# Heavy Baryons in the Large $N_c$ limit

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[research with C. Albertus (U. Huelva) and J.L. Goity (JLAB)]

# Introduction

- To what extent does the large $N_c$ expansion of t'Hooft work ?
- Many relations are based on quark-hadron duality $\to$ relations among hadronic observables with no explicit $N_c$ .
- Does $N_c$ appear explicitly in meson observables ? Yes !!. In the transition form factor of the photon at high energies

$$F_{\gamma\gamma^*\pi^0}(Q^2) = \frac{1}{4\pi^2 f_\pi} \frac{m_\rho^2}{m_\rho^2 + Q^2} \to \frac{6 f_\pi}{N_c Q^2} \qquad m_\rho^2 = \frac{24\pi^2 f_\pi^2}{N_c}$$

- LEC's from Short distance constraints (Ledwig,Pich,Ruiz de Elvira,Nieves,ERA)

$$2L_1 = L_2 = -\frac{L_3}{2} = \frac{L_5}{2} = \frac{L_8}{3} = \frac{L_9}{4} = -\frac{L_{10}}{3} = \frac{N_c}{192\pi^2}, \qquad L_4 = L_6 = L_7 = 0,$$

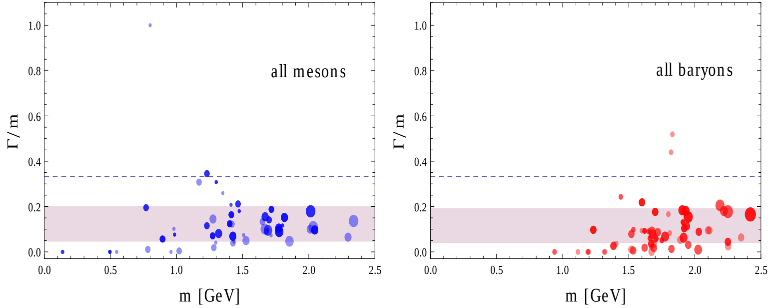
- Explict behaviour can also be found in heavy baryons

- The large $N_c$ expansion t'Hooft

$$N_c \to \infty \qquad \lambda = g^2 N_c \quad \text{fixed}$$

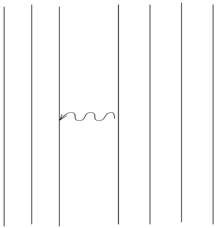
- Mesons and Baryons are stable

$$M_{q\bar{q}} \sim N_c \qquad M_B \sim N_c M \qquad \frac{\Gamma}{M} \sim 1/N_c$$

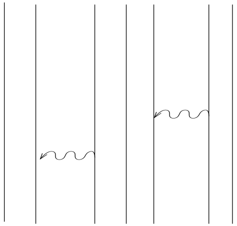



$$\frac{\Gamma}{M} \equiv \sum_{J,\alpha} (2J+1) \frac{\Gamma_{J,\alpha}}{M_{J,\alpha}} = 0.12(8) \qquad = \mathcal{O}({N_c}^{-1})$$

- Perturbative analisis for heavy baryons



$$\sim g^2 (N_c - 1) N_c / 2 \sim N_c$$



$$\sim g^4 [(N_c - 1) N_c / 2]^2 \sim N_c^2$$

- The Expansion of the evolution operator

$$e^{-iTMN_c(1+g^2 N_c)} = e^{-iTMN_c} \left[ 1 - iTg^2 N_c^2 + \frac{i^2}{2} g^4 N_c^4 + \dots \right]$$

- Baryon mass

$$M_B = N_c M (1 + g^2 N_c + \dots)$$

# Extensivity on the number of constituents

Does the limit of many particles implies means field ?

- Atomic nuclei: Short distance interactions $\mathcal{O}(1)$

$$M_A = AM_N - AB(A) \qquad B(A) \sim 8\text{MeV} \qquad (4 < A < 208)$$

  Next neighbors makes $n(n-1)/2 \sim 6$. Short distance correlations

- Baryons: Long distance interactions but $\mathcal{O}(1/N_c)$: all pairs interact

$$M_N = N_c \langle T \rangle + \frac{N_c(N_c-1)}{2} \langle V \rangle$$

- Ultrarelativistic quarks (neglect Coulomb, virial theorem)

$$M_B = N_c \left[ \langle p \rangle + \frac{1}{2} \langle r_{qq} \rangle \right] \to N_c \sigma \langle r_{qq} \rangle \equiv N_c M_q$$

$$M_{q\bar{q}} = \langle p \rangle + \sigma \langle r_{q\bar{q}} \rangle \to 2\sigma \langle r_{q\bar{q}} \rangle \equiv 2M_q$$

# Baryon spectrum at large $N_c$

- The masses of ground state baryons take the form of Rotational band

$$M_B(S) = N_c m_0 + \frac{C_{HF}}{N_c}(S(S+1) - \frac{3}{4}N_c) + \mathcal{O}(1/N_c^2)$$

  where $m_0$ and $C_{HF}$ are $N_c^0$ have an expansion in $1/N_c$, and depend on the quark mass $m_Q$.

