

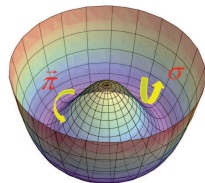
2. Chiral Perturbation Theory

- Goldstone Theorem
- Chiral Symmetry
- Effective Goldstone Theory
- Chiral Symmetry Breaking
- Phenomenology

Sigma Model

$$\Phi^T \equiv (\sigma, \vec{\pi})$$

$$\mathcal{L}_\sigma = \frac{1}{2} \partial_\mu \Phi^T \partial^\mu \Phi - \frac{\lambda}{4} (\Phi^T \Phi - v^2)^2$$



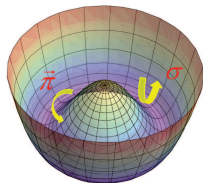
Global Symmetry: $O(4) \sim SU(2) \otimes SU(2)$

- $v^2 < 0$: $m_\Phi^2 = -\lambda v^2$
- $v^2 > 0$: $\langle 0 | \sigma | 0 \rangle = v$, $\langle 0 | \vec{\pi} | 0 \rangle = 0$

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SSB: $O(4) \rightarrow O(3)$ $[\frac{4 \times 3}{2} - \frac{3 \times 2}{2} = 3 \text{ broken generators}]$

$$\mathcal{L}_\sigma = \frac{1}{2} \{ \partial_\mu \hat{\sigma} \partial^\mu \hat{\sigma} + \partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - M^2 \hat{\sigma}^2 \} - \frac{M^2}{2v} \hat{\sigma} (\hat{\sigma}^2 + \vec{\pi}^2) - \frac{M^2}{8v^2} (\hat{\sigma}^2 + \vec{\pi}^2)^2$$

$$\hat{\sigma} \equiv \sigma - v \quad ; \quad M^2 = 2\lambda v^2$$

3 Massless Goldstone Bosons

$$1) \quad \mathbf{\Sigma}(x) \equiv \sigma(x) \mathbf{I}_2 + i \vec{\tau} \vec{\pi}(x) \quad ; \quad \langle \mathbf{A} \rangle \equiv \text{Tr}(\mathbf{A})$$

$$\mathcal{L}_\sigma = \frac{1}{4} \langle \partial_\mu \mathbf{\Sigma}^\dagger \partial^\mu \mathbf{\Sigma} \rangle - \frac{\lambda}{16} \left(\langle \mathbf{\Sigma}^\dagger \mathbf{\Sigma} \rangle - 2 v^2 \right)^2$$

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$$2) \quad \boldsymbol{\Sigma}(x) \equiv [v + S(x)] \mathbf{U}(x) \quad ; \quad \mathbf{U} \equiv \exp \left\{ \frac{i}{v} \vec{\tau} \vec{\phi} \right\} \rightarrow g_R \mathbf{U} g_L^\dagger$$

$$\mathcal{L}_\sigma = \frac{v^2}{4} \left(1 + \frac{S}{v} \right)^2 \langle \partial_\mu \mathbf{U}^\dagger \partial^\mu \mathbf{U} \rangle + \frac{1}{2} (\partial_\mu S \partial^\mu S - M^2 S^2) - \frac{M^2}{2v} S^3 - \frac{M^2}{8v^2} S^4$$

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Derivative Goldstone Couplings

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Derivative Goldstone Couplings

$$3) \quad E \ll M \sim v :$$

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Symmetry Realizations

Symmetry $\mathbf{G} \quad \{T_a\}$



Conserved charges Q_a

Noether Theorem: $\partial_\mu j_a^\mu = 0 \quad ; \quad Q_a = \int d^3x j_a^0(x) \quad ; \quad \frac{d}{dt} Q_a = 0$

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- Degenerate Multiplets
- Linear Representation

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Nambu–Goldstone

$$Q_a |0\rangle \neq 0$$

- Spontaneously Broken Symmetry
- Massless Goldstone Bosons
- Non-Linear Representation

$$\mathcal{L}_{QCD}^0 = -\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a + \bar{\mathbf{q}}_L i \gamma^\mu D_\mu \mathbf{q}_L + \bar{\mathbf{q}}_R i \gamma^\mu D_\mu \mathbf{q}_R$$

