

Color-Kinematics Duality and the Regge Limit

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13 October 2014

Work published in JHEP in collaboration with Agustín Sabio Vera,
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The limitations of perturbation theory

- The limitations of high energy physics are those of perturbation theory
- When it is well defined it is very difficult to make calculations at higher orders
- Interesting kinematic regions have ill defined perturbation expansions
- Interesting behaviour is lost in the expansion altogether

Effective Vertices means less diagrams

- The number of diagrams grows unmanageably with the order of a calculation
- This makes expansions asymptotic in general requiring a resummation
- By calculating effective vertices we can drastically cut down the number of contributing diagrams

High Energies and the need for resummations

- At high center of mass energies s , powers of logarithms of the energy can render this expansion invalid if $\kappa\sqrt{s}\ln s \approx \mathcal{O}(1)$
- Must work with an all order resummation.
- These large logarithms $\alpha_s^n \ln^n(s) \sim \alpha_s^n (y_A - y_B)^n$ are generated by final state configurations with strong ordering in the rapidities of the emitted particles, with comparable transverse momenta: **Multi-Regge Kinematics**

$$y_A \gg y_1 \gg y_2 \gg \dots \gg y_n \gg y_B,$$

$$k_i^2 \approx k^2.$$

Resumming the logarithms

$$\sigma_{tot}(s = e^{y_A - y_B}) \stackrel{s \rightarrow \infty}{\equiv} \frac{1}{s} \sum_{n=0}^{\infty} \left| \begin{array}{c} \text{Diagram: A central shaded circle with two straight lines entering from the left and two wavy lines exiting to the right. The top wavy line is labeled } y_i, k_{i\perp}^2 \text{ and the bottom wavy line is labeled } y_n, k_{i\perp}^2. \end{array} \right|^2$$

$$= \frac{1}{s} \sum_{n=0}^{\infty} \left| \begin{array}{c} \text{Diagram: A vertical stack of three shaded rectangular blocks connected by wavy lines. The top block is labeled A and the bottom block is labeled B. The middle section is labeled n.} \end{array} \right|^2 = \frac{1}{s} \text{Im}\{A_{\text{elast}}(s, t=0)\}$$

Effective Vertex in Multi-Regge Kinematics-1

The vertices in these diagrams are effective vertices

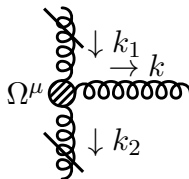


Figure: Lipatov Effective Vertex

Effective Vertex in Multi-Regge Kinematics-2

The effective vertex is given by

$$\Omega^\nu = \left[\left(\alpha_1 - \frac{2\beta_1}{\beta_2} \right) p^\nu + \left(\beta_2 + \frac{2\alpha_2}{\alpha_1} \right) q^\nu - (k_1 + k_2)_\perp^\nu \right]$$

Effective Vertex in Multi-Regge Kinematics-3

In gravity the vertex is given by

$$\Omega^\mu \Omega^\nu - \mathcal{N}^\mu \mathcal{N}^\nu$$

$$\mathcal{N}^\mu = 2\sqrt{k_1^2 k_2^2} \left(\frac{p^\mu}{s_{p'k}} - \frac{q^\mu}{s_{q'k}} \right) \simeq -2i\sqrt{\beta_1 \alpha_2} \left(\frac{p^\mu}{\beta_2} + \frac{q^\mu}{\alpha_1} \right)$$

The $\mathcal{N}^2$ part cancels double poles in overlapping channels in agreement with Steinmann's relations to preserve unitarity.

Gauge² $\equiv$ Gravity

- The relation between scattering amplitudes in gauge and gravity theories has been the subject of extensive research in recent years
- tree-level graviton (closed string) amplitudes can be written as linear combinations of the product of gluon (open string) amplitudes, as found by Kawai, Lewellen, and Tye (KLT)

$$\mathcal{M}_n^{grav} \sim A_{i,n}^{YM} K_{ij} A_{j,n}^{YM}.$$

Color-Kinematics duality-1

- The Bern, Carrasco, and Johansson (BCJ) prescription allows us to obtain graviton amplitudes from gluon amplitudes of two YM theories.
- The first step is to write the amplitude of a YM theory $\mathcal{F}$ as a sum over all Feynman diagram topologies Γ with only trivalent vertices:

$$\mathcal{A}(\mathcal{F}) = g^{n-2} \sum_{i \in \Gamma} \frac{c_i n_i(\mathcal{F})}{d_i}.$$

We can enforce Jacobi identity-like equations

$$j(n)_\alpha = n_i + n_j \pm n_k = 0$$

for the numerators.

