sqrt{a_2 - 2\sqrt{a_0a_4}}} \\
&\quad \times \left(\sqrt{a_0}a_3 + a_1\sqrt{a_4} + 4\sqrt{a_0a_4}\sqrt{a_2 - 2\sqrt{a_0a_4}}\right)
\end{aligned}$$

Two Non-Negative Scalars



- $\lambda_{40} = 0.125, \quad \lambda_{04} = 0.25, \quad \lambda_{22} = 0.125$

21 Two Real Scalars & the Higgs Boson

$$
\begin{aligned}
V &= \lambda_H |H|^4 + \left(\lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2\right) |H|^2 \\
&+ \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4 \\
&\equiv \lambda_H |H|^4 + M^2(\phi_1, \phi_2)|H|^2 + V(\phi_1, \phi_2)
\end{aligned}
$$

22 Two Real Scalars & the Higgs Boson

$$V_{\phi_1=\phi_2=0} = \lambda_H |H|^4 > 0$$

implies

$$\lambda_H > 0$$

and

$$V_{|H|^2=0} = V(\phi_1, \phi_2) > 0$$

implies

$$\lambda_{40} > 0, \quad \lambda_{04} > 0,$$
$$D_{|H|^2=0} > 0 \wedge (Q_{|H|^2=0} > 0 \vee R_{|H|^2=0} > 0)$$

23 Two Real Scalars & the Higgs Boson

$$V = \lambda_H |H|^4 + M^2 |H|^2 + V_{|H|^2=0}$$

$$0 = \frac{\partial V}{\partial |H|^2} = 2\lambda_H |H|^2 + M^2,$$

which gives

$$|H|^2_{\text{min}} = -\frac{1}{2\lambda_H} M^2$$

- $|H|^2_{\text{min}}$ is physical if $M^2 \leqslant 0$

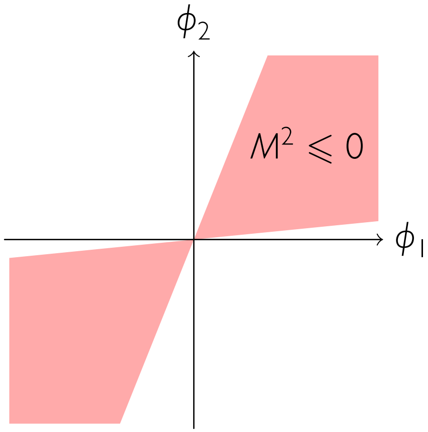
24 Two Real Scalars & the Higgs Boson

$$M^2 > 0 \quad \vee \quad V_{|H|^2=|H|^2_{\rm min}} = V_{|H|^2=0} - \frac{1}{4\lambda_H}M^4 > 0$$

is equivalent to

$$M^2 \leqslant 0 \quad \Rightarrow \quad V_{|H|^2=|H|^2_{\rm min}} = V_{|H|^2=0} - \frac{1}{4\lambda_H}M^4 > 0$$

25 Two Real Scalars & the Higgs Boson



$$M^2(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2$$

26 Two Real Scalars & the Higgs Boson

The coefficient matrix of

$$M^2(\phi_1, \phi_2) = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2$$

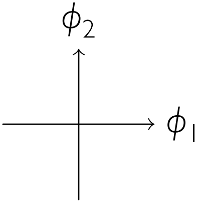
is given by

$$\mathbf{M}^2 = \begin{pmatrix} \lambda_{H20} & \frac{1}{2}\lambda_{H11} \\ \frac{1}{2}\lambda_{H11} & \lambda_{H02} \end{pmatrix}$$

27 Two Real Scalars & the Higgs Boson

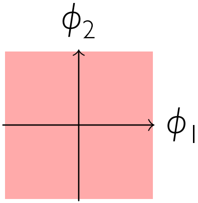
- Positive-definite if

$$\lambda_{H20} > 0, \quad \lambda_{H02} > 0, \quad 4\lambda_{H20}\lambda_{H02} > \lambda_{H11}^2$$



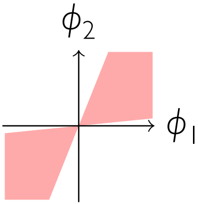
- Negative-definite if

$$\lambda_{H20} < 0, \quad \lambda_{H02} < 0, \quad 4\lambda_{H20}\lambda_{H02} > \lambda_{H11}^2$$