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$$q = \left(\frac{1 - \gamma_5}{2} \right) q + \left(\frac{1 + \gamma_5}{2} \right) q \equiv q_L + q_R$$

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- Only $\mathbf{SU}(3)_V$ in the hadronic spectrum: $(\pi, K, \eta)_{0-} ; (\rho, K^*, \omega)_{1-} ; \dots$

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- The 0^- octet is nearly massless: $m_\pi \approx 0$
- The vacuum is not invariant (SSB): $\langle 0 | (\bar{\mathbf{q}}_L \mathbf{q}_R + \bar{\mathbf{q}}_R \mathbf{q}_L) | 0 \rangle \neq 0$

8 Massless 0^- Goldstone Bosons

Effective Goldstone Theory

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$$\Phi \equiv \frac{\vec{\lambda}}{\sqrt{2}} \vec{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}} \eta \end{pmatrix}$$

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$$\mathbf{U} \longrightarrow \mathbf{g}_R \mathbf{U} \mathbf{g}_L^\dagger \quad ; \quad \mathbf{g}_{L,R} \in SU(3)_{L,R}$$

M_W

$$\begin{array}{c}
 W, Z, \gamma, g \\
 \tau, \mu, e, \nu_i \\
 t, b, c, s, d, u
 \end{array}$$

Standard Model

OPE

 $\lesssim m_c$

$$\begin{array}{c}
 \gamma, g; \mu, e, \nu_i \\
 s, d, u
 \end{array}$$
 $\mathcal{L}_{\text{QCD}}^{(n_f=3)}, \mathcal{L}_{\text{eff}}^{\Delta S=1,2}$ $N_C \rightarrow \infty$ M_K

$$\begin{array}{c}
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 χ^{PT}

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$$\mathcal{L}(\mathbf{U}) = \sum_n \mathcal{L}_{2n}$$

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- Expansion in powers of momenta $\longleftrightarrow$ derivatives

$$\text{Parity} \quad \longrightarrow \quad \text{even dimension} \quad ; \quad \mathbf{U} \mathbf{U}^\dagger = 1 \quad \longrightarrow \quad 2n \geq 2$$

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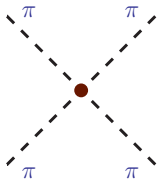
Derivative
Coupling

Goldstones become free at zero momenta

$$\begin{aligned}
\mathcal{L}_2 &= \frac{f^2}{4} \langle \partial_\mu \mathbf{U}^\dagger \partial^\mu \mathbf{U} \rangle = \partial_\mu \pi^- \partial^\mu \pi^+ + \frac{1}{2} \partial_\mu \pi^0 \partial^\mu \pi^0 + \dots \\
&+ \frac{1}{6f^2} \left\{ \left(\pi^+ \overleftrightarrow{\partial}_\mu \pi^- \right) \left(\pi^+ \overleftrightarrow{\partial}^\mu \pi^- \right) + 2 \left(\pi^0 \overleftrightarrow{\partial}_\mu \pi^+ \right) \left(\pi^- \overleftrightarrow{\partial}^\mu \pi^0 \right) + \dots \right\} \\
&+ O(\pi^6/f^4)
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Chiral Symmetry Determines the Interaction:



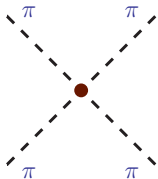
$$T(\pi^+ \pi^0 \rightarrow \pi^+ \pi^0) = \frac{t}{f^2}$$

$$t \equiv (p'_+ - p_+)^2$$

Weinberg

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Weinberg

Non-Linear Lagrangian:

$$2\pi \rightarrow 2\pi, 4\pi, \dots \quad \text{related}$$

Explicit Symmetry Breaking

$$\begin{aligned}
 \mathcal{L}_{QCD} &\equiv \mathcal{L}_{QCD}^0 + \bar{\mathbf{q}}(\not{\mathbf{y}} + \not{\mathbf{a}}\gamma_5)\mathbf{q} - \bar{\mathbf{q}}(\mathbf{s} - i\gamma_5\mathbf{p})\mathbf{q} \\
 &= \mathcal{L}_{QCD}^0 + \bar{\mathbf{q}}_L\not{\mathbf{q}}_L + \bar{\mathbf{q}}_R\not{\mathbf{q}}_R - \bar{\mathbf{q}}_R(\mathbf{s} + i\mathbf{p})\mathbf{q}_L - \bar{\mathbf{q}}_L(\mathbf{s} - i\mathbf{p})\mathbf{q}_R
 \end{aligned}$$

