Three-body radiative capture reactions and four-body CDCC calculations using the analytical THO method: Application to ^{9}Be

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J. Casal
M. Rodríguez-Gallardo
J. M. Arias
Departamento de Física Atómica, Molecular y Nuclear
Universidad de Sevilla
I. J. Thompson
Lawrence Livermore National Laboratory

1. Motivation
2. Formalism
   - Three-body model
   - Radiative capture reactions
   - THO method
3. Application: ^{9}Be
   - j^{π} states. Ground state
   - E1,M1 photodissociation cross section
   - Astrophysical reaction rate
   - ^{9}Be + ^{209}Pb elastic cross section
4. Conclusions and forthcoming work

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Astrophysical interest

Nucleosynthesis of light nuclei

^{12}C $(\alpha + \alpha + \alpha)$

In stars, where mainly alpha particles and nucleons are present, the triple alpha process allows the formation of heavier elements overcoming the $A = 5$ and $A = 8$ instability gaps.

^{6}He $(\alpha + n + n)$, **^{9}Be** $(\alpha + \alpha + n)$

In neutron rich environments, these nuclei can also overcome the instability gaps and they are important for the r-processes (rapid neutron capture)

Borromean systems

Three-body systems whose binary subsystems are unbound

Direct vs. sequential

Sequential picture

$$\alpha + \alpha \rightarrow \ ^{8}\text{Be} + \gamma_1 \ ; \qquad ^{8}\text{Be} + \alpha \rightarrow \ ^{12}\text{C} + \gamma_2$$

Direct picture (three-body)

$$\alpha + \alpha + \alpha \rightarrow \ ^{12}\text{C} + \gamma$$

At low temperatures the α particles have no access to intermediate resonances and therefore the three-body capture may be dominant

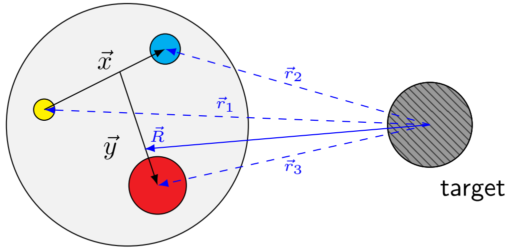
3-body scheme

$$a + b + c \rightarrow A + \gamma$$

Understanding of these processes requires:

- An accurate description of the j^{π} states (bound and continuum) of nucleus A in a three-body model
- The corresponding electromagnetic transition probabilities that govern the radiative capture reaction.

Treatment of continuum states $\Rightarrow$ Discretization Methods



3-body structure
discretization method
(THO)

4-body CDCC
expansion in projectile
internal states

(talk by M. Rodríguez-Gallardo)

$$a \; + \; b \; + \; c \; \rightarrow \; A \; + \; \gamma$$

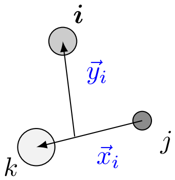
Reaction rate at a given energy $(\varepsilon = \varepsilon_\gamma + \varepsilon_B)$

$$\boldsymbol{R_{abc}(\varepsilon)} \propto \boldsymbol{\sigma_\gamma(\varepsilon_\gamma)}; \qquad \boldsymbol{\sigma_\gamma^{(\mathcal{O}\lambda)}(\varepsilon_\gamma)} \propto \boldsymbol{\frac{dB(\mathcal{O}\lambda)}{d\varepsilon}}$$

from inverse **photodissociation** process.