- Summing over spins we can extract the center of the multiplet (Lange interval's rule) $m_0$

$$m_0 = \frac{2}{N_c^2(N_c+1)(N_c+3)^2} \sum_{S=\frac{1}{2}}^{\frac{N_c}{2}} (3 + N_c(3N_c+2) - 8(N_c-3)S)\, M_B(S)$$

# Baryons and Mesons at $N_c$ on the lattice

- On the lattice we may fix the lattice spacing but in the limit $N_c \to \infty$ the lattice reduces to a point (Eguchi-Kawai theorem). Thus, the continuum limit is subtle.
- Panero and Bali have determined the dependence of meson masses on the lattice

$$N_c = 3, 4, 5, 6, 17$$

- Interpolating fields : Non relativistic quark model states (for instance)

$$B(x) = \prod_{a=1} N_c \left[ Q_{\uparrow,u,a}(x) \right] \qquad S = \frac{N_c}{2}$$

- DeGrand has done it for baryons

$$N_c = 3, 5, 7$$

# How to check large $N_c$?

There are two ways to check the validity of large $N_c$

- Fix $N_c$ and compute numerically the series

$$O = \sum_i Q_i \sqrt{N_c}$$

- Changing from $N_c = 3$ to any value $N_c$ but Fixing two quantities

$$(\sigma, m_q) \qquad (m_\pi, m_\rho) \qquad (m_\pi, f_\pi) \qquad (m_\pi, f_0) \qquad \ldots$$

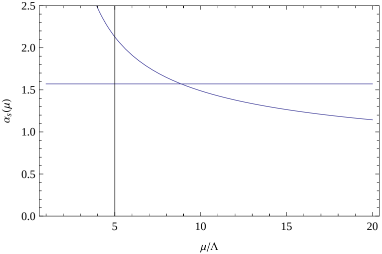
- On the lattice

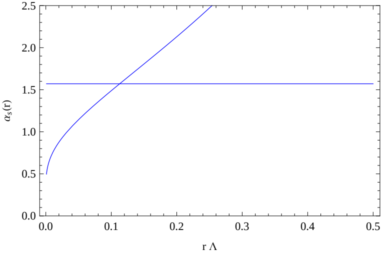
$$V_{Q\bar{Q}}^{N_c}(r) = V_{Q\bar{Q}}^{N_c=3}(r)$$

- Light quarks vs heavy quarks

$$0 \le \sqrt{\sigma}/m_q \le \infty$$

# Running of the coupling contant




Running coupling

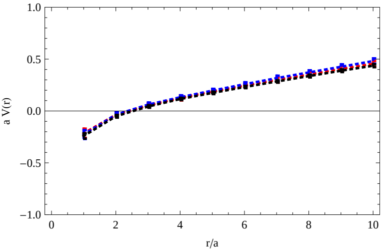
$$\alpha_s = \frac{4\pi}{\beta_0 \log(\mu^2/\Lambda^2)}$$

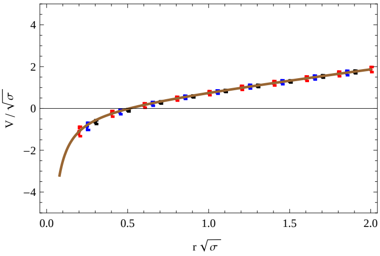
The bare coupling ( $0.5\%$ accuracy)

$$\frac{\Lambda_{\overline{\rm MS}}}{\sqrt{\sigma}} = 0.503(2)(40) + \frac{0.33(3)(3)}{N_c^2} + \mathcal{O}(N_c^{-4})$$

Casimir scaling of the *bare* t'Hooft coupling $N_c = 3, 5, 7$

$$(\alpha_s N_c)_{\rm Bare} \sim \frac{N_c^2}{N_c^2 - 1} = 0.24(0.24), 0.23(0.22), 0.22(0.22)$$




