

THE ROLE OF RENORMALIZATION IN CURVED SPACETIME

Adrián del Río Vega

Department of Theoretical Physics, IFIC. University of Valencia

20.11.2014

OUTLINE

1 BRIEF INTRODUCTION TO QFT IN CURVED SPACETIME

- Foundations. Inherent ambiguity
- The stress-energy tensor

2 EXPECTATION VALUES QUADRATIC IN FIELDS

- Adiabatic renormalization
- Renormalization in general space-times
- Effective action. Renormalized coupling constants

3 APPLICATIONS

- Inflationary cosmology
- Determination of the renormalized value of C_ℓ

FOUNDATIONS

Let $\varphi(x)$ be a general free field (boson or fermion) propagating in a curved space-time $[C^\infty, 4\text{-dimensional, globally hyperbolic manifold, equipped with a lorentzian metric } g]$,

DYNAMICS OF THE FIELD THEORY

$$S = \int \mathcal{L}(x) d^4x, \quad \delta S = 0 \quad \implies \quad F\varphi(x) = 0$$

The action is constructed by requiring to be scalar under general coordinate transformations $\rightarrow$ minimal coupling prescription:

$$\partial_\mu \Rightarrow \nabla_\mu \quad \eta_{\mu\nu} \Rightarrow g_{\mu\nu} \quad d^n x \Rightarrow d^n x \sqrt{-g(x)}$$

Examples,

- K-G (spin 0) field $(g^{\mu\nu} \nabla_\mu \nabla_\nu + m^2 + \xi R)\varphi(x) = 0$
- Dirac (spin 1/2) field $(i\underline{\gamma}^\mu \nabla_\mu - m)\varphi(x) = 0$

FOUNDATIONS

Let φ_1, φ_2 be any two solutions of the field equations, **define an inner product**

$$(\varphi_1, \varphi_2) = -i \int_{\Sigma} \varphi_1^\dagger(x) n^\mu \nabla_\mu \varphi_2(x) d\Sigma$$

Introduce ONE complete set (over an infinity of possibilities) of conjugate solutions u_i and u_i^* satisfying orthonormality conditions

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0,$$

Expand the field in modes

$$\varphi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right]$$

Quantization: Impose canonical (anti)commutation relations for the fields, then

$$[a_i, a_j^\dagger]_{\pm} = \delta_{ij} \quad [a_i, a_j]_{\pm} = 0 \quad [a_i^\dagger, a_j^\dagger]_{\pm} = 0$$

This algebra serves to construct a Fock space by defining a "vacuum" state as

$$a_i |0\rangle = 0, \quad \forall i.$$

FOUNDATIONS

Let φ_1, φ_2 be any two solutions of the field equations, **define an inner product**

$$(\varphi_1, \varphi_2) = -i \int_{\Sigma} \varphi_1^\dagger(x) n^\mu \nabla_\mu \varphi_2(x) d\Sigma$$

Introduce ONE complete set (over an infinity of possibilities) of conjugate solutions u_i and u_i^* satisfying orthonormality conditions

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0,$$

Expand the field in modes

$$\varphi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right]$$

Quantization: Impose canonical (anti)commutation relations for the fields, then

$$[a_i, a_j^\dagger]_{\pm} = \delta_{ij} \quad [a_i, a_j]_{\pm} = 0 \quad [a_i^\dagger, a_j^\dagger]_{\pm} = 0$$

This algebra serves to construct a Fock space by defining a "vacuum" state as

$$a_i |0\rangle = 0, \quad \forall i.$$

FOUNDATIONS

Let φ_1, φ_2 be any two solutions of the field equations, **define an inner product**

$$(\varphi_1, \varphi_2) = -i \int_{\Sigma} \varphi_1^\dagger(x) n^\mu \nabla_\mu \varphi_2(x) d\Sigma$$

Introduce ONE complete set (over an infinity of possibilities) of conjugate solutions u_i and u_i^* satisfying orthonormality conditions

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0,$$

Expand the field in modes

$$\varphi(x) = \sum_i \left[a_i u_i(x) + a_i^\dagger u_i^*(x) \right]$$

Quantization: Impose canonical (anti)commutation relations for the fields, then

$$[a_i, a_j^\dagger]_{\pm} = \delta_{ij} \quad [a_i, a_j]_{\pm} = 0 \quad [a_i^\dagger, a_j^\dagger]_{\pm} = 0$$

This algebra serves to construct a Fock space by defining a "vacuum" state as

$$a_i |0\rangle = 0, \quad \forall i.$$

INHERENT AMBIGUITY

- **Question:** in **QFT** one inherently has **infinite vacuum states**, which one should I consider to describe the physics?
- In **Minkowski** $(\mathbb{R}^4, \eta)$ there is a natural set of modes given by the **Poincare Group**.
Maximal symmetry $\rightarrow$ maximal set of killing vectors generating a Lie algebra. Choose u_i to yield irreducible representations of that algebra.
- In a **general spacetime** $(\mathbb{M}^4, g)$ **no symmetry is guaranteed** $\rightarrow$ no preferred vacuum state. Additional physics may point out any particular choice for u_i .
- The choice of the quantum state is crucial to compute observables.