Color-Kinematics duality-2

When these equations hold we obtain a graviton amplitude for a theory $\mathcal{Z}$ from two gluon amplitudes $\mathcal{A}_n(\mathcal{F}), \mathcal{A}_n(\mathcal{Y})$ in theories $\mathcal{F}, \mathcal{Y}$ with a YM sector and the same color structure using the following formula

$$\mathcal{M}(\mathcal{Z}) = i \left(\frac{\kappa}{2} \right)^{n-2} \sum_{i \in \Gamma} \frac{n_i(\mathcal{Y}) n_i(\mathcal{F})}{d_i}.$$

Crucially these work for loops at the integrand level.

Perturbative Einstein-Hilbert Coupled to Scalars and the Regge Limit

Feynman amplitudes

We calculate exact effective vertices in QCD, and take the Regge Limit of the process

$$2 \text{ quarks} \longrightarrow 2 \text{ quarks} + \text{gluon}$$

- We calculate amplitudes with quarks and scalars, this simplifies the calculation dramatically.
- This also enables us to check whether the Multi-Regge limit is blind the incoming particles
- And explore BCJ in the presence of scalars

Diagrams

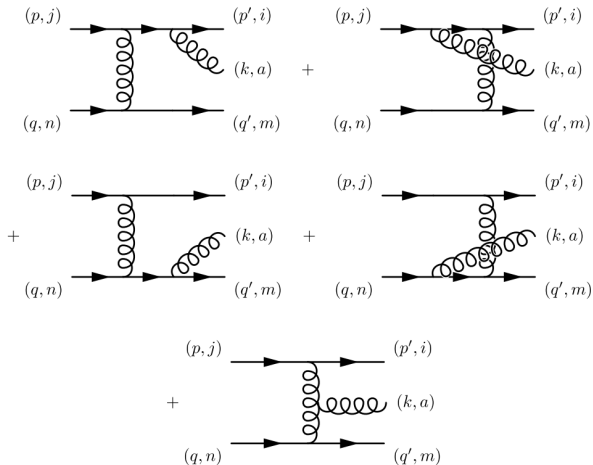


Figure: 2 quarks $\rightarrow$ 2 quarks + gluon

Gauge invariant subsets-1

We can split the amplitude into two gauge invariant subsets

$$M_{\uparrow} = M_1 + M_2 + \frac{t}{t - t'} M_5$$

$$M_{\downarrow} = M_3 + M_4 + \frac{t'}{t' - t} M_5,$$

where, for future reference

$$t = (p - p')^2, \quad t' = (q - q')^2$$

Gauge invariant subsets-2

This split allows us to write the amplitude using a non-local, gauge-invariant effective vertex

$$M =$$

The figure shows two Feynman diagrams representing the amplitude M . The first diagram on the left shows an incoming fermion line with momentum p and an incoming gluon line with momentum q meeting at a shaded circular effective vertex. A gluon line with momentum k is emitted from this vertex, and an outgoing fermion line with momentum p' continues. The second diagram on the right shows an incoming fermion line with momentum p and an incoming gluon line with momentum q meeting at a shaded circular effective vertex. A gluon line with momentum k is emitted from this vertex, and an outgoing fermion line with momentum q' continues. The two diagrams are summed, as indicated by the plus sign between them.

