Vilkovisky–De Witt Effective Action Approach to Scalar-Curvature Inflation Theories

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based on

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Introduction: Motivation

- Inflation is one possibility to solve the horizon and flatness problems in cosmology. It also serves as an explanation for anisotropies in the CMB.
- In the search for viable inflation models, one particular class of theories have emerged: **scalar-curvature theories**.
- We have developed a new formalism, the **frame-covariant formalism**, to calculate cosmological observables with greater ease at the classical level for this class.
- The computation of cosmological observables requires the quantisation of the metric and inflaton perturbations. Due to the quantisation, there may be **non-negligible radiative corrections** to the observables.
- The **Vilkovisky–De Witt formalism** is a starting point for the analysis of frame transformations at the quantum-loop level.

Scalar-curvature theories

- Defined by class of actions:

$$S[g_{\mu\nu},\varphi,f(\varphi),k(\varphi),V(\varphi)] \equiv \int d^4x\sqrt{-g}\left[-\frac{f(\varphi)}{2}R+\frac{k(\varphi)}{2}g^{\mu\nu}(\nabla_\mu\varphi)(\nabla_\nu\varphi)-V(\varphi)\right],$$

- Free to make conformal transformations and inflaton reparametrizations (*frame transformations*):

$$\tilde{g}_{\mu\nu}=\Omega^2(\varphi)g_{\mu\nu}\,, \qquad \left(\frac{d\tilde{\varphi}}{d\varphi}\right)^2 = K(\varphi)\,.$$

- Result of transformation [Flanagan, 2004; Jarv et al, 2014]

$$\tilde{f}(\tilde{\varphi})=\Omega^{-2}f, \quad \tilde{k}(\tilde{\varphi})=\frac{\Omega^{-2}}{K}\left(k-\,6f\,\Omega^{-2}\Omega_{,\varphi}^2\,+\,6\,\Omega^{-1}f_{,\varphi}\,\Omega_{,\varphi}\right), \quad \widetilde{V}(\tilde{\varphi})=\Omega^{-4}\,V$$

- Hence, one finds form invariance:

$$S[g_{\mu\nu},\varphi,f(\varphi),k(\varphi),V(\varphi)] \;=\; S[\tilde{g}_{\mu\nu},\tilde{\varphi},\tilde{f}(\tilde{\varphi}),\tilde{k}(\tilde{\varphi}),\widetilde{V}(\tilde{\varphi})]\,.$$

- This is the starting point to the understanding of the frame-covariance of the cosmological observables. See companion talk by S. Karamitsos (full title in Summary slide).

Beyond the tree-level approximation

- Beyond the tree-level, we require a similar starting point to the classical case.
- With the inclusion of radiative corrections, the classical action is promoted to the effective action Γ.
- For illustration, we shall neglect quantum gravitational corrections and inflaton couplings to other matter fields.
- Focus on inflaton reparametrisations for this talk.
- Functional integro-differential equation for Γ:

$$\exp\left(\frac{i}{\hbar}\Gamma[g_{\mu\nu},\varphi]\right) = \int \mathcal{D}\varphi^Q \mathcal{M}[\varphi^Q] \exp\left(\frac{i}{\hbar}\left[S[g_{\mu\nu},\varphi^Q] - \int d^4x\,(\varphi-\varphi^Q)\frac{\delta\Gamma[g_{\mu\nu},\varphi]}{\delta\varphi}\right]\right),$$

- Solving this perturbatively in $\hbar$ to $O(\hbar)$, we obtain

$$\Gamma_0[g_{\mu\nu},\varphi] = S[g_{\mu\nu},\varphi], \qquad \Gamma_1[g_{\mu\nu},\varphi] = \ln\mathcal{M}[\varphi] - \frac{1}{2}\ln\det\left(\frac{\delta^2 S[g_{\mu\nu},\varphi]}{\delta\varphi(x)\delta\varphi(y)}\right).$$