Explicit Symmetry Breaking

$$\begin{aligned}\mathcal{L}_{QCD} &\equiv \mathcal{L}_{QCD}^0 + \bar{\mathbf{q}}(\not{\mathbf{v}} + \not{\mathbf{a}}\gamma_5)\mathbf{q} - \bar{\mathbf{q}}(\mathbf{s} - i\gamma_5\mathbf{p})\mathbf{q} \\ &= \mathcal{L}_{QCD}^0 + \bar{\mathbf{q}}_L\not{\mathbf{q}}_L + \bar{\mathbf{q}}_R\not{\mathbf{q}}_R - \bar{\mathbf{q}}_R(\mathbf{s} + i\mathbf{p})\mathbf{q}_L - \bar{\mathbf{q}}_L(\mathbf{s} - i\mathbf{p})\mathbf{q}_R\end{aligned}$$

$$\mathbf{l}_\mu \equiv \mathbf{v}_\mu - \mathbf{a}_\mu = e\mathcal{Q}A_\mu + \dots$$

$$\mathbf{r}_\mu \equiv \mathbf{v}_\mu + \mathbf{a}_\mu = e\mathcal{Q}A_\mu + \dots$$

$$\mathbf{s} = \mathcal{M} + \dots \quad ;$$

$$\mathcal{Q} \equiv \frac{1}{3} \text{diag}(2, -1, -1)$$

$$\mathcal{M} \equiv \text{diag}(m_u, m_d, m_s)$$

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$$\mathcal{Q} \equiv \frac{1}{3} \text{diag}(2, -1, -1)$$

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$$\mathbf{s} = \mathcal{M} + \dots \quad ; \quad \mathcal{M} \equiv \text{diag}(m_u, m_d, m_s)$$

Local $\text{SU}(3)_L \otimes \text{SU}(3)_R$ Symmetry:

$$\begin{aligned}\mathbf{q}_L &\rightarrow g_L \mathbf{q}_L & \mathbf{l}_\mu &\rightarrow g_L \mathbf{l}_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger \\ \mathbf{q}_R &\rightarrow g_R \mathbf{q}_R & \mathbf{r}_\mu &\rightarrow g_R \mathbf{r}_\mu g_R^\dagger + i g_R \partial_\mu g_R^\dagger \\ & & (\mathbf{s} + i\mathbf{p}) &\rightarrow g_R (\mathbf{s} + i\mathbf{p}) g_L^\dagger\end{aligned}$$

Lowest-Order Effective Lagrangian:

$$\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U} D^\mu \mathbf{U}^\dagger + \chi \mathbf{U}^\dagger + \mathbf{U} \chi^\dagger \rangle$$

$$D_\mu \mathbf{U} = \partial_\mu \mathbf{U} - i \mathbf{r}_\mu \mathbf{U} + i \mathbf{U} \mathbf{l}_\mu$$

$$\chi \equiv 2 B_0 (\mathbf{s} + i \mathbf{p})$$

Lowest-Order Effective Lagrangian:

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Dashen
Theorem

$$(M_{K^0}^2 - M_{K^\pm}^2)_{\text{em}} = (M_{\pi^0}^2 - M_{\pi^\pm}^2)_{\text{em}} + \mathcal{O}(e^2 p^2)$$

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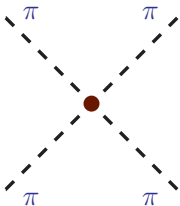


$$m_u : m_d : m_s = 0.55 : 1 : 20.3$$

Weinberg

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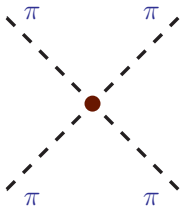


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$\mathcal{L}_2 \longleftrightarrow$ Current Algebra 60's