Figure: Reconstructing the amplitude with gauge-invariant effective vertices

Exact Effective Vertex

- This split gives exact non-local effective vertices
- Useful for loop amplitudes via unitarity cuts to reduce the number of topologies

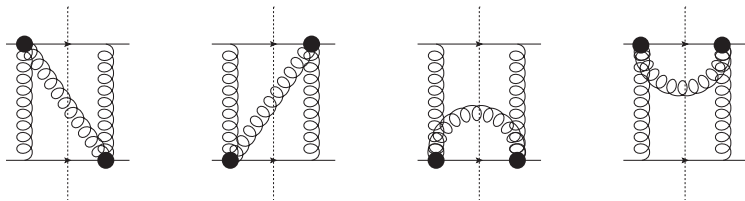


Figure: e.g. 49 of the 2-loop 2-fermion QCD elastic amplitude diagrams can be obtained from the ones above

Sudakov Variables

- The multi-Regge limit is defined by a strong ordering in the rapidity of the outgoing particles.
- Sudakov particles are tailored for this because they separate the transverse and longitudinal momenta while implementing momentum conservation

The mapping is

$$p' = p - k_1, \quad q' = q + k_2, \quad k = k_1 - k_2$$

The new momenta k_1, k_2 are off-shell and are parametrised as

$$k_1 = \alpha_1 p + \beta_1 q + \vec{k}_{1\perp}$$
$$k_2 = \alpha_2 p + \beta_2 q + \vec{k}_{2\perp}$$

Multi-Regge Kinematics

In these variables, the multi-Regge Limit is very cleanly implemented

$$\begin{aligned}1 \gg \alpha_1 \gg \alpha_2 &= \frac{-t'}{s} \\1 \gg |\beta_2| \gg |\beta_1| &= \frac{-t}{s} \\ \frac{-t'}{s} &\approx \frac{-t}{s}\end{aligned}$$

Regge Limit

In the Regge Limit the amplitude is given by

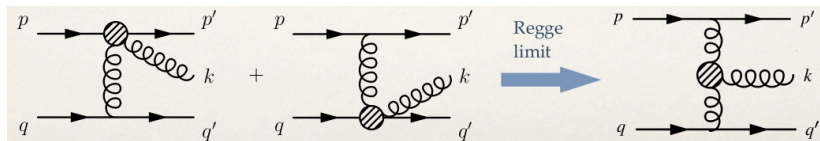


Figure: Reggeized Effective Vertex

$$M = \varepsilon_\nu(k_1 - k_2) \left(2g p^\mu T_{ij}^c \right) \left(\frac{-i}{t} \right) f^{cab} \Gamma_{\mu\sigma}^\nu \left(\frac{-i}{t'} \right) \left(2g q^\sigma T_{mn}^b \right)$$

with Γ being Lipatov's effective vertex

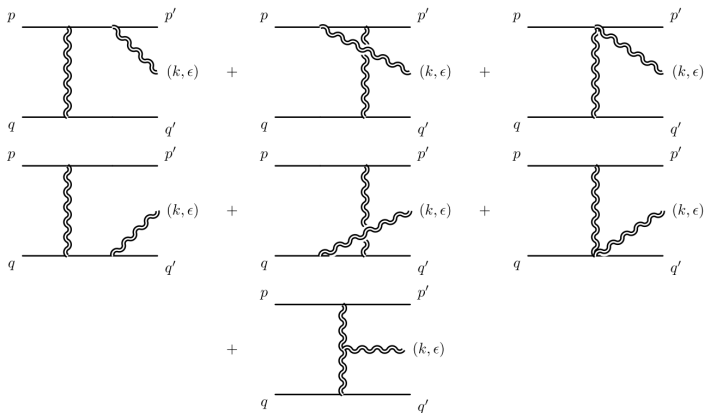
$$\Gamma_{\mu\sigma}^\nu = ig \eta_{\mu\sigma} \left[\left(\alpha_1 - \frac{2t}{s\beta_2} \right) p^\nu + \left(\beta_2 - \frac{2t'}{s\alpha_1} \right) q^\nu - (k_1 + k_2)_\perp^\nu \right]$$

Effective Vertex in Einstein-Hilbert Gravity

To minimize the number of contributing Feynman diagrams and the complexity of the amplitude, we analyze the gravitational scattering of two distinct scalars

$$2 \text{ scalars} \longrightarrow 2 \text{ scalars} + \text{graviton}$$

Diagrams



Gauge invariant subsets-1

As in the QCD case analyzed above, our aim is to decompose the total amplitude in terms of gauge invariant combinations. Here we define

$$\mathcal{M}_{\uparrow} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}' \quad \text{and} \quad \mathcal{M}_{\downarrow} = \mathcal{M}_4 + \mathcal{M}_5 + \mathcal{M}'',$$

where

$$\mathcal{M}' = \frac{t}{t-t'} (\mathcal{M}_3 + \mathcal{M}_7), \quad \mathcal{M}'' = \frac{t'}{t'-t} (\mathcal{M}_6 + \mathcal{M}_7).$$