Energy-averaged reaction rate at a given temperature

$$\langle R_{abc}(\varepsilon)\rangle(T) = \int \boldsymbol{R_{abc}(\varepsilon)} F(\varepsilon, T) d\varepsilon$$



Jacobi and hyperspherical coordinates (6D)

$$\{\vec{x},\vec{y}\} \Rightarrow \{\rho,\Omega\}\ (\Omega=\{\alpha,\widehat{x},\widehat{y}\})$$

Pseudo-State (PS) method

$$\psi_{i\beta j\mu}(\rho,\Omega) = R_{i\beta}(\rho)\mathcal{Y}_{\beta j\mu}(\Omega); \quad \beta=\{K,...\}$$

$$|nj\mu\rangle = \sum_{\beta}\sum_{i=0}^{i_{max}} C_n^{i\beta j}\psi_{i\beta j\mu}(\rho,\Omega)$$

$$B(\mathcal{O}\lambda; n_0 j_0 \to nj) = |\langle n_0 j_0 || \widehat{\mathcal{O}}_\lambda || nj\rangle|^2$$

Analytical transformed harmonic oscillator (THO) method

HO gaussian asymptotic behavior ⇒ THO exponential behavior

$$R_{i\beta}^{\text{THO}}(\rho) = \sqrt{\frac{ds}{d\rho}} R_{iK}^{\text{HO}}[s(\rho)]; \qquad s(\rho) = \frac{1}{\sqrt{2}\boldsymbol{b}} \left[\frac{1}{\left(\frac{1}{\rho}\right)^4 + \left(\frac{1}{\boldsymbol{\gamma}\sqrt{\rho}}\right)^4} \right]^{\frac{1}{4}}$$

Asymptotically,

$$s(\rho) \longrightarrow \frac{\gamma}{b}\sqrt{\frac{\rho}{2}} \qquad \Rightarrow \qquad R_{i\beta}^{\text{THO}}(\rho) \longrightarrow \exp\left(-\frac{\gamma^2}{b^2}\frac{\rho}{2}\right)$$

The ratio γ/b governs the hyper-radial extension of the basis, and so controls the density of PS as a function of the excitation energy.

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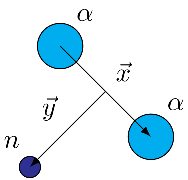
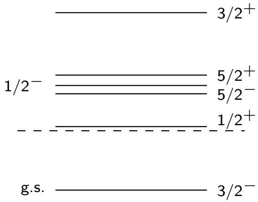
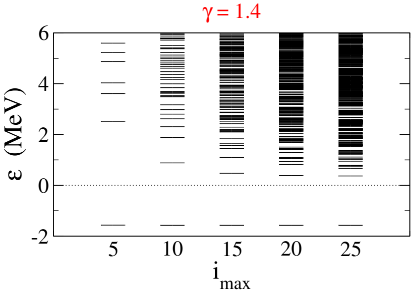




Interactions:

- Binary $V_{\alpha\text{-}n}$ and $V_{\alpha\text{-}\alpha}$; (experimental two-body phase shifts)
- Effective 3-b force; $V_{3b}(\rho)$; (position of the known states)



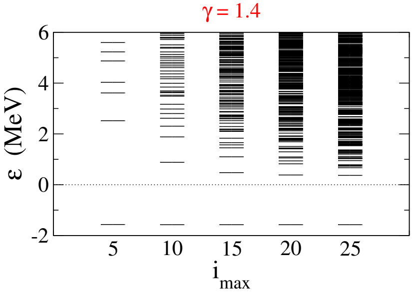
$3/2^-$ states (g.s.)

THO basis with low density of PSs

$\gamma = 1.4$

Fast convergence of the g.s.

$i_{\max}$	ε_B (MeV)	r_{mat} (fm)	r_{ch} (fm)
5	-1.5659	2.453	2.502
10	-1.5734	2.465	2.507
15	-1.5736	2.466	2.508
20	-1.5736	2.466	2.508
25	-1.5736	2.466	2.508
exp.	-1.5736	$\sim$2.4-2.6	2.519$\pm$0.012



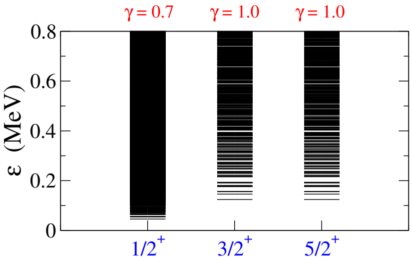
$3/2^-$ states (g.s.)