- Neither if

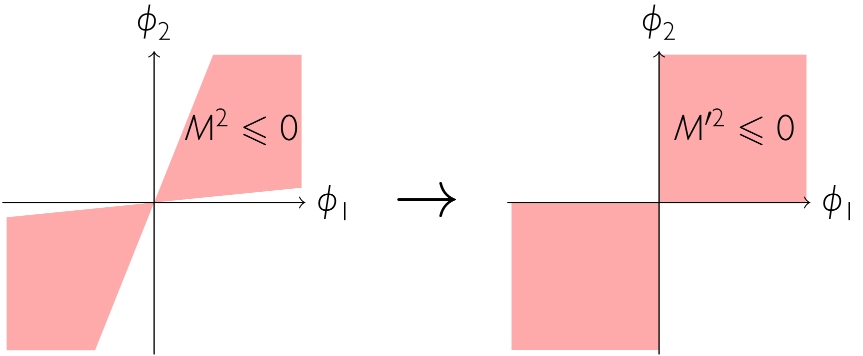
$$\det \mathbf{M}^2 < 0 \quad \text{or} \quad 4\lambda_{H20}\lambda_{H02} < \lambda_{H11}^2$$



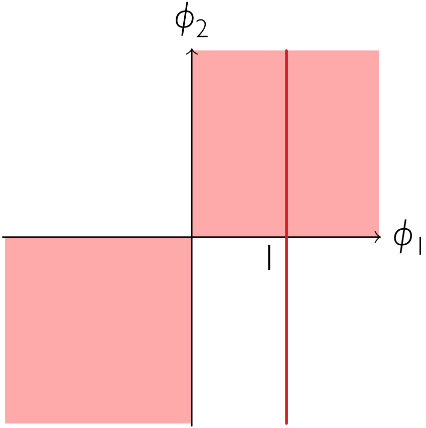
Two Real Scalars & the Higgs Boson

If $\det \mathbf{M}^2 < 0$, we can always bring M^2 into the anti-diagonal form

$$M^2 = \lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2 \quad \longrightarrow \quad M'^2 = \lambda'_{H11}\phi_1\phi_2$$



Two Real Scalars & the Higgs Boson



- Use positivity of $P(x), x \geqslant 0$ for $V'_{|H|^2=|H|^2_{\text{min}}}(\phi_1 = 1, \phi_2)$

$$\mathbf{M}^2 = \begin{pmatrix} \lambda_{H20} & \frac{1}{2}\lambda_{H11} \\ \frac{1}{2}\lambda_{H11} & \lambda_{H02} \end{pmatrix} \xrightarrow{\mathbf{U}_\theta} \begin{pmatrix} \lambda'_{H2,0} & 0 \\ 0 & \lambda'_{H0,2} \end{pmatrix}$$
$$\xrightarrow{\mathbf{S}} \frac{1}{2}\,\mathrm{sgn}(\lambda_{H02} - \lambda_{H20}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\xrightarrow{\mathbf{U}_{\frac{\pi}{4}}} \frac{1}{2}\,\mathrm{sgn}(\lambda_{H02} - \lambda_{H20}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Altogether, we rotate the fields by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \mathbf{U}_\theta^T \mathbf{S}\, \mathbf{U}_{\frac{\pi}{4}} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

obtaining $V_{|H|^2=|H|^2_{\mathrm{min}}} \to V'_{|H|^2=|H|^2_{\mathrm{min}}}$

Two Real Scalars & the Higgs Boson

In the new basis

$$M'^2 = \lambda'_{H1,1}\phi_1\phi_2 = \mathrm{sgn}(\lambda_{H02} - \lambda_{H20})\phi_1\phi_2$$

and the minimised potential has the form

$$V'_{|H|^2=|H|^2_{\min}} \equiv \lambda'_{40}\phi_1^4 + \lambda'_{31}\phi_1^3\phi_2 + \lambda'_{22}\phi_1^2\phi_2^2 + \lambda'_{13}\phi_1\phi_2^3 + \lambda'_{04}\phi_2^4$$

- If $\lambda_{H02} = \lambda_{H20} = 0$, then $\lambda'_{H11} = \lambda_{H11}$ and $V'_{|H|^2=|H|^2_{\min}} = V_{|H|^2=|H|^2_{\min}}$
- If $\lambda_{H11} = 0$, then $\lambda'_{H11} = \mathrm{sgn}(\lambda_{H02} - \lambda_{H20}) = \mathrm{sgn}\,\lambda_{H02}$ and $V'_{|H|^2=|H|^2_{\min}}$ has a simpler form