- Casimir scaling

$$V_{AB}^{N_c}(r) = F_A \cdot F_B \frac{\alpha_s}{r}$$

- $N_c$ Independence of the potential

$$V_{Q\bar{Q}}^{N_c}(r) = V_{Q\bar{Q}}^{N_c=3}(r) \equiv -\frac{\pi}{\mathbf{12r}} + \sigma \mathbf{r} \quad \textbf{(Bosonic string model)}$$

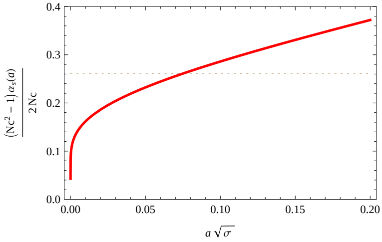
- Rigourous Convexity properties fail in perturbation theory (reflection positivity)

$$V'_{Q\bar{Q}}(r) \geq 0 \qquad V''_{Q\bar{Q}}(r) \geq 0$$

- Hyperfine splitting is beyond the lattice since it involves too short distances $1/m_Q$

# Running of the coupling on the lattice

What is the scale where $\alpha_s = \alpha_{\rm BSM}$ ?




From lattice to continuum (Tepper)

$$\frac{\mu}{\Lambda_{\rm SF}} \equiv \frac{1}{a\Lambda_{\rm SF}} \in [23.9, 58.6] \qquad \Lambda_{\rm SF} \sim 0.48811\Lambda_{\overline{\rm MS}}$$

Perturbative Scale Running will NOT be seen since $r\Lambda \ll 1$ is required The connection between the BSM and perturbation theory is obscure

# The Large $N_c$ Hamiltonian

- Hamiltonian with Quark Fields

$$\begin{aligned} H &= \int d^3x \left[ -\frac{1}{2m_Q} Q^\dagger(x)\, \Delta Q(x) + m_Q\, Q^\dagger(x) Q(x) \right] \\ &+ \frac{1}{2} \int d^3x d^3x' Q^\dagger(x) \frac{\lambda_a}{2} Q(x) \cdot Q^\dagger(x') \frac{\lambda^a}{2} Q(x') V(x-x') \end{aligned} \quad (2)$$

- Configuration space representation

$$H = \sum_i \left[ m_Q + \frac{p_i^2}{2m_Q} \right] + \frac{1}{4} \sum_{i<j}^{N_c} \lambda_a(i) \otimes \lambda^a(j) V(x_i - x_j) \quad (3)$$

- Use this Hamiltonian in Hartree mean field for $N_c = 3, 5, 7$, and corrections to it

# The mean field Hamiltonian

- Baryon wave function (ground state)

$$\Psi(x_1,\ldots,x_N)=\psi(x_1,\ldots,x_N)\chi_{SF}, \tag{4}$$

where $\chi_{SF}$ is the spin-flavor wave function.

- Mean field energy

$$M_{\rm B}=N_c m_Q+N_c\int d^3x\frac{1}{2m_Q}\left[\nabla\phi(x)\right]^2+ \tag{5}$$

$$\frac{N_c(N_c-1)}{2}\int d^3x d^3x'\left[\phi(x)\right]^2 V_{QQ}(x-x')\left[\phi(x')\right]^2 \tag{6}$$

- Use Casimir scaling

$$V_{QQ}(r)=\frac{V_{\bar{Q}Q}(r)}{N_c-1}=-\frac{1}{N_c-1}\left[-\frac{\pi}{12r}+\sigma r\right]$$

- Baryon mass scales

$$M_B=N_c m_Q+N_c\frac{\langle p^2\rangle}{2m_Q}+\frac{1}{2}N_c\langle v\rangle\,, \qquad v(x-x`)=V_{\bar{Q}Q}(x-x')$$

# The mean field Equations

- Minimizing with respect to normalized $\phi(x)$ one gets Hartree equation

$$-\frac{1}{2m_q}\nabla^2\phi(x) + \int d^3x' v(x-x')|\phi(x')|^2\phi(x) = \epsilon\phi(x) \quad (7)$$

- Equivalent Schrodinger potential

$$-\frac{1}{2m_q}\nabla^2\phi_0(x) + \bar{V}(x)\phi_0(x) = \epsilon\phi_0(x) \quad (8)$$