INHERENT AMBIGUITY

Consider therefore a second complete orthonormal set of modes $\bar{u}_i(x)$,

$$\varphi(x) = \sum_i \left[\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \right], \quad \bar{a}_i |\bar{0}\rangle = 0, \quad \forall i, \quad \bar{a}_i |0\rangle \neq 0, \quad \forall i$$

Since both set are complete $\implies$ *Bogolubov transformations*

$$\bar{u}_j(x) = \sum_i [\alpha_{ji} u_i(x) + \beta_{ji} u_i^*(x)] \quad u_i(x) = \sum_j [\alpha_{ji}^* \bar{u}_j(x) - \beta_{ji} \bar{u}_j^*(x)]$$

$$a_i = \sum_j (\alpha_{ji} \bar{a}_j + \beta_{ji} \bar{a}_j^\dagger) \quad \bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger)$$

Number of u_i -mode particles in the state $|\bar{0}\rangle$:

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2 \neq 0!$$

Particles: observer-dependent concept

Parker, Hawking, Unruh effects

INHERENT AMBIGUITY

Consider therefore a second complete orthonormal set of modes $\bar{u}_i(x)$,

$$\varphi(x) = \sum_i \left[\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \right], \quad \bar{a}_i |\bar{0}\rangle = 0, \quad \forall i, \quad \bar{a}_i |0\rangle \neq 0, \quad \forall i$$

Since both set are complete $\implies$ *Bogolubov transformations*

$$\bar{u}_j(x) = \sum_i [\alpha_{ji} u_i(x) + \beta_{ji} u_i^*(x)] \quad u_i(x) = \sum_j [\alpha_{ji}^* \bar{u}_j(x) - \beta_{ji} \bar{u}_j^*(x)]$$

$$a_i = \sum_j (\alpha_{ji} \bar{a}_j + \beta_{ji} \bar{a}_j^\dagger) \quad \bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger)$$

Number of u_i -mode particles in the state $|\bar{0}\rangle$:

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2 \neq 0!$$

Particles: observer-dependent concept

Parker, Hawking, Unruh effects

INHERENT AMBIGUITY

Consider therefore a second complete orthonormal set of modes $\bar{u}_i(x)$,

$$\varphi(x) = \sum_i \left[\bar{a}_i \bar{u}_i(x) + \bar{a}_i^\dagger \bar{u}_i^*(x) \right], \quad \bar{a}_i |\bar{0}\rangle = 0, \quad \forall i, \quad \bar{a}_i |0\rangle \neq 0, \quad \forall i$$

Since both set are complete $\implies$ *Bogolubov transformations*

$$\bar{u}_j(x) = \sum_i [\alpha_{ji} u_i(x) + \beta_{ji} u_i^*(x)] \quad u_i(x) = \sum_j [\alpha_{ji}^* \bar{u}_j(x) - \beta_{ji} \bar{u}_j^*(x)]$$

$$a_i = \sum_j (\alpha_{ji} \bar{a}_j + \beta_{ji} \bar{a}_j^\dagger) \quad \bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger)$$

Number of u_i -mode particles in the state $|\bar{0}\rangle$:

$$\langle \bar{0} | N_i | \bar{0} \rangle = \sum_j |\beta_{ji}|^2 \neq 0!$$

Particles: observer-dependent concept

Parker, Hawking, Unruh effects

THE STRESS-ENERGY TENSOR

- **The concept of particles is rather ambiguous.** Global concept in terms of the field modes (a_k) or not even well-defined for non-stationary space times.

$$[H, N] = 0 \longleftrightarrow \exists \xi^\mu \text{ (timelike)} / \mathcal{L}_\xi g = 0$$

- Better to investigate **physical quantities** that are **defined locally**: $T_{\mu\nu}(x)$.

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$$

- Plays an important role in the **back-reaction** problem

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$$

- Expectation values of quantities quadratic in the field **diverge in the UV**. In Minkowski normal-ordering is applied.
- In curved space-time the gravitational interaction introduces additional divergences. Furthermore, vacuum energy can give rise to gravitational effects. **New regularization methods should be studied.**

ADIABATIC RENORMALIZATION

Consider a general free K-G field determined by the dynamics

$$(\square + m^2 + \xi R)\phi(x) = 0$$

propagating in a FLRW spacetime described by the line element

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2$$

Expand the field in modes

$$\phi(x) = \int d^3k \left[a_{\vec{k}} f_{\vec{k}}(t, \vec{x}) + a_{\vec{k}}^\dagger f_{\vec{k}}^*(t, \vec{x}) \right], \quad f_{\vec{k}}(t, \vec{x}) = \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{2(2\pi)^3 a^3(t)}} h_k(t)$$

where the modes obey the equation of motion

$$\ddot{h}_k + (\omega_k^2 + \sigma) h_k(t) = 0, \quad \sigma = \left(6\xi - \frac{3}{4}\right) \frac{\dot{a}^2}{a^2} + \left(6\xi - \frac{3}{2}\right) \frac{\ddot{a}}{a}$$

For $\omega_k^2 \gg \sigma \rightarrow$ Minkowski...

ADIABATIC RENORMALIZATION

...take an WKB-type asymptotic expansion

$$h_k^{Mink}(t) = \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k t} \longrightarrow h_k(t) \sim \frac{1}{\sqrt{W_k(t)}} e^{-i \int^t W_k(x) dx}$$

then the equation of motion provides an implicit equation for W_k

$$W_k^2 = \omega_k^2 + \sigma + W_k^{1/2} \frac{d^2}{dt^2} W_k^{-1/2}$$

which can be solved iteratively expanding W_k in *adiabatic* series [by means of derivatives of the metric, i.e. of $a(t)$]

$$W_k(t) = \omega^{(0)} + \omega^{(2)} + \omega^{(4)} \dots, \quad \omega^{(0)} = \omega_k = \sqrt{\frac{k^2}{a^2(t)} + m^2}$$

We recover the Minkowskian-type solutions in the adiabatic limit ($\dot{a}(t) \rightarrow 0$)

ADIABATIC RENORMALIZATION

IMPORTANT REMARKS:

- The 0^{th} **order (Minkowskian) contribution uniquely determines the adiabatic expansion** of $h_k(t)$. However, the expansion yields asymptotic, not convergent, series.
- The **adiabatic solution only provides the asymptotic behaviour** (in large k) of the general solution of $h_k(t)$. It may have a non-perturbative contribution. Therefore, it **cannot be regarded as a formal solution** to the equation of motion.
- The **adiabatic solution is geometrical and unique**, while the non-perturbative one contains the information about the choice of the quantum state. All the possible solutions for $h_k(t)$ share the same adiabatic behaviour.