The total amplitude can be written as

$$\mathcal{M} = \mathcal{M}_{\uparrow} + \mathcal{M}_{\downarrow} + \left(\frac{t'}{t'-t} \mathcal{M}_3 + \frac{t}{t-t'} \mathcal{M}_6 \right).$$

Remarkably, after a long calculation one can show that the last term cancels

$$\frac{t'}{t'-t} \mathcal{M}_3 + \frac{t}{t-t'} \mathcal{M}_6 = 0.$$

Gauge invariant subsets-2

$$t' \times \left(\begin{array}{c} p \text{ --- } p' \\ | \text{ wavy } \diagup \\ q \text{ --- } q' \end{array} (k, \epsilon) \right) = t \times \left(\begin{array}{c} p \text{ --- } p' \\ | \text{ wavy } \diagdown \\ q \text{ --- } q' \end{array} (k, \epsilon) \right)$$

Multi-Regge limit

If we factor out the coefficient of the transverse momenta,

$$-i\mathcal{M} \equiv -iA_{kk}\mathcal{M}^{\mu\nu}\epsilon_{\mu\nu}(k)$$

we obtain the expected behaviour for a gravity amplitude in the multi-Regge limit

$$\mathcal{M}^{\mu\nu} = \Omega^\mu\Omega^\nu - \mathcal{N}^\mu\mathcal{N}^\nu$$

Color-Kinematics Duality and Scalar QCD

BCJ with Scalars-1

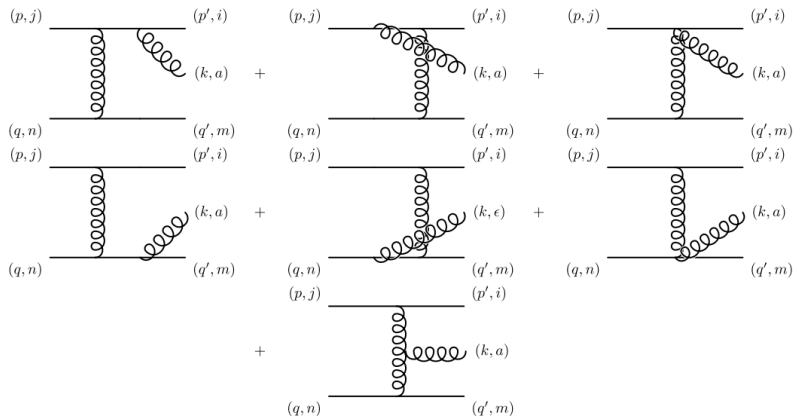


Figure: Contributing diagrams to the five point sQCD amplitude with distinguishable scalars

BCJ with Scalars-2

The amplitude

$$\mathcal{A} = g^3 \left(\frac{C_1 N_1}{t' s_{p'k}} + \frac{C_2 N_2}{t' s_{pk}} + \frac{C_3 N_3}{t'} + \frac{C_4 N_4}{t s_{q'k}} + \frac{C_5 N_5}{t s_{qk}} + \frac{C_6 N_6}{t} + \frac{C_7 N_7}{t t'} \right),$$

has numerators with different dimensionality that we must absorb using the relations between color factors

$$C_1 - C_2 + C_7 = 0,$$

$$C_1 + C_2 - C_3 = 0,$$

$$C_4 - C_5 - C_7 = 0,$$

$$C_4 + C_5 - C_6 = 0.$$

BCJ with Scalars-3

$$\mathcal{A} = g^3 \left(\frac{C_1 N'_1}{t' s_{p'k}} + \frac{C_2 N'_2}{t' s_{pk}} + \frac{C_4 N'_4}{t s_{q'k}} + \frac{C_5 N'_5}{t s_{qk}} + \frac{C_7 N'_7}{t t'} \right).$$