- Effective mean field potential $\bar{V}(x)$ is given by

$$\bar{V}(x) = \int d^3x' V_{Q\bar{Q}}(x-x')|\phi_0(x')|^2 \quad (9)$$

# The Ground State

- In terms of the reduced wave functions

$$\phi(\vec{x}) = \frac{1}{\sqrt{4\pi}} \frac{u(r)}{r} \tag{10}$$

$$\int d^3x |\phi(\vec{x})|^2 = \int_0^\infty dr |u(r)|^2 \tag{11}$$

- For a spherically symmetric single particle wave function we can compute the integral over angles explicitly

$$\int_{-1}^{1} \frac{dz}{2} |\vec{x} - \vec{x}'| = \frac{r_<^2}{3r_>} + r_> \tag{12}$$

$$\int_{-1}^{1} \frac{dz}{2} \frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r_>} \tag{13}$$

where $r_< = \min\{r, r'\}$ and $r_> = \max\{r, r'\}$ and thus

$$\int_{-1}^{1} \frac{dz}{2} V_{QQ}(\vec{x} - \vec{x}') = V_{QQ}(r_>, r_>) \tag{14}$$

# The Ground State

- Using rotational invariance the mean field energy reads

$$\begin{aligned} E_\Psi &= N_c m_q + \frac{N_c}{2m_q}\int_0^\infty dr|u'(r)|^2 \\ &+ N_c \int_0^\infty dr \int_0^r ds|u(r)|^2|u(s)|^2\left[-\frac{\pi}{12r}+\sigma(\frac{s^2}{3r}+r)\right] \end{aligned} \tag{15}$$

whereas the effective potential becomes

$$\bar{V}(r) = \int_0^\infty ds|u(s)|^2 \int_{-1}^1 \frac{dz}{2} v\left(\sqrt{r^2+s^2-2rsz}\right) \tag{16}$$

which after integrating reads

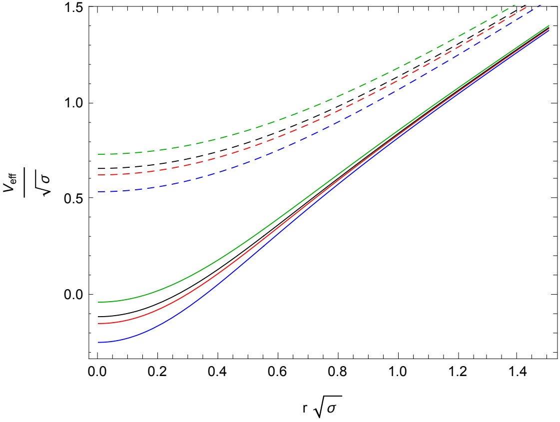
$$\begin{aligned} \bar{V}(r) &= \int_0^r ds|u(s)|^2\left[-\frac{\pi}{12r}+\sigma\left(\frac{s^2}{3r}+r\right)\right] \\ &+ \int_r^\infty ds|u(s)|^2\left[-\frac{\pi}{12s}+\sigma\left(\frac{r^2}{3s}+s\right)\right] \end{aligned} \tag{17}$$

Asymptotically one has at large distances

$$\bar{V}(r) \to -\frac{\pi}{12r}+\sigma r+\frac{\sigma\langle r^2\rangle}{3r} \tag{18}$$

Note that the Coulomb contribution cancels for $\sigma\langle r^2\rangle = \frac{\pi}{4}$ which for $\sqrt{\sigma} = 0.44\text{GeV}$ yields $\langle r^2\rangle = (0.38\text{fm})^2$ a quite reasonable value

# Effective Mean Field potential



# Results with gaussian wave functions

- At the mean field level gaussians are very accurate

$$\phi(r) = \frac{e^{-r^2/2b^2}}{(\sqrt{\pi}b)^{\frac{3}{2}}} \qquad \langle r^2 \rangle = \frac{3}{2}b^2 \tag{22}$$

- The Fourier transformation is defined as

$$\phi_0(p) = \int d^3x e^{i\vec{x}\cdot\vec{p}} \phi(\vec{x}) = (2b\sqrt{\pi})^{\frac{3}{2}} e^{-b^2p^2/2} \tag{23}$$

- Simple relations

$$\int d^3x |\phi(\vec{x})|^2 = \int \frac{d^3p}{(2\pi)^3} |\phi_0(\vec{p})|^2 \tag{24}$$

$$\int d^3x |\nabla\phi(\vec{x})|^2 = \int \frac{d^3p}{(2\pi)^3} p^2 |\phi_0(\vec{p})|^2 = \frac{3}{2b^2} \tag{25}$$