Chiral Power Counting

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$\mathcal{O}(p^0)$

$D_\mu \mathbf{U}, \mathbf{l}_\mu, \mathbf{r}_\mu$

$\mathcal{O}(p^1)$

$\chi, \mathbf{F}_{L,R}^{\mu\nu}$

$\mathcal{O}(p^2)$

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General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

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 $L = 1$, $d = 2$

i) $\mathcal{L}_4$ at tree level (Gasser–Leutwyler)

$$\begin{aligned}
 \mathcal{L}_4 = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\
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 & + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\
 & + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle \\
 & - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle
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$\mathcal{O}(p^4)$ χ PT

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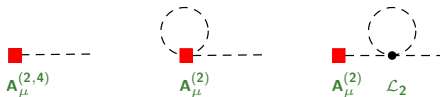
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iii) Wess–Zumino–Witten term (chiral anomaly): $\pi^0, \eta \rightarrow \gamma\gamma$

Meson Decay Constants:



$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2} \log \left(\frac{M_P^2}{\mu^2} \right)$$

$$f_\pi = f \left\{ 1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu) \right\}$$

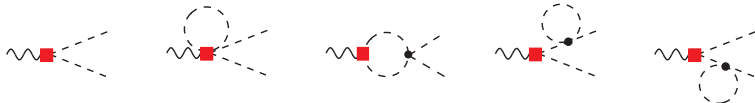
$$f_K = f \left\{ 1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu) \right\}$$

$$f_{\eta_8} = f \left\{ 1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu) \right\}$$

$$\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \quad \Rightarrow \quad L_5^r(M_\rho) = (1.4 \pm 0.5) \cdot 10^{-3} \quad \Rightarrow \quad \frac{f_{\eta_8}}{f_\pi} = 1.3 \pm 0.05$$

Vector Form Factor:

$$\langle \pi^+ \pi^- | J_{\text{em}}^\mu | 0 \rangle = (p_+ - p_-)^\mu F_\pi^V(s)$$

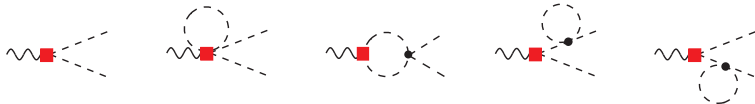


$$\begin{aligned} F_\pi^V(s) &= 1 + \frac{2L_9^r(\mu)}{f^2} s - \frac{s}{96\pi^2 f^2} \left[A\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2} A\left(\frac{m_K^2}{s}, \frac{m_K^2}{\mu^2}\right) \right] \\ &= 1 + \frac{1}{6} \langle r^2 \rangle_\pi^V s + \dots \end{aligned}$$

$$A\left(\frac{m_P^2}{s}, \frac{m_P^2}{\mu^2}\right) = \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3 \log\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \quad , \quad \sigma_P \equiv \sqrt{1 - \frac{4m_P^2}{s}}$$

Vector Form Factor:

$$\langle \pi^+ \pi^- | J_{\text{em}}^\mu | 0 \rangle = (p_+ - p_-)^\mu F_\pi^V(s)$$



$$\begin{aligned} F_\pi^V(s) &= 1 + \frac{2L_9^r(\mu)}{f^2} s - \frac{s}{96\pi^2 f^2} \left[A\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2} A\left(\frac{m_K^2}{s}, \frac{m_K^2}{\mu^2}\right) \right] \\ &= 1 + \frac{1}{6} \langle r^2 \rangle_\pi^V s + \dots \end{aligned}$$

$$A\left(\frac{m_P^2}{s}, \frac{m_P^2}{\mu^2}\right) = \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3 \log\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \quad , \quad \sigma_P \equiv \sqrt{1 - \frac{4m_P^2}{s}}$$

$$\langle r^2 \rangle_\pi^V = \frac{12 L_9^r(\mu)}{f^2} - \frac{1}{32\pi^2 f^2} \left\{ 2 \log\left(\frac{M_\pi^2}{\mu^2}\right) + \log\left(\frac{M_K^2}{\mu^2}\right) + 3 \right\}$$

$$\langle r^2 \rangle_\pi^V = (0.439 \pm 0.008) \text{ fm}^2 \quad \longrightarrow \quad L_9^r(M_\rho) = (6.9 \pm 0.7) \cdot 10^{-3}$$