32 Two Real Scalars & the Higgs Boson

$$
\begin{aligned}
&\lambda_{40} > 0 \wedge \lambda_{04} > 0 \wedge \lambda_H > 0 \wedge D_{|H|^2=0} \\
&\ \wedge \left(Q_{|H|^2=0} > 0 \vee R_{|H|^2=0} > 0\right), \\
&\lambda_{H20} \leqslant 0 \wedge \lambda_{H02} \leqslant 0 \wedge \lambda_{H11}^2 \leqslant 4\lambda_{H20}\lambda_{H02} \Rightarrow \\
&4\lambda_H\lambda_{40} - \lambda_{H20}^2 > 0 \wedge 4\lambda_H\lambda_{04} - \lambda_{H02}^2 > 0 \wedge D_{|H|^2=|H|^2_{\min}} \\
&\ \wedge \left(Q_{|H|^2=|H|^2_{\min}} > 0 \vee R_{|H|^2=|H|^2_{\min}} > 0\right), \\
&\ \ldots
\end{aligned}
$$

33 Two Real Scalars & the Higgs Boson

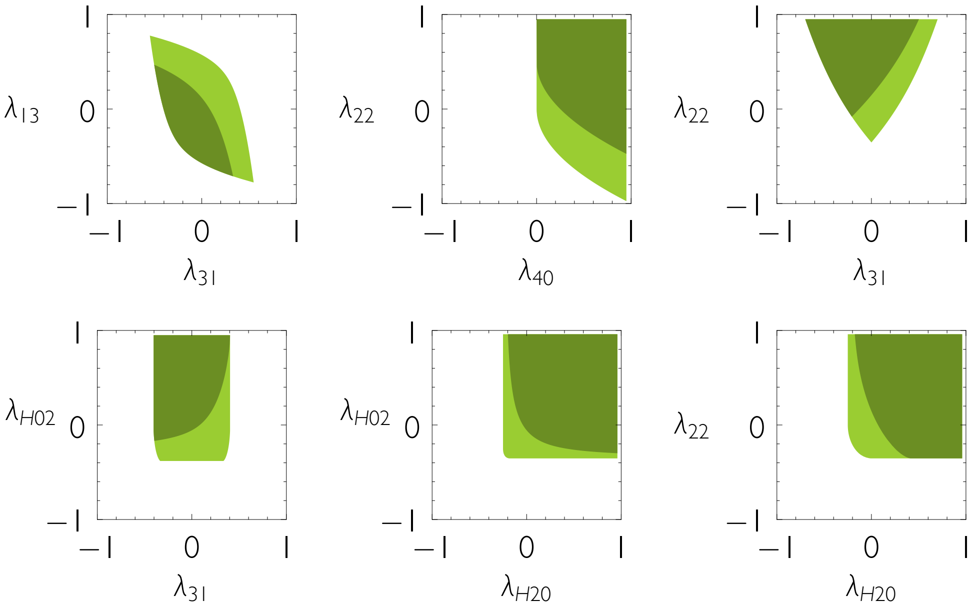
$$
\begin{aligned}
&\lambda_{H11}^2 > 4\lambda_{H20}\lambda_{H02} \Rightarrow \lambda'_{04} > 0 \wedge \lambda'_{40} > 0 \\
&\wedge \Bigg[\bigg(D_{|H|^2=|H|^2_{\min}} \leqslant 0 \wedge \left(\lambda'_{31}\sqrt{\lambda'_{04}} + \lambda'_{13}\sqrt{\lambda'_{40}}\right) > 0\bigg) \\
&\vee \bigg(-2\sqrt{\lambda'_{04}\lambda'_{40}} < \lambda'_{22} < 6\sqrt{\lambda'_{04}\lambda'_{40}} \wedge D_{|H|^2=|H|^2_{\min}} \geqslant 0 \\
&\wedge L'_{1\;|H|^2=|H|^2_{\min}} \leqslant 0\bigg) \vee \bigg(6\sqrt{\lambda'_{04}\lambda'_{40}} < \lambda'_{22} \\
&\wedge [(\lambda'_{13} > 0 \wedge \lambda'_{31} > 0) \\
&\vee (D_{|H|^2=|H|^2_{\min}} \geqslant 0 \wedge L'_{2\;|H|^2=|H|^2_{\min}} \leqslant 0)]\bigg)\Bigg]
\end{aligned}
$$