- Accuracy at the mean field level (Coulomb like: Schrödinger-Newton)

$$M_{\rm exact} = N_c m_q \left[1 - 0.4625\alpha_s^2\right] \qquad M_{\rm gauss} = N_c m_q \left[1 - 0.4201\alpha_s^2\right]$$

# Restoration of translational symmetry

- The Hamiltonian is invariant under translations

$$[H, U(a)] = 0 \to [H, P] = 0 \to P|\Psi\rangle = p|\Psi\rangle$$

- The Mean Field Hartree solution breaks the symmetry

$$\langle P\rangle_\Psi = 0\,, \qquad \langle P^2\rangle_\Psi = N_c\langle p^2\rangle \equiv N_c \int d^3x (\nabla\phi)^2$$

- Thus we have a superposition of plane waves

$$|\Psi\rangle = \int \frac{d^3p}{(2\pi)^3} c_p |p\rangle \to$$

- Peierls-Yoccoz projection method

$$|\Psi_p\rangle = \int d^3a e^{ip\cdot a} U(a)|\Psi\rangle$$

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# Projected energy and overlapps

- Projected energy

$$E_p = \frac{\langle p|H|p\rangle}{\langle p|p\rangle} = \frac{\langle \phi|\mathcal{P}_p^\dagger H \mathcal{P}_p|\phi\rangle}{\langle \phi|\mathcal{P}_p^\dagger \mathcal{P}_p|\phi\rangle} = \frac{\langle \phi|H \mathcal{P}_p|\phi\rangle}{\langle \phi|\mathcal{P}_p|\phi\rangle} = \frac{\int e^{ipa} h(a)}{\int e^{ipa} n(a)}$$

- Norm and Hamiltonian overlapps

$$n(a) = \langle \phi|U(a)|\phi\rangle \qquad h(a) = \langle \phi|HU(a)|\phi\rangle$$

- Cluster separation condition

$$\lim_{a\to\pm\infty} n(a) = 0 \qquad \lim_{a\to\pm\infty} h(a) = 0$$

- Sum rules

$$1 = \langle \phi|\phi\rangle = \int dp\, \langle \phi|\mathcal{P}_p|\phi\rangle = \int dp\, n_p \qquad \text{Norm}$$

$$E = \langle \phi|H|\phi\rangle = \int dp\, \langle \phi|H\mathcal{P}_p|\phi\rangle = \int dp\, E_p\, n_p \qquad \text{Energy}$$

- Correlation energy $1/N_c$-correction

$$E_0 = E_0 \int dp\, n_p < \int dp E_p n_p = \langle \Psi|H|\Psi\rangle$$

# Galilean invariance

- Galilei transformation

$$\psi_v(x_1,\ldots,x_N) \equiv U(v)\psi(x_1,\ldots,x_N) = e^{-iXMv}\psi(x_1,\ldots,x_N) \qquad X \equiv \frac{1}{N_c}(x_1 + \cdot\cdot$$

- Boosted energy

$$E_v \equiv \langle\psi_v|H|\psi_v\rangle = \langle\psi|H|\psi\rangle + \frac{1}{2}Mv^2 \qquad ,\langle\psi|P|\psi\rangle = 0 \qquad ,\langle\psi_v|\psi_v\rangle = 1$$

- Projected state

$$P\left[e^{-iXMv}\psi_0(x_1,\ldots,x_N)\right] = Mv\left[e^{-iXMv}\psi_0(x_1,\ldots,x_N)\right]$$

- Projected vs boosted energy

$$E_p = E_0 + \frac{p^2}{2M^*} + \ldots \qquad ,M^* \neq mA$$

# Multiquark forces

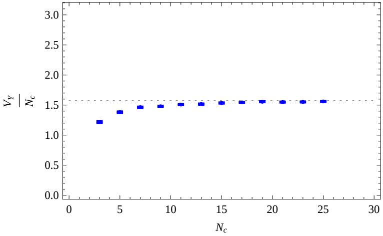
- The Y-Junction vs $\Delta$-Junction

$$V_Y(x_1,\ldots,x_N) = \sigma_Y \min_{x_0} \sum_{i=1}^{N} |x_i - x_0| \to \sigma_Y \sum_{i=1}^{N} |x_i - X| \qquad X = \frac{1}{N_c}\sum_{i=1}^{N_c} x_i$$