which leaves the following BCJ equations

$$N'_1 - N'_2 + N'_7 = 0, \quad N'_4 - N'_5 - N'_7 = 0.$$

We apply now color-kinematics duality to construct the amplitude:

$$-i\mathcal{M} = \left(\frac{\kappa}{2}\right)^3 \left(\frac{N'_1 \tilde{N}'_1}{t' s_{p'k}} + \frac{N'_2 \tilde{N}'_2}{t' s_{pk}} + \frac{N'_4 \tilde{N}'_4}{t s_{q'k}} + \frac{N'_5 \tilde{N}'_5}{t s_{qk}} + \frac{N'_7 \tilde{N}'_7}{t t'} \right),$$

BCJ with Scalars-4

We can also absorb N_7

$$\mathcal{A} = g^3 \left(\frac{C_1 N_1''}{t' s_{p'k}} + \frac{C_2 N_2''}{t' s_{pk}} + \frac{C_4 N_4''}{t s_{q'k}} + \frac{C_5 N_5''}{t s_{qk}} \right),$$

and impose

$$N_1'' - N_2'' + N_3'' - N_4'' = 0.$$

$$-i\mathcal{M} = \left(\frac{\kappa}{2}\right)^3 \left(\frac{N_1'' \tilde{N}_1''}{t' s_{p'k}} + \frac{N_2'' \tilde{N}_2''}{t' s_{pk}} + \frac{N_4'' \tilde{N}_4''}{t s_{q'k}} + \frac{N_5'' \tilde{N}_5''}{t s_{qk}} \right).$$

Multi-Regge limit

If we factor out the coefficient of the transverse momenta,

$$-i\mathcal{M} \equiv -iA_{kk}\mathcal{M}^{\mu\nu}\epsilon_{\mu\nu}(k)$$

we would expect a gravity amplitude to have the multi-Regge limit

$$\mathcal{M}^{\mu\nu} = \Omega^\mu\Omega^\nu - \mathcal{N}^\mu\mathcal{N}^\nu$$

However, in both cases we only capture $\mathcal{M}^{\mu\nu} = \Omega^\mu\Omega^\nu$ which is universal

BCJ and Dimensional Reduction

BCJ with scalars at five points-1

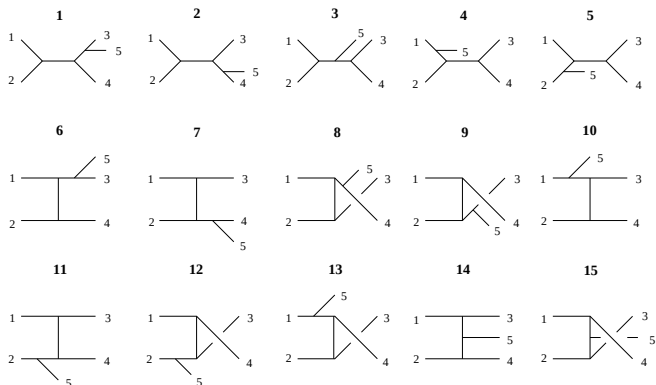


Figure: The fifteen three-vertex topologies contributing to the sum in the amplitude (1). The labels 1-4 correspond to the scalars and 5 to the gluon.

BCJ with scalars at five points-2

$$\mathcal{A}_5 = g^3 \sum_{i=1}^{15} \frac{c_i n_i}{d_i},$$

$$j_1 \equiv c_{12} - c_9 + c_{15} = 0,$$

$$j_2 \equiv c_{11} - c_7 + c_{14} = 0,$$

$$j_3 \equiv -c_4 + c_5 + c_3 = 0,$$

$$j_4 \equiv c_1 - c_2 - c_3 = 0,$$

$$j_5 \equiv -c_{10} + c_6 - c_{14} = 0,$$

$$j_6 \equiv -c_{13} + c_8 - c_{15} = 0,$$

$$j_7 \equiv c_4 - c_{10} + c_{13} = 0,$$

$$j_8 \equiv c_8 + c_7 - c_2 = 0,$$

$$j_9 \equiv c_6 + c_9 - c_1 = 0.$$

BCJ with scalars at five points-3

$$\mathcal{A}_5 = \sum_{i=1}^{15} \frac{c_i n_i}{d_i} + \sum_{\alpha=1}^9 \gamma_{\alpha} j_{\alpha} = \sum_{i=1}^{15} \frac{c_i n'_i}{d_i},$$

$$-i\mathcal{M} = \left(\frac{\kappa}{2}\right)^3 \sum_{i=1}^{15} \frac{n'_i \tilde{n}'_i}{d_i},$$

Distinguishable scalars.