ADIABATIC RENORMALIZATION

In the adiabatic scheme the physically relevant result is obtained from the formal one (UV) by subtracting the adiabatic expansion up to the last order containing UV divergences (general ξ and m)

$$\begin{aligned}\langle 0 | \phi^2(x) | 0 \rangle &= \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 |h_k(t)|^2 \sim \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 W_k^{-1} \\ &= \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \left[\omega_k^{-1} + (W_k^{-1})^{(2)} + (W_k^{-1})^{(4)} + \dots \right]\end{aligned}$$

From the iterative solutions we find $(W_k^{-1})^{(n)} \sim k^{-(n+1)}$, so two different divergences appear. The physical quantity should read then

$$\langle 0 | \phi^2(x) | 0 \rangle_{phys} = \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \left[|h_k(t)|^2 - \omega_k^{-1} - (W_k^{-1})^{(2)} \right] \sim 0 + O(T^{-4})$$

ADIABATIC RENORMALIZATION

In the adiabatic scheme the physically relevant result is obtained from the formal one (UV) by subtracting the adiabatic expansion up to the last order containing UV divergences (general ξ and m)

$$\begin{aligned}\langle 0 | \phi^2(x) | 0 \rangle &= \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 |h_k(t)|^2 \sim \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 W_k^{-1} \\ &= \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \left[\omega_k^{-1} + (W_k^{-1})^{(2)} + (W_k^{-1})^{(4)} + \dots \right]\end{aligned}$$

From the iterative solutions we find $(W_k^{-1})^{(n)} \sim k^{-(n+1)}$, so two different divergences appear. The physical quantity should read then

$$\langle 0 | \phi^2(x) | 0 \rangle_{phys} = \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \left[|h_k(t)|^2 - \omega_k^{-1} - (W_k^{-1})^{(2)} \right] \sim 0 + O(T^{-4})$$

ADIABATIC RENORMALIZATION

Stress-energy tensor for a K-G field,

$$T_{\mu\nu}(x) = \nabla_\mu \phi(x) \nabla_\nu \phi(x) - \frac{1}{2} g_{\mu\nu} \nabla_\rho \phi(x) \nabla^\rho \phi(x) + \frac{1}{2} g_{\mu\nu} m^2 \phi^2(x) \\ - \xi G_{\mu\nu} \phi^2(x) + \xi [g_{\mu\nu} \square \phi^2(x) - \nabla_\mu \nabla_\nu \phi^2(x)]$$

For simplicity consider the case $\xi = 1/6$,

$$T_\mu^\mu(x) = m^2 \phi^2(x)$$

A finite quantity implies subtracting up to fourth adiabatic order. Renormalized solution reads then

$$\langle 0 | T_\mu^\mu(x) | 0 \rangle_{phys} = m^2 \left[\langle 0 | \phi^2(x) | 0 \rangle \right. \\ \left. - \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \left(\omega_k^{-1} + (W_k^{-1})^{(2)} + (W_k^{-1})^{(4)} \right) \right] \\ = m^2 \left[\langle 0 | \phi^2(x) | 0 \rangle_{phys} - \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 (W_k^{-1})^{(4)} \right]$$

ADIABATIC RENORMALIZATION

Stress-energy tensor for a K-G field,

$$T_{\mu\nu}(x) = \nabla_\mu \phi(x) \nabla_\nu \phi(x) - \frac{1}{2} g_{\mu\nu} \nabla_\rho \phi(x) \nabla^\rho \phi(x) + \frac{1}{2} g_{\mu\nu} m^2 \phi^2(x) \\ - \xi G_{\mu\nu} \phi^2(x) + \xi [g_{\mu\nu} \square \phi^2(x) - \nabla_\mu \nabla_\nu \phi^2(x)]$$

For simplicity consider the case $\xi = 1/6$,

$$T_\mu^\mu(x) = m^2 \phi^2(x)$$

A finite quantity implies subtracting up to fourth adiabatic order. Renormalized solution reads then

$$\langle 0 | T_\mu^\mu(x) | 0 \rangle_{phys} = m^2 \left[\langle 0 | \phi^2(x) | 0 \rangle \right. \\ \left. - \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 \left(\omega_k^{-1} + (W_k^{-1})^{(2)} + (W_k^{-1})^{(4)} \right) \right] \\ = m^2 \left[\langle 0 | \phi^2(x) | 0 \rangle_{phys} - \frac{1}{4\pi^2 a^3(t)} \int_0^\infty dk k^2 (W_k^{-1})^{(4)} \right]$$

ADIABATIC RENORMALIZATION

For a conformal field ($m \rightarrow 0$),

$$\begin{aligned}\langle 0 | T^\mu_\mu | 0 \rangle_{Phys} (m \rightarrow 0) &= \frac{1}{480\pi^2} \left[\frac{\ddot{a}^2}{a^2} + \frac{\dddot{a}}{a} - 3 \frac{\ddot{a}^2 \dot{a}^2}{a^3} + 3 \frac{\ddot{a} \dddot{a}}{a^2} \right] \\ &= \frac{1}{2880\pi^2} \left[\square R - \left(R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R^2 \right) \right]\end{aligned}$$

we do **NOT** get a **vanishing trace** $\Rightarrow$ **conformal trace anomaly**