Two Real Scalars & the Higgs Boson

$$V = \lambda_H |H|^4 + \left(\lambda_{H20}\phi_1^2 + \lambda_{H11}\phi_1\phi_2 + \lambda_{H02}\phi_2^2\right)\,|H|^2$$
$$+ \lambda_{40}\phi_1^4 + \lambda_{31}\phi_1^3\phi_2 + \lambda_{22}\phi_1^2\phi_2^2 + \lambda_{13}\phi_1\phi_2^3 + \lambda_{04}\phi_2^4$$

- $\lambda_H = 0.125$, $\lambda_{40} = 0.125$, $\lambda_{04} = 0.25$, $\lambda_{31} = \lambda_{13} = \lambda_{H20} = \lambda_{H02} = 0$, except where they vary
- $\lambda_{H11} = 0$: light & dark green
- $\lambda_{H11} = 0.5$: dark green

35 Two Real Scalars & the Higgs Boson



36 Conclusions

- Whether a polynomial is bounded from below is a hard question of algebraic geometry
- Analytical vacuum stability conditions can be derived if the number of fields is small or the theory has a large symmetry
- We derived analytical conditions for SM Higgs & two real singlets
- The methods can applied to a wide range of potentials with higher multiplets as well

37 Thank You!

38 Two Higgs Doublet Model

$$V = \lambda_1 |H_1|^4 + \lambda_2 |H_2|^4 + \lambda_3 |H_1|^2 |H_2|^2 \\ + \lambda_4 (H_1^\dagger H_2)(H_2^\dagger H_1) + \frac{1}{2}\lambda_5 \left[(H_1^\dagger H_2)^2 + (H_2^\dagger H_1)^2 \right] \\ + \lambda_6 |H_1|^2 (H_1^\dagger H_2 + H_2^\dagger H_1) + \lambda_7 |H_2|^2 (H_1^\dagger H_2 + H_2^\dagger H_1)$$

- We assume no explicit CP violation and take all the couplings real
- General vacuum stability conditions given in 'light-cone' formalism

  Maniatis et al., Eur. Phys. J. C48 (2006) 805–823 [hep-ph/0605184]; Ivanov, Phys. Rev. D75 (2007) 035001 [hep-ph/0609018]

Two Higgs Doublet Model

$$|H_1|^2 = h_1^2$$
$$|H_2|^2 = h_2^2$$
$$H_1^\dagger H_2 = h_1 h_2 \rho e^{i\phi}$$

- $\rho \in [0, 1]$ due to $0 \leqslant |H_1^\dagger H_2| \leqslant |H_1||H_2|$
- $\phi \in [0, 2\pi]$

Ginzburg & Krawczyk, Phys. Rev. D72 (2005) 115013 [hep-ph/0408011]

Two Higgs Doublet Model

$$
\begin{aligned}
V = {} & \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 \\
& + \lambda_4 \rho^2 \, h_1^2 h_2^2 + \lambda_5 \rho^2 \cos 2\phi \, h_1^2 h_2^2 \\
& + 2\lambda_6 \rho \cos\phi \, h_1^3 h_2 + 2\lambda_7 \rho \cos\phi \, h_1 h_2^3
\end{aligned}
$$

40 Two Higgs Doublet Model

$$V = \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 + \lambda_4 \rho^2 h_1^2 h_2^2 + \lambda_5 \rho^2 \cos 2\phi \, h_1^2 h_2^2 + 2\lambda_6 \rho \cos\phi \, h_1^3 h_2 + 2\lambda_7 \rho \cos\phi \, h_1 h_2^3$$

Two Higgs Doublet Model

- Apply positivity conditions for quartic $P(x)$ with $x \geqslant 0$?
- Practically impossibly to minimise D, Λ_1, Λ_2 with respect to orbit space variables ρ and ϕ
- Minimise with respect to all variables
- Use the positivity conditions of a general quartic for edge cases

42 Two Higgs Doublet Model

$$\begin{aligned} V = {} & \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2 \\ & + (\lambda_4 + \lambda_5 \cos 2\phi)\rho^2 h_1^2 h_2^2 \\ & + 2\lambda_6 \rho \cos\phi\, h_1^3 h_2 + 2\lambda_7 \rho \cos\phi\, h_1 h_2^3 \\ & + \lambda(1 - h_1^2 - h_2^2) \end{aligned}$$