$O(p^4)$ χ PT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	Source	Γ_i
1	0.4 ± 0.3	$K_{e4}, \pi\pi \rightarrow \pi\pi$	3/32
2	1.4 ± 0.3	$K_{e4}, \pi\pi \rightarrow \pi\pi$	3/16
3	-3.5 ± 1.1	$K_{e4}, \pi\pi \rightarrow \pi\pi$	0
4	-0.3 ± 0.5	Zweig rule	1/8
5	1.4 ± 0.5	F_K/F_π	3/8
6	-0.2 ± 0.3	Zweig rule	11/144
7	-0.4 ± 0.2	GMO, $L_{5,8}$	0
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}, L_5, (m_s - \hat{m})/(m_d - m_u)$	5/48
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	1/4
10	-5.5 ± 0.7	$\pi \rightarrow e\nu\gamma$	-1/4

- $$L_i = L_i^r(\mu) + \Gamma_i \frac{\mu^{D-4}}{32\pi^2} \left\{ \frac{2}{D-4} + \gamma_E - \log(4\pi) - 1 \right\}$$

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$\mathcal{O}(p^6)$ χ PT

i) $\mathcal{L}_6 = \sum_i c_i \mathcal{O}_i^{p^6}$ at tree level

Bijnens-Colangelo-Ecker, Fearing-Scherer

90 + 4 [53 + 4] terms in SU(3) [SU(2)] χ PT (even-intrinsic parity only)

ii) $\mathcal{L}_4$ at one loop, $\mathcal{L}_2$ at two loops

Bijnens-Colangelo-Ecker

Double chiral logarithms

Many Calculations: $M_\phi, f_\phi, \gamma\gamma \rightarrow \pi\pi, \pi\pi \rightarrow \pi\pi, \pi K \rightarrow \pi K, K_{l4},$
 $\pi \rightarrow e \bar{\nu}_e \gamma, F_V(s), F_S(s), \Pi_{V,A}(s), \dots$

Amoros-Bijnens-Dhonte-Talavera, Ananthanarayan-Colangelo-Gasser-Leutwyler, Bellucci-Gasser-Sainio, Bürgui, Bijnens et al, Descotes-Genon et al, Golowich-Kambor, Post-Schilcher...

Theoretical Challenge: QCD calculation of the χ PT couplings

$$K^+ \rightarrow \pi^0 \ell^+ \nu_\ell, \quad K^0 \rightarrow \pi^- \ell^+ \nu_\ell:$$

$$C_{K^+\pi^0} = \frac{1}{\sqrt{2}}, \quad C_{K^0\pi^-} = 1$$

$$\langle \pi | \bar{s} \gamma^\mu u | K \rangle = C_{K\pi} [(P_K + P_\pi)^\mu f_+^{K\pi}(t) + (P_K - P_\pi)^\mu f_-^{K\pi}(t)]$$

• **Lowest order** [$\mathcal{O}(p^2)$]: $f_+^{K\pi}(t) = 1$, $f_-^{K\pi}(t) = 0$

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Gasser-Leutwyler '85

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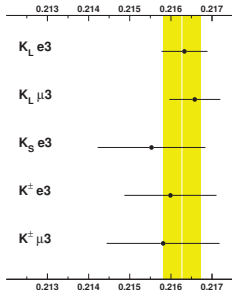
Needed to determine V_{us}

$$K \rightarrow \pi \ell \nu_\ell$$

$$|\mathbf{V}_{us} f_+(0)| = 0.2163 \pm 0.0005$$

Flavianet Kaon WG, arXiv:1005.2323 [hep-ph]

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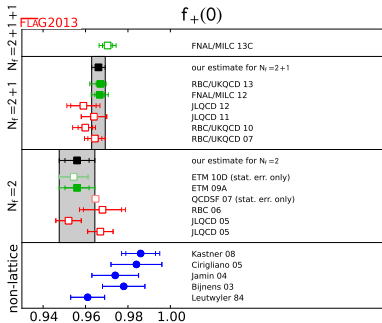
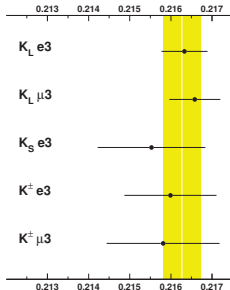