RENORMALIZATION IN GENERAL SPACE-TIMES

Consider the line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Expand the field in modes

$$\phi(x) = \sum_j \left[a_j f_j(x) + a_j^\dagger f_j(x)^* \right], \quad a_j |0\rangle = 0, \forall j$$

Analogue of the adiabatic condition specified in the Feynman Green function

$$[\square_x + m^2 + \xi R(x)] G_F(x, x') = -|g(x)|^{-1/2} \delta(x - x')$$

with

$$G_F(x, x') \sim \frac{\Delta^{1/2}}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-is\left(m^2 + \frac{\sigma(x, x')}{2s^2}\right)}, \quad \sigma(x, x') = \frac{1}{2} \tau(x, x')^2$$

RENORMALIZATION IN GENERAL SPACE-TIMES

The rest of the terms of the proper-time expansion are univocally determined from the zeroth order by iteration (coupled differential eqs.)

$$G_F(x, x') \sim \frac{\Delta^{1/2}}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-is\left(m^2 + \frac{\sigma(x, x')}{2s^2}\right)} F(x, x', is)$$

with

$$F(x, x', is) = \sum_{n=0}^{\infty} a_n(x, x')(is)^n$$

For renormalization only the coincidence limits are important

$$a_0(x) = 1, \quad a_1(x) = \left(\frac{1}{6} - \xi\right) R,$$

$$a_2(x) = \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{6} \left(\frac{1}{5} - \xi\right) \square R + \frac{1}{2} \left(\frac{1}{6} - \xi\right)^2 R^2$$

DeWitt-Schwinger (proper-time) formalism: generally covariant way of expressing the divergent terms

RENORMALIZATION IN GENERAL SPACE-TIMES

Divergences at $s = 0$ are the short distance UV divergences that must be subtracted because they appear when $\sigma(x, x') \rightarrow 0$. For the two point function...

$$\langle 0|\phi^2(x)|0\rangle_{phys} = \lim_{x \rightarrow x'} \left\{ \langle 0|\phi(x)\phi(x')|0\rangle - iG_F^{(2)}(x, x') \right\}$$

and for the trace of the stress-energy tensor...

$$\begin{aligned} \langle T^\mu_\mu(x) \rangle_{Phys} &= m^2 \left\{ \langle \phi^2(x) \rangle - i \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-ism^2} [1 + a_1(is) + a_2(is)^2] \right\}_{\xi=0} \\ &\quad m^2 \left\{ \langle \phi^2(x) \rangle_{phys} - i \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-ism^2} a_2(x)(is)^2 \right\}_{\xi=1/6} \end{aligned}$$

RENORMALIZATION IN GENERAL SPACE-TIMES

The conformal anomaly is identical to that obtained via the adiabatic scheme

$$\begin{aligned}\langle T_{\mu}^{\mu}(x) \rangle_{Phys} (m \rightarrow 0) &= -\frac{a_2(x)_{\xi=1/6}}{16\pi^2} \\ &= \frac{1}{2880\pi^2} \left[\square R - \left(R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R^2 \right) \right]\end{aligned}$$

this suggests a duality about both schemes...

EFFECTIVE ACTION. RENORMALIZED COUPLING CONSTANTS

Using path-integral quantization one can show that the effective action for the quantum field in this gravitational scenario is given by

$$W = \int \sqrt{-g(x)} L_{\text{eff}}(x) d^4x = \frac{i}{2} \lim_{x' \rightarrow x} \int_{m^2}^{\infty} dm^2 G_F(x, x')$$

Taking the asymptotic geometrical structure of the Green function one can identify the effective lagrangian as

$$L_{\text{eff}} \sim \lim_{x' \rightarrow x} \frac{\Delta^{1/2}(x, x')}{32\pi^2} \sum_{j=0}^{\infty} a_j(x, x') \int_0^{\infty} ds (is)^{j-1-n/2} e^{-i(m^2 s - \frac{\sigma}{2s^2})}$$

Using dimensional regularization and taking the coincidence limit we get

$$L_{\text{eff}} \sim \frac{1}{32\pi^2} \left(\frac{m}{\mu} \right)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma\left(j - \frac{n}{2}\right)$$

EFFECTIVE ACTION. RENORMALIZED COUPLING CONSTANTS

As $n \rightarrow 4$ the leading three contributions diverge. General behaviour for the divergent lagrangian:

$$L_{div} \sim \frac{1}{n-4} [A m^4 a_0(x) + B m^2 a_1(x) + C a_2(x)]$$

... which is purely geometrical $\implies$ absorb it in the gravitational lagrangian!

$$L_{Tot} = L_g + L_{eff} = \frac{1}{16\pi G}(-2\Lambda + R) + L_{div} + L_{ren}$$

$a_0(x)$: 0^{th} order $\implies$ renormalized cosmological constant Λ_{ren}

$a_1(x)$: 2^{nd} order $\implies$ renormalized Newton's constant G_{ren}

$a_2(x)$: 4^{th} order $\implies$ higher order corrections to GR!

Renormalization predicts **new gravitational physics** [lagrangian terms R^2 , $R_{\mu\nu}R^{\mu\nu}$]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha^{(1)} H_{\mu\nu} + \beta^{(2)} H_{\mu\nu} = 8\pi G T_{\mu\nu}$$

EFFECTIVE ACTION. RENORMALIZED COUPLING CONSTANTS

As $n \rightarrow 4$ the leading three contributions diverge. General behaviour for the divergent lagrangian:

$$L_{div} \sim \frac{1}{n-4} [A m^4 a_0(x) + B m^2 a_1(x) + C a_2(x)]$$

... which is purely geometrical $\Rightarrow$ absorb it in the gravitational lagrangian!

$$L_{Tot} = L_g + L_{eff} = \frac{1}{16\pi G}(-2\Lambda + R) + L_{div} + L_{ren}$$

$a_0(x) : 0^{th}$ order $\Rightarrow$ renormalized cosmological constant Λ_{ren}

$a_1(x) : 2^{nd}$ order $\Rightarrow$ renormalized Newton's constant G_{ren}

$a_2(x) : 4^{th}$ order $\Rightarrow$ higher order corrections to GR!