43 2HDM: Minimisation

$$
\begin{aligned}
0 &= h_1 h_2 \rho \left(2\lambda_5 \rho h_1 h_2 \cos\phi + \lambda_6 h_1^2 + \lambda_7 h_2^2\right) \sin\phi, \\
0 &= h_1 h_2 \left[\left(\lambda_4 + \lambda_5 \cos 2\phi\right) \rho h_1 h_2 + \left(\lambda_6 h_1^2 + \lambda_7 h_2^2\right) \cos\phi\right], \\
\lambda h_1 &= 2\lambda_1 h_1^3 + \left[\lambda_3 + \left(\lambda_4 + \lambda_5 \cos 2\phi\right) \rho^2\right] h_1 h_2^2 \\
&\quad + 3\lambda_6 \rho \cos\phi \, h_1^2 h_2 + \lambda_7 \rho \cos\phi \, h_2^3, \\
\lambda h_2 &= 2\lambda_2 h_2^3 + \left[\lambda_3 + \left(\lambda_4 + \lambda_5 \cos 2\phi\right) \rho^2\right] h_1^2 h_2 \\
&\quad + \lambda_6 \rho \cos\phi \, h_1^3 + 3\lambda_7 \rho \cos\phi \, h_1 h_2^2, \\
1 &= h_1^2 + h_2^2
\end{aligned}
$$

2HDM: Minimisation

$$
\begin{aligned}
0 &= h_1 h_2 \rho \left(2\lambda_5 \rho h_1 h_2 \cos\phi + \lambda_6 h_1^2 + \lambda_7 h_2^2\right) \sin\phi, \\
0 &= h_1 h_2 \left[\left(\lambda_4 + \lambda_5 \cos 2\phi\right) \rho h_1 h_2 + \left(\lambda_6 h_1^2 + \lambda_7 h_2^2\right) \cos\phi\right], \\
\lambda h_1 &= 2\lambda_1 h_1^3 + \left[\lambda_3 + \left(\lambda_4 + \lambda_5 \cos 2\phi\right) \rho^2\right] h_1 h_2^2 \\
&\quad + 3\lambda_6 \rho \cos\phi \, h_1^2 h_2 + \lambda_7 \rho \cos\phi \, h_2^3, \\
\lambda h_2 &= 2\lambda_2 h_2^3 + \left[\lambda_3 + \left(\lambda_4 + \lambda_5 \cos 2\phi\right) \rho^2\right] h_1^2 h_2 \\
&\quad + \lambda_6 \rho \cos\phi \, h_1^3 + 3\lambda_7 \rho \cos\phi \, h_1 h_2^2, \\
1 &= h_1^2 + h_2^2
\end{aligned}
$$

2HDM: Solution for ϕ

The equation

$$\sin\phi = 0$$

yields

$$\phi = 0 \quad \text{or} \quad \phi = \pi$$

or

$$\cos\phi = \pm 1$$

45 2HDM: Extremum

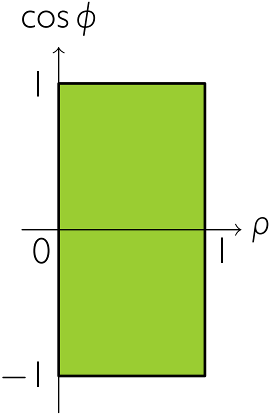
$$\rho^2 = \frac{(\lambda_6 h_1^2 + \lambda_7 h_2^2)^2}{h_1^2 h_2^2 (\lambda_4 + \lambda_5)^2},$$

$$h_1^2 = \frac{1}{2}\frac{(2\lambda_2 - \lambda_3)(\lambda_4 + \lambda_5) + 2\lambda_7(\lambda_6 - \lambda_7)}{(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_4 + \lambda_5) - (\lambda_6 - \lambda_7)^2},$$