$$V_\Delta(x_1,\ldots,x_N) = \sigma_\Delta \sum_{i<j}^{N} |x_i - x_j|$$

- The expectation value is for $N_c \to \infty$

$$\langle V_Y \rangle / N_c \to \sigma_Y \frac{\pi}{2} b$$




Other observables needed to disentangle

# Energy variance and large $N_c$

Accuracy of the mean field state

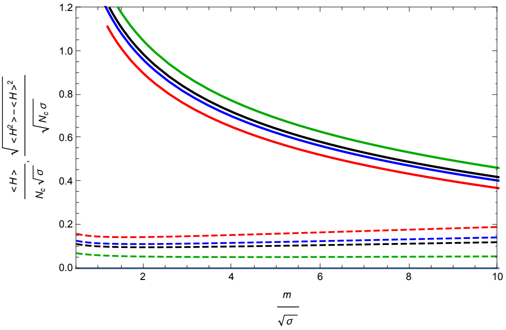



Figure: Energy variance and energy scaled by their respective large $N_c$ scaling factor. Same color code as in Fig. (1).

# Relativistic corrections

- On the lattice $\sqrt{\sigma} > m_q$ relativistic corrections are important
- We try a Simple substitution

$$m_q + p^2/2m_q \to \sqrt{p^2 + m_q^2}$$

- Baryon mass

$$M_B = N_c \langle \sqrt{m_Q^2 + p^2} \rangle + \frac{N_c}{2} \langle \sigma r - \frac{\pi}{12 r} \rangle$$

- Linear momentum projection more difficult (relativistic boost)

# Projected Results with gaussian wave functions

- At the mean field level gaussians are very accurate. Relativistic mean field energy

$$M = \frac{b m_q^2 N_c e^{\frac{b^2 m_q^2}{4}} K_1\left(\frac{b^2 m_q^2}{4}\right)}{\sqrt{2\pi}} + \frac{b N_c \sigma}{\sqrt{\pi}} - \frac{\sqrt{\pi} N_c}{12 b} \tag{26}$$

$$M_B = N_c \left[ \langle p \rangle + \frac{1}{2} \langle \sigma r_{qq} \rangle \right]$$

- Peierls-Yoccz Linear momentum projection

$$M_0 = -\frac{\pi N_c/12}{\sqrt{2\pi} b} + \frac{b m_q^2 N_c^{3/2} e^{\frac{b^2 m_q^2 N_c}{2(N_c-1)}} K_1\left(\frac{b^2 m_q^2 N_c}{2(N_c-1)}\right)}{\sqrt{\pi}\sqrt{N_c - 1}} + \sqrt{\frac{2}{\pi}} b N_c \sigma \tag{27}$$

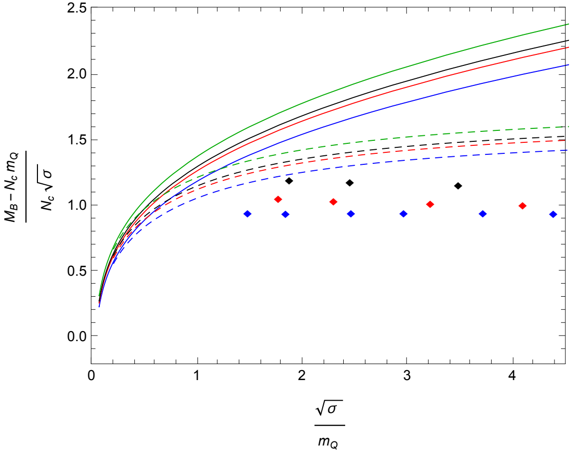
- Masless quarks

$$\frac{M_0}{\sqrt{\sigma}} = \frac{\sqrt{N_c}\sqrt{48\sqrt{N_c-1}\sqrt{N_c} - \sqrt{2}\pi N_c}}{\sqrt[4]{2}\sqrt{3\pi}} \tag{28}$$

- In a $1/N_c$ expansion

$$\frac{M_0}{\sqrt{\sigma}} = 1.80 N_c - 0.49 - \frac{0.19}{N_c} - \frac{0.11}{{N_c}^2} - \frac{0.08}{{N_c}^3} + \mathcal{O}\left({N_c}^{-7/2}\right) \tag{29}$$

# Baryon Energy compared with lattice



# Conclusions

- Large $N_c$ baryon masses can be estimated by mean field methods for heavy baryons
- Lattice calculations with LARGER quark masses are needed to set up the framework on a more solid ground
- Excited states, radii, etc.