Renormalization predicts **new gravitational physics** [lagrangian terms $R^2, R_{\mu\nu}R^{\mu\nu}$]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha^{(1)} H_{\mu\nu} + \beta^{(2)} H_{\mu\nu} = 8\pi G T_{\mu\nu}$$

EFFECTIVE ACTION. RENORMALIZED COUPLING CONSTANTS

As $n \rightarrow 4$ the leading three contributions diverge. General behaviour for the divergent lagrangian:

$$L_{div} \sim \frac{1}{n-4} [A m^4 a_0(x) + B m^2 a_1(x) + C a_2(x)]$$

... which is purely geometrical $\implies$ absorb it in the gravitational lagrangian!

$$L_{Tot} = L_g + L_{eff} = \frac{1}{16\pi G}(-2\Lambda + R) + L_{div} + L_{ren}$$

$a_0(x) : 0^{th}$ order $\implies$ renormalized cosmological constant Λ_{ren}

$a_1(x) : 2^{nd}$ order $\implies$ renormalized Newton's constant G_{ren}

$a_2(x) : 4^{th}$ order $\implies$ higher order corrections to GR!

Renormalization predicts **new gravitational physics** [lagrangian terms $R^2, R_{\mu\nu}R^{\mu\nu}$]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha^{(1)} H_{\mu\nu} + \beta^{(2)} H_{\mu\nu} = 8\pi G T_{\mu\nu}$$

EFFECTIVE ACTION. RENORMALIZED COUPLING CONSTANTS

As $n \rightarrow 4$ the leading three contributions diverge. General behaviour for the divergent lagrangian:

$$L_{div} \sim \frac{1}{n-4} [A m^4 a_0(x) + B m^2 a_1(x) + C a_2(x)]$$

... which is purely geometrical $\Rightarrow$ absorb it in the gravitational lagrangian!

$$L_{Tot} = L_g + L_{eff} = \frac{1}{16\pi G}(-2\Lambda + R) + L_{div} + L_{ren}$$

$a_0(x)$: 0^{th} order $\Rightarrow$ renormalized cosmological constant Λ_{ren}

$a_1(x)$: 2^{nd} order $\Rightarrow$ renormalized Newton's constant G_{ren}

$a_2(x)$: 4^{th} order $\Rightarrow$ higher order corrections to GR!

Renormalization predicts **new gravitational physics** [lagrangian terms R^2 , $R_{\mu\nu}R^{\mu\nu}$]:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha^{(1)} H_{\mu\nu} + \beta^{(2)} H_{\mu\nu} = 8\pi G T_{\mu\nu}$$

INFLATIONARY COSMOLOGY

MAIN IDEA

The very early universe underwent a period of very rapid expansion described by a (quasi) *de Sitter* spacetime, called Inflation

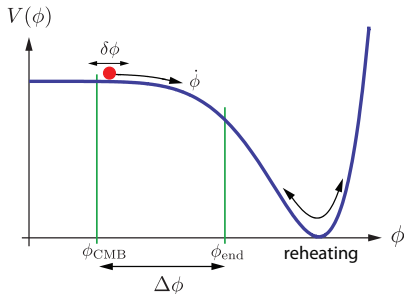
$$\text{FLRW metric : } ds^2 = dt^2 - a^2(t)d\vec{x}^2, \quad a(t) \sim e^{Ht}, \quad H \equiv \frac{\dot{a}}{a} \approx \text{cst}$$

[Guth (1981); Starobinsky (1980); Linde (1982), (1983); Albrecht, Steinhardt (1982); Sato (1981)]

Powered by a source of constant vacuum energy, $H \sim \sqrt{\Lambda} \equiv \sqrt{V(0)}$, provided by a field theory.

Slow roll scenario,

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \delta \equiv \frac{\ddot{H}}{2\dot{H}H^2}$$



INFLATION + QUANTUM FLUCTUATIONS

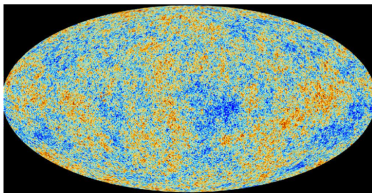
EXCITING FEATURE

Spontaneous production of perturbations/particles during inflation as the **origin of the primordial cosmological fluctuations**

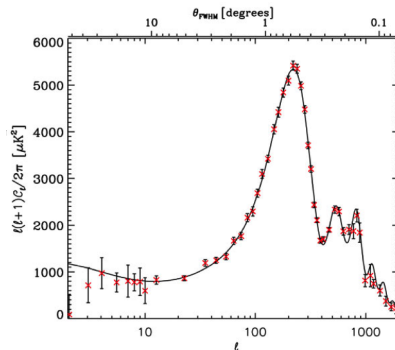
[Mukhanov, Chibisov (1981); Hawking (1982); Guth, Pi (1982); Starobinsky (1982); Bardeen, Steinhardt, Turner (1983)]

Scalar perturbations:

$$\phi(x) = \phi_0(t) + \delta\phi(t, \vec{x})$$



CMB anisotropies in temperature



SCALAR PERTURBATIONS

Scalar fluctuations generated during inflation:

$$\phi(x) = \phi_0(t) + \delta\phi(t, \vec{x}) \quad + \quad ds^2 = (1 + 2\Psi)dt^2 - a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j$$