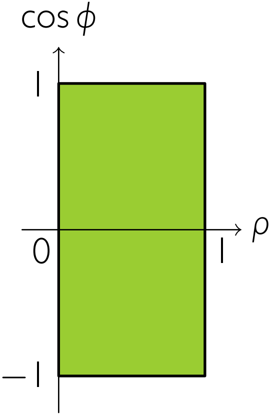
$$h_2^2 = \frac{1}{2}\frac{(2\lambda_1 - \lambda_3)(\lambda_4 + \lambda_5) - 2\lambda_6(\lambda_6 - \lambda_7)}{(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_4 + \lambda_5) - (\lambda_6 - \lambda_7)^2},$$

$$V_{\min} = \frac{1}{4}\frac{(\lambda_4 + \lambda_5)(4\lambda_1\lambda_2 - \lambda_3^2) - 4(\lambda_1\lambda_7^2 + \lambda_2\lambda_6^2 - \lambda_3\lambda_6\lambda_7)}{(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_4 + \lambda_5) - (\lambda_6 - \lambda_7)^2}$$

46 2HDM: Edges of Orbit Space



46 2HDM: Edges of Orbit Space



- We have to check $\rho = 0$ and $\cos\phi = \pm 1$, $\rho = 1$

47 2HDM: $\rho = 0$

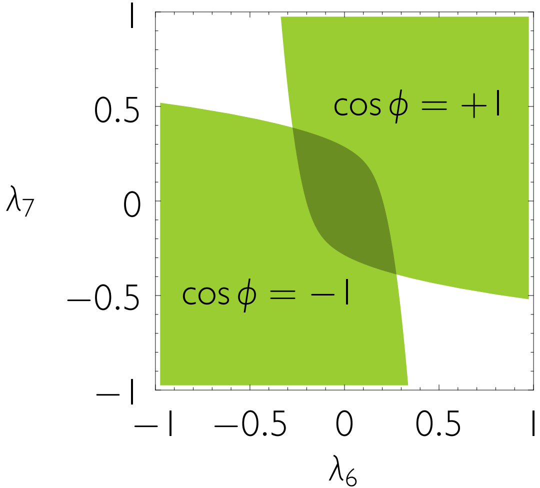
$$V_{\rho=0} = \lambda_1 h_1^4 + \lambda_2 h_2^4 + \lambda_3 h_1^2 h_2^2$$

yields

$$\lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_3 + 2\sqrt{\lambda_1 \lambda_2} > 0$$

2HDM: $\cos\phi = \pm 1, \rho = 1$

$$V_{\cos\phi=\pm 1,\rho=1} = \lambda_1 h_1^4 + \lambda_2 h_2^4 + (\lambda_3+\lambda_4+\lambda_5) h_1^2 h_2^2 \pm \left(2\lambda_6 h_1^3 h_2 + 2\lambda_7 h_1 h_2^3\right)$$



49 2HDM: $\cos\phi = \pm 1, \rho = 1$

$$V_{\cos\phi=\pm 1,\rho=1} = \lambda_1 h_1^4 + \lambda_2 h_2^4 + (\lambda_3 + \lambda_4 + \lambda_5) h_1^2 h_2^2 \\ \pm \left(2\lambda_6 h_1^3 h_2 + 2\lambda_7 h_1 h_2^3\right)$$

yields

$$D_{\cos\phi=\pm 1,\rho=1} \wedge (Q_{\cos\phi=\pm 1,\rho=1} > 0 \vee R_{\cos\phi=\pm 1,\rho=1} > 0)$$

2HDM: $\cos\phi = \pm 1, \rho = 1$

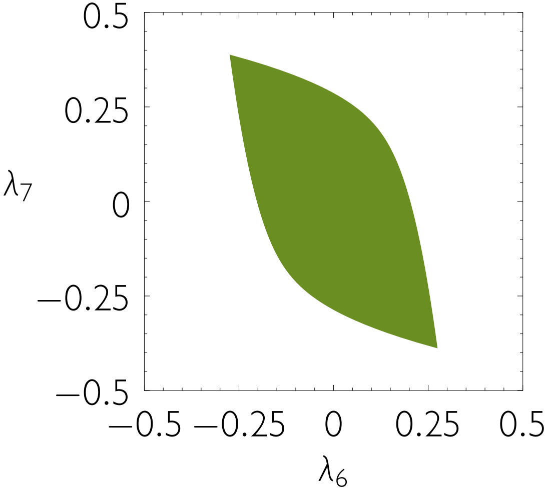
$$\begin{aligned}
D_{\cos\phi=\pm 1,\rho=1} &= 16[16\lambda_1^3\lambda_2^3 + \lambda_1\lambda_2\lambda_{345}^4 - 27\lambda_2^2\lambda_6^4 \\
&- 48\lambda_1^2\lambda_2^2\lambda_6\lambda_7 - 6\lambda_1\lambda_2\lambda_6^2\lambda_7^2 \\
&- 16\lambda_6^3\lambda_7^3 - 27\lambda_1^2\lambda_7^4 - \lambda_{345}^3(\lambda_2\lambda_6^2 + \lambda_1\lambda_7^2) \\
&+ 18\lambda_{345}(2\lambda_1\lambda_2 + \lambda_6\lambda_7)(\lambda_2\lambda_6^2 + \lambda_1\lambda_7^2) \\
&+ \lambda_{345}^2(-8\lambda_1^2\lambda_2^2 - 20\lambda_1\lambda_2\lambda_6\lambda_7 + \lambda_6^2\lambda_7^2)], \\
Q_{\cos\phi=\pm 1,\rho=1} &= 8\lambda_1\lambda_{345} - 12\lambda_6^2, \\
R_{\cos\phi=\pm 1,\rho=1} &= 16[4\lambda_1^3\lambda_2 + 4\lambda_1\lambda_{345}\lambda_6^2 - 3\lambda_6^4 \\
&- \lambda_1^2(\lambda_{345}^2 + 4\lambda_6\lambda_7)],
\end{aligned}$$