We can find a solution to the BCJ equations by taking both scalars to transform in the adjoint representation and introducing a matter super-symmetric self-coupling for the two scalars of the form

$$\Delta\mathcal{L} = \frac{g^2}{2} \text{Tr} \left([\Phi, \Phi']^2 \right).$$

Indistinguishable scalars.

- We can also relax distinguishability.
- Four scalar interaction not needed in this case.
- Remarkably, these numerators obtained from the application of the Feynman rules automatically satisfy the Jacobi-like identities
- This happens with four point YM amplitudes but it is the first example at five points.
- Could provide insight into the structure of the BCJ equations.

Dimensional Reduction-1

- The two cases can be thought of as the bosonic sectors of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ SYM theory.
- Alternatively, the interaction term is generated by dimensionally reducing $D = 6$ pure Yang-Mills to $D = 4$.

$$\begin{aligned}\Phi &\equiv A_4, & \Phi' &\equiv A_5 \\ F_{\mu 4} &= D_\mu \Phi, & F_{\mu 5} &= D_\mu \Phi', & F_{45} &= -ig[\Phi, \Phi'],\end{aligned}$$

The four-dimensional Lagrangian is then

$$\mathcal{L} = -\frac{1}{4}\text{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{1}{2}\text{Tr}\left(D_\mu\Phi D^\mu\Phi\right) + \frac{1}{2}\text{Tr}\left(D_\mu\Phi' D^\mu\Phi'\right) + \frac{g^2}{2}\text{Tr}\left([\Phi, \Phi']^2\right).$$

Dimensional Reduction-2

The existence of BCJ solutions guarantees that the double copy of these theories are indeed gravity theories. Reassuringly these have the correct Regge limit

$$\mathcal{M}^{\mu\nu} = \Omega^\mu \Omega^\nu - \mathcal{N}^\mu \mathcal{N}^\nu$$

The four scalar interaction has a BCJ dual of the form

$$\Delta\mathcal{L} \sim \kappa^2 g^{\mu\nu} \Phi'^2 \partial_\mu \partial_\nu \Phi^2.$$

This term typically appears in the bosonic sector of SUGRA theories, or in dimensional reductions of $D > 4$ gravities.

Conclusions and Outlook

Conclusions-1

- We have confirmed that the Regge limit of a gravity theory coupled to scalars has the expected double copy structure $\Omega^\mu \Omega^\nu - \mathcal{N}^\mu \mathcal{N}^\nu$.
- Found exact gauge invariant effective vertices that could be used to efficiently generate higher loop amplitudes in gravity, drastically cutting down the number of topologies.

Conclusions-2

- Introduced BCJ into the picture motivated by the relation between BCJ, the double copy and the multi-Regge Limit.
- Naive attempts to apply BCJ to scalar QCD amplitudes only recover the $\Omega^\mu \Omega^\nu$ term in the Regge limit.
- To be expected given that we are essentially squaring the term that dominates the gauge theory amplitude.

Conclusions-3

- Fine tune the scalar QCD theory to find examples of amplitudes that have solutions to the BCJ equations
- Their double copy gives the desired $\Omega^\mu \Omega^\nu - \mathcal{N}^\mu \mathcal{N}^\nu$, giving further evidence that these are gravity amplitudes.
- Can be explained in hindsight by dimensional reduction.
- First trivial five-point BCJ solution.

Outlook

- Most promising avenue lies in exploring the invariant space of solutions under the relabellings transforming the trivalent color factors of the amplitude into each other.
- Finding solutions to the BCJ equations is difficult both at tree- and loop-level.
- The problem of the existence of these solutions for loop amplitudes remains open.
- Understanding into the relation between gravity and Yang Mills
- Might point to some modification of gravity that is finite and/or renormalizable.