Curvature perturbations (gauge invariant):

$$\mathcal{R} = \Psi + H \frac{\delta\phi}{\dot{\phi}_0}, \quad \text{study of a quantum massless "scalar" field}$$

For modes k outside the horizon scale ($k/a \ll H$):

$$\mathcal{R}_k(t) \rightarrow \mathcal{R}_k^0 \approx \text{cst}, \quad \mathcal{P}_{\mathcal{R}}(k) \sim k^3 |\mathcal{R}_k^0|^2 \sim k^{-4\epsilon - 2\delta}$$

- Origin of the primordial perturbations of matter density
- CMB temperature fluctuations and large structure formation

SCALAR PERTURBATIONS

Scalar fluctuations generated during inflation:

$$\phi(x) = \phi_0(t) + \delta\phi(t, \vec{x}) \quad + \quad ds^2 = (1 + 2\Psi)dt^2 - a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j$$

Curvature perturbations (gauge invariant):

$$\mathcal{R} = \Psi + H \frac{\delta\phi}{\dot{\phi}_0}, \quad \text{study of a quantum massless "scalar" field}$$

For modes k outside the horizon scale ($k/a \ll H$):

$$\mathcal{R}_k(t) \rightarrow \mathcal{R}_k^0 \approx \text{cst}, \quad \mathcal{P}_{\mathcal{R}}(k) \sim k^3 |\mathcal{R}_k^0|^2 \sim k^{-4\epsilon - 2\delta}$$

- Origin of the primordial perturbations of matter density
- CMB temperature fluctuations and large structure formation

SCALAR PERTURBATIONS

Scalar fluctuations generated during inflation:

$$\phi(x) = \phi_0(t) + \delta\phi(t, \vec{x}) \quad + \quad ds^2 = (1 + 2\Psi)dt^2 - a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j$$

Curvature perturbations (gauge invariant):

$$\mathcal{R} = \Psi + H \frac{\delta\phi}{\dot{\phi}_0}, \quad \text{study of a quantum massless "scalar" field}$$

For modes k outside the horizon scale ($k/a \ll H$):

$$\mathcal{R}_k(t) \rightarrow \mathcal{R}_k^0 \approx cst, \quad \mathcal{P}_{\mathcal{R}}(k) \sim k^3 |\mathcal{R}_k^0|^2 \sim k^{-4\epsilon-2\delta}$$

- Origin of the primordial perturbations of matter density
- CMB temperature fluctuations and large structure formation

SCALAR PERTURBATIONS

Two-point function of scalar perturbations in a *slow-roll* scenario,

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle = \frac{GH^2}{4\pi\epsilon} \Gamma\left(\frac{3}{2} + \nu\right) \Gamma\left(\frac{3}{2} - \nu\right) {}_2F_1\left(\frac{3}{2} + \nu, \frac{3}{2} - \nu; 2; 1 - \frac{H^2 a^2 |\Delta \vec{x}|^2}{4}\right)$$

For **large separations**, $a|\vec{x} - \vec{x}'| \gg H^{-1}$,

$$\nu = \frac{3}{2} + \frac{2\epsilon + \delta}{1 - \epsilon}$$

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim |\Delta \vec{x}|^{2(\nu - 3/2)},$$

The scalar amplitude is

- **nearly scale invariant** ($\nu \approx 3/2$)
- **time independent**

Define $t_{\Delta x}$ as, $a(t_{\Delta x})|\Delta \vec{x}| = H^{-1}(t_{\Delta x})$

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim -\frac{4\pi G}{(1-n)\epsilon} \left(\frac{H(t_{\Delta x})}{2\pi} \right)^2$$

$n = 4 - 2\nu \lesssim 1$: scalar spectral index

SCALAR PERTURBATIONS

Two-point function of scalar perturbations in a *slow-roll* scenario,

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle = \frac{GH^2}{4\pi\epsilon} \Gamma\left(\frac{3}{2} + \nu\right) \Gamma\left(\frac{3}{2} - \nu\right) {}_2F_1\left(\frac{3}{2} + \nu, \frac{3}{2} - \nu; 2; 1 - \frac{H^2 a^2 |\Delta \vec{x}|^2}{4}\right)$$

For **large separations**, $a|\vec{x} - \vec{x}'| \gg H^{-1}$,

$$\nu = \frac{3}{2} + \frac{2\epsilon + \delta}{1 - \epsilon}$$

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim |\Delta \vec{x}|^{2(\nu - 3/2)},$$

The scalar amplitude is

- **nearly scale invariant** ($\nu \approx 3/2$)
- **time independent**

Define $t_{\Delta x}$ as, $a(t_{\Delta x})|\Delta \vec{x}| = H^{-1}(t_{\Delta x})$

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim -\frac{4\pi G}{(1-n)\epsilon} \left(\frac{H(t_{\Delta x})}{2\pi} \right)^2$$

$n = 4 - 2\nu \lesssim 1$: scalar spectral index

SCALAR PERTURBATIONS

Two-point function of scalar perturbations in a *slow-roll* scenario,

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle = \frac{GH^2}{4\pi\epsilon} \Gamma\left(\frac{3}{2} + \nu\right) \Gamma\left(\frac{3}{2} - \nu\right) {}_2F_1\left(\frac{3}{2} + \nu, \frac{3}{2} - \nu; 2; 1 - \frac{H^2 a^2 |\Delta \vec{x}|^2}{4}\right)$$

For **large separations**, $a|\vec{x} - \vec{x}'| \gg H^{-1}$,

$$\nu = \frac{3}{2} + \frac{2\epsilon + \delta}{1 - \epsilon}$$

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim |\Delta \vec{x}|^{2(\nu - 3/2)},$$