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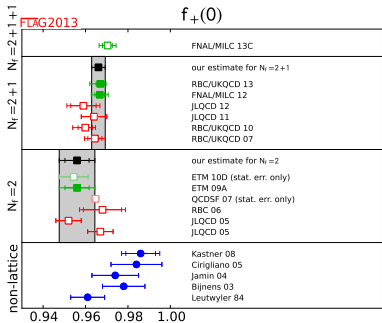
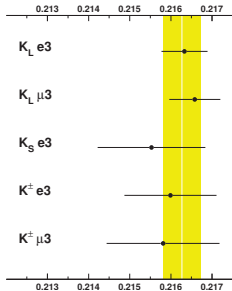


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$$f_+(0) = \begin{cases} 0.9704(32) & (N_f = 2+1+1) \\ 0.9661(32) & (N_f = 2+1) \end{cases}$$

0.9677 new RBC-UKQCD

$$\rightarrow |V_{us}| = \begin{cases} 0.2229(9) \\ 0.2239(9) \end{cases}$$

0.2235

$$f_+(0) = 1 + f_2 + f_4 + \dots$$

Large $\mathcal{O}(p^6)$ χ PT correction

Backup Slides

Goldstone Theorem

$$\mathcal{Q} = \int d^3x j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists \mathcal{O} : v(t) \equiv \langle 0 | [\mathcal{Q}(t), \mathcal{O}] | 0 \rangle \neq 0$$



$$\exists |n\rangle : \langle 0 | \mathcal{O} | n \rangle \langle n | j^0 | 0 \rangle \neq 0 \quad ; \quad E_n \delta^{(3)}(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

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$$\begin{aligned} \frac{d}{dt} v(t) &= 0 = -i (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) E_n \{ e^{-iE_n t} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle \\ &\quad + e^{iE_n t} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \} \end{aligned}$$

□

Noether QCD Currents:

$$G \equiv SU(3)_L \otimes SU(3)_R$$

$$J_X^{a\mu} = \bar{\mathbf{q}}_X \gamma^\mu \frac{\lambda^a}{2} \mathbf{q}_X \quad ; \quad Q_X^a = \int d^3x J_X^{a0}(x) \quad (a = 1, \dots, 8 ; X = L, R)$$

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$$\langle 0 | [Q_A^a, \mathcal{O}^b] | 0 \rangle = -\frac{1}{2} \langle 0 | \bar{\mathbf{q}} \{ \lambda^a, \lambda^b \} \mathbf{q} | 0 \rangle = -\frac{2}{3} \langle 0 | \bar{\mathbf{q}} \mathbf{q} | 0 \rangle$$

 $\langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle = \langle 0 | \bar{s} s | 0 \rangle \neq 0$

Noether QCD Currents: $G \equiv SU(3)_L \otimes SU(3)_R$

$$J_X^{a\mu} = \bar{\mathbf{q}}_X \gamma^\mu \frac{\lambda^a}{2} \mathbf{q}_X \quad ; \quad Q_X^a = \int d^3x J_X^{a0}(x) \quad (a = 1, \dots, 8 ; X = L, R)$$

Current Algebra ('60) : $[Q_X^a, Q_Y^b] = i \delta_{XY} f^{abc} Q_X^c$

Dynamical Symmetry Breaking:

- 8 Pseudoscalar Goldstones $\pi^a = (\pi, K, \eta)$

- $Q_A^a = Q_R - Q_L \quad ; \quad \mathcal{O}^b = \bar{\mathbf{q}} \gamma_5 \lambda^b \mathbf{q}$

$$\langle 0 | [Q_A^a, \mathcal{O}^b] | 0 \rangle = -\frac{1}{2} \langle 0 | \bar{\mathbf{q}} \{ \lambda^a, \lambda^b \} \mathbf{q} | 0 \rangle = -\frac{2}{3} \langle 0 | \bar{\mathbf{q}} \mathbf{q} | 0 \rangle$$