The one-loop effective action

- Under an inflaton reparametrization:

$$\Gamma_1[\tilde{\varphi}, \tilde{k}(\tilde{\varphi})] = \ln \widetilde{\mathcal{M}}[\tilde{\varphi}] - \frac{1}{2} \ln \det \left(\frac{\delta^2 S[\tilde{\varphi}, \tilde{k}(\tilde{\varphi})]}{\delta\tilde{\varphi}(x)\delta\tilde{\varphi}(y)} \right)$$

where

$$\widetilde{\mathcal{M}}[\tilde{\varphi}] \equiv \det\left(K_x^{-1/2}\delta(x-y)\right)\mathcal{M}[\varphi(\tilde{\varphi})],$$

$$\frac{\delta^2 S[\tilde{\varphi}, \tilde{k}(\tilde{\varphi})]}{\delta\tilde{\varphi}(x)\delta\tilde{\varphi}(y)} = K_x^{-1/2}K_y^{-1/2}\left[\frac{\delta^2 S[\varphi, k(\varphi)]}{\delta\varphi(x)\delta\varphi(y)} - \frac{K_x^{-1/2}}{2}(\ln K_x)_{,\varphi}\frac{\delta S[\varphi, k(\varphi)]}{\delta\varphi(x)}\delta(x-y)\right].$$

$$K_x \equiv K(\varphi(x))$$

- Second functional derivative does not transform covariantly under inflaton reparametrizations. Hence

$$\Gamma_1[\tilde{\varphi}, \tilde{k}(\tilde{\varphi})] \neq \Gamma_1[\varphi, k(\varphi)]$$

The functional covariant derivative

- How do we fully parametrize the non-covariant term?
- Take inspiration from differential geometry and introduce a connection term via a version of the covariant derivative [Vilkovisky, 1983]:

$$\frac{D^2 S}{D\varphi(x) D\varphi(y)} \equiv \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} - \Gamma^z_{xy}\frac{\delta S}{\delta\varphi(z)} ,$$

- Utilise Einstein–De Witt summation convention: repeated spacetime indices are integrated over all spacetime.
- If we require that covariant derivatives transform covariantly under inflaton reparametrizations, then the connection must transform as

$$\widetilde{\Gamma}^z_{xy} = K_z^{1/2} K_x^{-1/2} K_y^{-1/2}\left[\Gamma^z_{xy} - \frac{1}{2}(\ln K_x)_{,\varphi}\delta(x-y)\delta(y-z)\right]$$

therefore

$$\frac{\widetilde{D}^2 S[\tilde{\varphi}, \tilde{k}(\tilde{\varphi})]}{\widetilde{D}\varphi(x)\widetilde{D}\varphi(y)} = K_x^{-1/2} K_y^{-1/2}\frac{D^2 S[\varphi]}{D\varphi(x) D\varphi(y)} .$$

The Vilkovisky–De Witt effective action

- Using the covariant derivative, define the 1-loop **Vilkovisky–De Witt effective action**

$$\Gamma_1^{\mathrm{VD}}[\varphi, k(\varphi), \Gamma^z_{xy}] \;\equiv\; \ln \mathcal{M}[\varphi] \;-\; \frac{1}{2} \ln \det \left(\frac{D^2 S[\varphi, k(\varphi)]}{D\varphi(x) D\varphi(y)} \right) .$$

- With the added connection term, we have successfully parametrized the non-covariant term into a transformation of the connection. Consequently, we have

$$\Gamma_1^{\mathrm{VD}}[\tilde{\varphi}, \tilde{k}(\tilde{\varphi}), \tilde{\Gamma}^z_{xy}] \;=\; \Gamma_1^{\mathrm{VD}}[\varphi, k(\varphi), \Gamma^z_{xy}] \; .$$

- The next question is: what are the forms of the measure functional $\mathcal{M}$ and the connection Γ^z_{xy}?

The field space metric

- In differential geometry, the invariant integral measure and the affine connection are written in terms of a metric tensor.
- Therefore, we may construct $\mathcal{M}$ and Γ^z_{xy} in terms of some metric $\mathcal{G}_{xy}$ for the field space. These expressions then take the form

$$\mathcal{M}[\varphi] \equiv \sqrt{\det \mathcal{G}_{xy}}\,, \qquad \Gamma^z_{xy} \equiv \frac{1}{2}\mathcal{G}^{zw}\left(\frac{\delta \mathcal{G}_{wx}}{\delta\varphi(y)} + \frac{\delta \mathcal{G}_{wy}}{\delta\varphi(x)} - \frac{\delta \mathcal{G}_{xy}}{\delta\varphi(w)}\right),$$

with $\mathcal{G}^{yz}\mathcal{G}_{zx} = \delta(x-y)$.
- Metric must transform as

$$\widetilde{\mathcal{G}}_{xy} = K_x^{-1/2}K_y^{-1/2}\mathcal{G}_{xy}\,, \qquad \widetilde{\mathcal{G}}^{xy} = K_x^{1/2}K_y^{1/2}\mathcal{G}^{xy}\,,$$