The scalar amplitude is

- **nearly scale invariant** ($\nu \approx 3/2$)
- **time independent**

Define $t_{\Delta x}$ as, $a(t_{\Delta x})|\Delta \vec{x}| = H^{-1}(t_{\Delta x})$

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim -\frac{4\pi G}{(1-n)\epsilon} \left(\frac{H(t_{\Delta x})}{2\pi} \right)^2$$

$n = 4 - 2\nu \lesssim 1$: scalar spectral index

DETERMINATION OF C_ℓ

Take the **large-distance behaviour**,

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim |\Delta \vec{x}|^{1-n}$$

Angular power spectrum in the Sachs-Wolfe regime

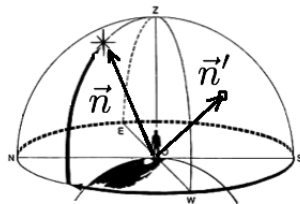
$$C_\ell = 2\pi \int_{-1}^1 d(\cos \theta) P_\ell(\cos \theta) \langle \Delta T(\vec{n}) \Delta T(\vec{n}') \rangle ,$$

$$\langle \Delta T(\vec{n}) \Delta T(\vec{n}') \rangle_{SW} = \frac{T_0^2}{25} \langle \mathcal{R}(r_L \vec{n}) \mathcal{R}(r_L \vec{n}') \rangle$$

Final result:

$$C_\ell^{SW} = \frac{8\pi T_0^2}{25} \frac{4\pi G}{\epsilon} \frac{H^2(1-\epsilon)^2}{16\pi^2} \frac{\Gamma(3-n)\Gamma(\ell+\frac{n-1}{2})}{\Gamma(\ell+2-\frac{n-1}{2})} \bar{r}_L^{1-n} \approx \frac{2GH^2 T_0^2}{25\epsilon} \frac{\bar{r}_L^{1-n}}{\ell(\ell+1)}$$

Expected identical result using the momentum representation of the modes (power spectrum)



$$\vec{n}' = \frac{\vec{x}'}{|\vec{x}'|} , \quad \vec{n} = \frac{\vec{x}}{|\vec{x}|}$$

$$|\vec{x}| = |\vec{x}'| = r_L , \quad \cos \theta = \vec{n}' \cdot \vec{n}$$

DETERMINATION OF C_ℓ

Take the **large-distance behaviour**,

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim |\Delta \vec{x}|^{1-n}$$

Angular power spectrum in the Sachs-Wolfe regime

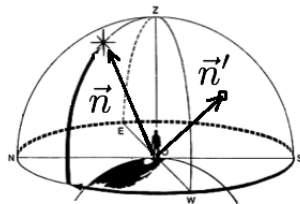
$$C_\ell = 2\pi \int_{-1}^1 d(\cos \theta) P_\ell(\cos \theta) \langle \Delta T(\vec{n}) \Delta T(\vec{n}') \rangle ,$$

$$\langle \Delta T(\vec{n}) \Delta T(\vec{n}') \rangle_{SW} = \frac{T_0^2}{25} \langle \mathcal{R}(r_L \vec{n}) \mathcal{R}(r_L \vec{n}') \rangle$$

Final result:

$$C_\ell^{SW} = \frac{8\pi T_0^2}{25} \frac{4\pi G}{\epsilon} \frac{H^2(1-\epsilon)^2}{16\pi^2} \frac{\Gamma(3-n)\Gamma(\ell+\frac{n-1}{2})}{\Gamma(\ell+2-\frac{n-1}{2})} \bar{r}_L^{1-n} \approx \frac{2GH^2 T_0^2}{25\epsilon} \frac{\bar{r}_L^{1-n}}{\ell(\ell+1)}$$

Expected identical result using the momentum representation of the modes (power spectrum)



$$\vec{n}' = \frac{\vec{x}'}{|\vec{x}'|} , \quad \vec{n} = \frac{\vec{x}}{|\vec{x}|}$$

$$|\vec{x}| = |\vec{x}'| = r_L , \quad \cos \theta = \vec{n}' \cdot \vec{n}$$

DETERMINATION OF C_ℓ

Take the **large-distance behaviour**,

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim |\Delta \vec{x}|^{1-n}$$

Angular power spectrum in the Sachs-Wolfe regime

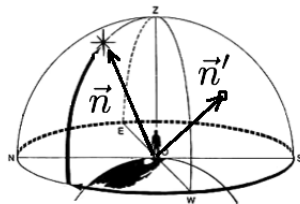
$$C_\ell = 2\pi \int_{-1}^1 d(\cos \theta) P_\ell(\cos \theta) \langle \Delta T(\vec{n}) \Delta T(\vec{n}') \rangle ,$$

$$\langle \Delta T(\vec{n}) \Delta T(\vec{n}') \rangle_{SW} = \frac{T_0^2}{25} \langle \mathcal{R}(r_L \vec{n}) \mathcal{R}(r_L \vec{n}') \rangle$$

Final result:

$$C_\ell^{SW} = \frac{8\pi T_0^2}{25} \frac{4\pi G}{\epsilon} \frac{H^2(1-\epsilon)^2}{16\pi^2} \frac{\Gamma(3-n)\Gamma(\ell+\frac{n-1}{2})}{\Gamma(\ell+2-\frac{n-1}{2})} \bar{r}_L^{1-n} \approx \frac{2GH^2 T_0^2}{25\epsilon} \frac{\bar{r}_L^{1-n}}{\ell(\ell+1)}$$