 $\langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle = \langle 0 | \bar{s} s | 0 \rangle \neq 0$

- $\langle 0 | J_A^{a\mu} | \pi^b(p) \rangle = i \delta^{ab} \sqrt{2} f_\pi p^\mu$

Chiral Anomaly:

$$\delta Z[v, a, s, p] = -\frac{N_c}{16\pi^2} \int d^4x \langle \delta\beta(x) \Omega(x) \rangle$$

$$g_{L,R} \approx 1 + i\delta\alpha \mp i\delta\beta$$

$$\Omega(x) = \varepsilon^{\mu\nu\sigma\rho} \left[v_{\mu\nu} v_{\sigma\rho} + \frac{4}{3} \nabla_\mu a_\nu \nabla_\sigma a_\rho + \frac{2}{3} i \{ v_{\mu\nu}, a_\sigma a_\rho \} + \frac{8}{3} i a_\sigma v_{\mu\nu} a_\rho + \frac{4}{3} a_\mu a_\nu a_\sigma a_\rho \right]$$
$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i [v_\mu, v_\nu] \quad , \quad \nabla_\mu a_\nu = \partial_\mu a_\nu - i [v_\mu, a_\nu] \quad , \quad \varepsilon_{0123} = 1$$

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$$S[U, \ell, r]_{\text{wzw}} = -\frac{iN_C}{240\pi^2} \int d\sigma^{ijklm} \langle \Sigma_i^L \Sigma_j^L \Sigma_k^L \Sigma_l^L \Sigma_m^L \rangle$$

$$- \frac{iN_C}{48\pi^2} \int d^4x \varepsilon_{\mu\nu\alpha\beta} (W(U, \ell, r)^{\mu\nu\alpha\beta} - W(\mathbf{1}, \ell, r)^{\mu\nu\alpha\beta})$$

Wess–Zumino–Witten

$$W(U, \ell, r)_{\mu\nu\alpha\beta} = \langle U \ell_\mu \ell_\nu \ell_\alpha U^\dagger r_\beta + \frac{1}{4} U \ell_\mu U^\dagger r_\nu U \ell_\alpha U^\dagger r_\beta + i U \partial_\mu \ell_\nu \ell_\alpha U^\dagger r_\beta$$

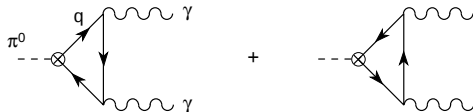
$$+ i \partial_\mu r_\nu U \ell_\alpha U^\dagger r_\beta - i \Sigma_\mu^L \ell_\nu U^\dagger r_\alpha U \ell_\beta + \Sigma_\mu^L U^\dagger \partial_\nu r_\alpha U \ell_\beta - \Sigma_\mu^L \Sigma_\nu^L U^\dagger r_\alpha U \ell_\beta$$

$$+ \Sigma_\mu^L \ell_\nu \partial_\alpha \ell_\beta + \Sigma_\mu^L \partial_\nu \ell_\alpha \ell_\beta - i \Sigma_\mu^L \ell_\nu \ell_\alpha \ell_\beta + \frac{1}{2} \Sigma_\mu^L \ell_\nu \Sigma_\alpha^L \ell_\beta - i \Sigma_\mu^L \Sigma_\nu^L \Sigma_\alpha^L \ell_\beta \rangle$$

$$- (L \leftrightarrow R)$$

$$\Sigma_\mu^L = U^\dagger \partial_\mu U \quad , \quad \Sigma_\mu^R = U \partial_\mu U^\dagger$$

$$\pi^0 \rightarrow \gamma\gamma:$$



$$A_3^\mu \equiv \bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$$

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \left(\frac{N_c}{3}\right)^2 \frac{\alpha^2 M_\pi^3}{64 \pi^3 f_\pi^2} = 7.73 \text{ eV}$$

$$\text{Exp: } (7.7 \pm 0.6) \text{ eV}$$

There are no QCD corrections

The chiral anomaly contributes to: $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$

$\gamma 3\pi$, $\gamma \pi^+ \pi^- \eta$, $K \bar{K} 3\pi$, ...