- Path-integral quantization via the Hamiltonian gives the measure functional:

$$\mathcal{M}[\varphi] = \det\left(k^{1/2}(\varphi)\delta(x-y)\right)$$

- This gives the **unique** choice

$$\mathcal{G}_{xy} = k(\varphi)\delta(x-y)$$

- Consequently, the connection becomes

$$\Gamma^z_{xy} = \frac{1}{2}\big(\ln k(\varphi)\big)_{,\varphi}\,\delta(x-y)\delta(y-z)$$

Beyond one-loop: De Witt's modification

- One can find expressions for higher orders of the effective action beyond one-loop.
- Every term in the perturbative expansion depends upon field derivatives of the classical action S.
- As demonstrated at the one-loop level, the way to parametrize higher orders is to introduce a covariant derivative.
- Consequently, the connection accounts for all non-covariant terms in the perturbative expansion and it may be shown that

$$\Gamma^{VD}[\tilde{\varphi}, \tilde{k}(\varphi), \tilde{\Gamma}^{z}_{xy}] \;=\; \Gamma^{VD}[\varphi, k(\varphi), \Gamma^{z}_{xy}]$$

  if the covariant derivative procedure is performed at each order in $\hbar$.
- Rigorously, one can obtain the same result by replacing $\varphi - \varphi^{Q}$ in the functional integro-differential equation with a general function $\sigma_x(\varphi, \varphi^{Q})$ which transforms as

$$\sigma_x(\tilde{\varphi}, \varphi^{Q}) = K_x^{1/2} \sigma_x(\varphi, \varphi^{Q})$$

Summary

- Form-invariance exists at the level of the classical action and this is used to show frame-covariance of the cosmological observables (see **”Frame-Covariant Formalism of Inflation in the Slow-Roll Approximation”** by **S. Karamitsos**)
- Corresponding effective action is **not automatically form-invariant**: there are non-covariant terms that are not parametrised explicitly.
- The introduction of the **covariant derivative** and the connection term allows for parametrisation of the non-covariant terms.
- Given that the measure functional is explicitly calculated during the path-integral quantisation procedure, identify this with the metric expression to determine the metric.
- It is possible to generalise this beyond the one-loop level using De Witt’s approach.
- The result is a form-invariant effective action, as long as the connection transforms appropriately. This creates a viable starting point for the study of frame transformations at the quantum-loop level.

Backup slide: Hamiltonian derivation of the measure

- The Hamiltonian density for scalar-curvature theories:

$$\mathcal{H} = \dot{\varphi}(\Pi)\Pi - \frac{f(\varphi)}{2}R - \frac{k(\varphi)}{2}\dot{\varphi}^2(\Pi) - k(\varphi)g^{0i}\dot{\varphi}(\Pi)\partial_i\varphi - \frac{k(\varphi)}{2}g^{ij}\partial_i\varphi\partial_j\varphi + V(\varphi)$$

$$\dot{\varphi}(\Pi) = \frac{1}{g^{00}}\left(\frac{\Pi}{k(\varphi)} - g^{0i}\partial_i\varphi\right)$$

- Hamiltonian version of the functional integro-differential equation:

$$\exp\left(\frac{i}{\hbar}\Gamma\right) = \int \mathcal{D}\varphi^Q \mathcal{D}\Pi \, \exp\left(\frac{i}{\hbar}\left[\int d^4x\sqrt{-g}(\Pi\dot{\varphi} - \mathcal{H}) - \int d^4x\,(\varphi - \varphi^Q)\frac{\delta\Gamma}{\delta\varphi}\right]\right),$$

- Gaussian in Π. To integrate out, transform $\Pi \to \Pi' + \dot{\varphi}(\Pi)$:

$$\exp\left(\frac{i}{\hbar}\Gamma\right) = \int \mathcal{D}\varphi^Q \left(\int \mathcal{D}\Pi' e^{\frac{i}{\hbar}\int d^4x \frac{\Pi^2}{g^{00}k(\varphi)}}\right) \exp\left(\frac{i}{\hbar}\left[\int d^4x\sqrt{-g}\mathcal{L} - \int d^4x\,(\varphi - \varphi^Q)\frac{\delta\Gamma}{\delta\varphi}\right]\right)$$

$$\int \mathcal{D}\Pi' e^{\frac{i}{\hbar}\int d^4x \frac{\Pi^2}{g^{00}k(\varphi)}} \propto \det\left(k^{1/2}(\varphi)\delta(x-y)\right)$$