Expected identical result using the momentum representation of the modes (power spectrum)



$$\vec{n}' = \frac{\vec{x}'}{|\vec{x}'|} , \quad \vec{n} = \frac{\vec{x}}{|\vec{x}|}$$

$$|\vec{x}| = |\vec{x}'| = r_L , \quad \cos \theta = \vec{n}' \cdot \vec{n}$$

RENORMALIZED OBSERVABLE

- However, if we also consider the **short-distance behaviour...**

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim \frac{A(t)}{|\Delta \vec{x}|^2} + B |\Delta \vec{x}|^{1-n} \Rightarrow \text{divergent contribution to } C_\ell^{SW}$$

- We propose adiabatic renormalization

[in momentum space see Parker (2007); Agulló, Navarro-Salas, Olmo, Parker (2012)]

$$\begin{aligned} \langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle_{ren} &\equiv \langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle - G_{Ad}^{(2)}[(t, \vec{x}'), (t, \vec{x})] \\ G_{Ad}^{(2)}[(t, \vec{x}'), (t, \vec{x})] &\sim \frac{A(t)}{|\Delta \vec{x}|^2} + C H^2 \log |\Delta \vec{x}|^2 \end{aligned}$$

- So the renormalized observable reads

$$C_{\ell, ren}^{SW} \approx \frac{4\pi G}{\epsilon} \frac{8\pi T_0^2}{25} \frac{H^2 (1-\epsilon)^2 \bar{r}_L^{1-n}}{16\pi^2} \left\{ \frac{\Gamma(\ell + \frac{n-1}{2})}{\Gamma(\ell + 2 - \frac{n-1}{2})} - \frac{\bar{r}_L^{n-1}}{\ell(\ell+1)} \right\}$$

[A. del Río, J. Navarro-Salas, PRD 2014; JPCS 2014]

RENORMALIZED OBSERVABLE

- However, if we also consider the **short-distance behaviour...**

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim \frac{A(t)}{|\Delta \vec{x}|^2} + B |\Delta \vec{x}|^{1-n} \Rightarrow \text{divergent contribution to } C_\ell^{SW}$$

- We propose adiabatic renormalization

[in momentum space see Parker (2007); Agulló, Navarro-Salas, Olmo, Parker (2012)]

$$\begin{aligned} \langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle_{ren} &\equiv \langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle - G_{Ad}^{(2)}[(t, \vec{x}'), (t, \vec{x})] \\ G_{Ad}^{(2)}[(t, \vec{x}'), (t, \vec{x})] &\sim \frac{A(t)}{|\Delta \vec{x}|^2} + C H^2 \log |\Delta \vec{x}|^2 \end{aligned}$$

- So the renormalized observable reads

$$C_{\ell, ren}^{SW} \approx \frac{4\pi G}{\epsilon} \frac{8\pi T_0^2}{25} \frac{H^2 (1-\epsilon)^2 \tilde{r}_L^{1-n}}{16\pi^2} \left\{ \frac{\Gamma(\ell + \frac{n-1}{2})}{\Gamma(\ell + 2 - \frac{n-1}{2})} - \frac{\tilde{r}_L^{n-1}}{\ell(\ell+1)} \right\}$$

[A. del Río, J. Navarro-Salas, PRD 2014; JPCS 2014]

RENORMALIZED OBSERVABLE

- However, if we also consider the **short-distance behaviour**...

$$\langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle \sim \frac{A(t)}{|\Delta \vec{x}|^2} + B |\Delta \vec{x}|^{1-n} \Rightarrow \text{divergent contribution to } C_\ell^{SW}$$

- We propose adiabatic renormalization

[in momentum space see Parker (2007); Agulló, Navarro-Salas, Olmo, Parker (2012)]

$$\begin{aligned} \langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle_{ren} &\equiv \langle \mathcal{R}(t, \vec{x}) \mathcal{R}(t, \vec{x}') \rangle - G_{Ad}^{(2)}[(t, \vec{x}'), (t, \vec{x})] \\ G_{Ad}^{(2)}[(t, \vec{x}'), (t, \vec{x})] &\sim \frac{A(t)}{|\Delta \vec{x}|^2} + C H^2 \log |\Delta \vec{x}|^2 \end{aligned}$$

- So the renormalized observable reads

$$C_{\ell, ren}^{SW} \approx \frac{4\pi G}{\epsilon} \frac{8\pi T_0^2}{25} \frac{H^2(1-\epsilon)^2 \tilde{r}_L^{1-n}}{16\pi^2} \left\{ \frac{\Gamma(\ell + \frac{n-1}{2})}{\Gamma(\ell + 2 - \frac{n-1}{2})} - \frac{\tilde{r}_L^{n-1}}{\ell(\ell+1)} \right\}$$

[A. del Río, J. Navarro-Salas, PRD 2014; JPCS 2014]

ADDITIONAL IMPLICATIONS

- Compatible with observations (**scale invariance**), but may reduce the amplitude (H) significantly
- The renormalized version provides $C_\ell = 0$ for perfect *de Sitter* space-time. The correction is justified in terms of symmetries, since no particle production/cosmological fluctuations (eventual temperature fluctuations) are expected.
- C_ℓ is definite positive according to its original definition, $C_\ell = \langle |a_{\ell m}|^2 \rangle$, which implies that primordial perturbations occurs when

$$a(t)r_L \gtrsim H^{-1}$$

and the effective beginning of inflation can be regarded to take place at this moment.

[A. del Río, J. Navarro-Salas, *PRD* 2014; *JPCS* 2014]

CONCLUSIONS

- Renormalization: highly powerful and predictive theory
- Possibility of extension of these methods to higher spin fields. New features.
- Adiabatic vs DeWitt-Schwinger. Equivalence.

Thank you very much for your attention!