where $\lambda_{345} \equiv \lambda_3 + \lambda_4 + \lambda_5$

2HDM: Full Conditions

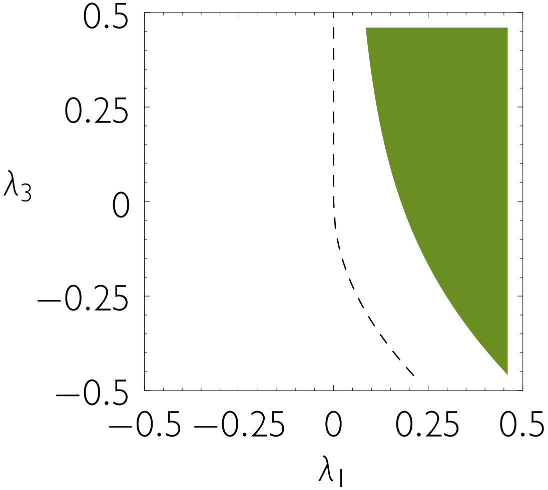
$$\lambda_1 > 0 \;\wedge\; \lambda_2 > 0 \;\wedge\; \lambda_3 + 2\sqrt{\lambda_1 \lambda_2} > 0$$
$$\wedge\, D_{\cos\phi=\pm 1,\rho=1} \wedge \left(Q_{\cos\phi=\pm 1,\rho=1} > 0 \vee R_{\cos\phi=\pm 1,\rho=1} > 0\right)$$
$$\wedge \left(0 < h_1^2 < 1 \wedge 0 < h_2^2 < 1 \wedge 0 < \rho^2 < 1 \Rightarrow V_{\min} > 0\right)$$

52 Two Higgs Doublet Model



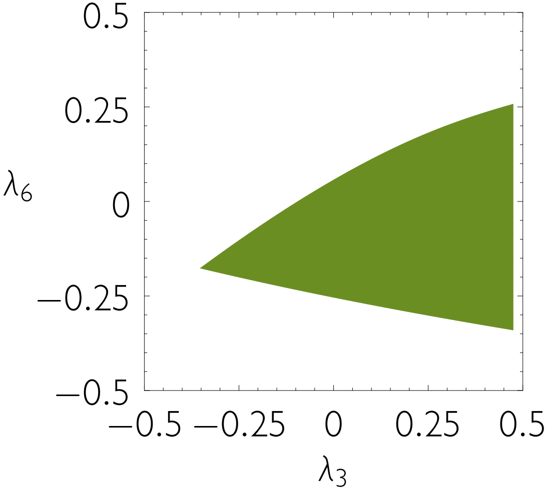
- $\lambda_1 = 0.125, \lambda_2 = 0.25, \lambda_3 = 0, \lambda_4 = 0.25, \lambda_5 = 0$

Two Higgs Doublet Model



- $\lambda_2 = 0.25, \lambda_4 = 0.25, \lambda_5 = 0, \lambda_6 = 0.25, \lambda_7 = 0$

Two Higgs Doublet Model



- $\lambda_1 = 0.125$, $\lambda_2 = 0.25$, $\lambda_4 = 0.25$, $\lambda_5 = 0$, $\lambda_7 = 0.25$