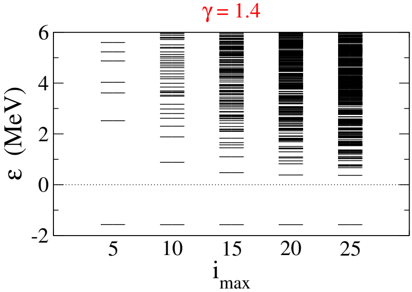
THO basis with low density of PSs
$\gamma = 1.4$

E1

g.s. $\longrightarrow$ $1/2^+$
g.s. $\longrightarrow$ $3/2^+$
g.s. $\longrightarrow$ $5/2^+$

smaller γ for higher density of PSs





$3/2^-$ states (g.s.)

THO basis with low density of PSs
$\gamma = 1.4$

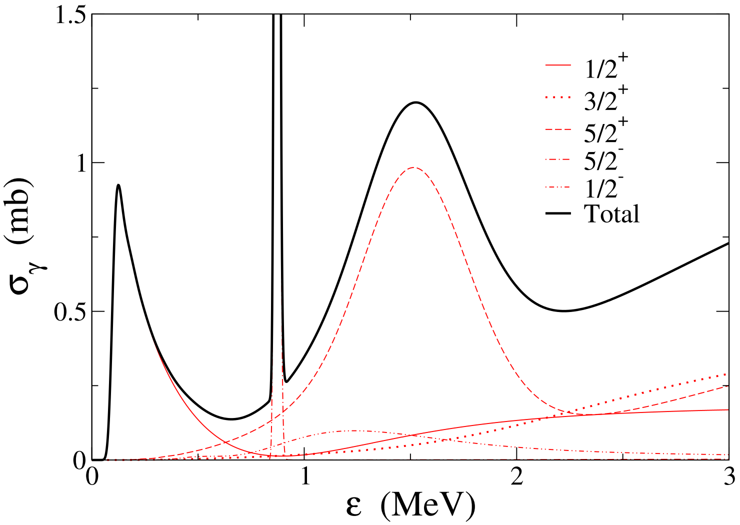
E1

g.s. $\longrightarrow 1/2^+$
g.s. $\longrightarrow 3/2^+$
g.s. $\longrightarrow 5/2^+$

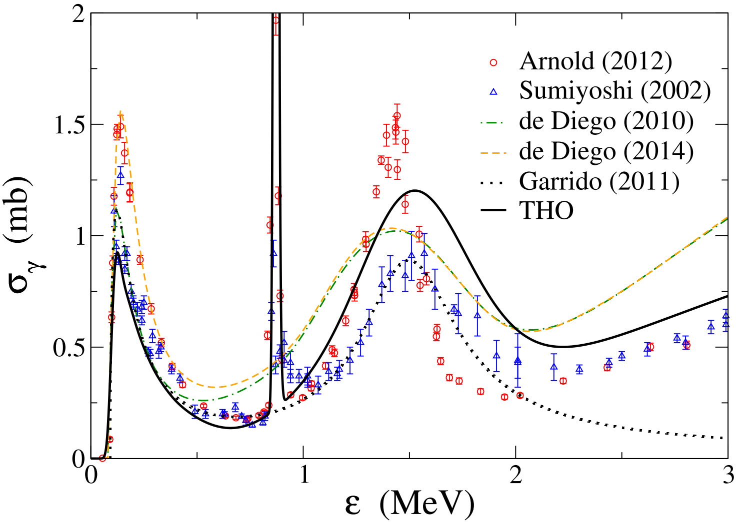
M1

g.s. $\longrightarrow 1/2^-$
g.s. $\longrightarrow 5/2^-$

$\Rightarrow$ Transition probability distribution for dipolar transitions

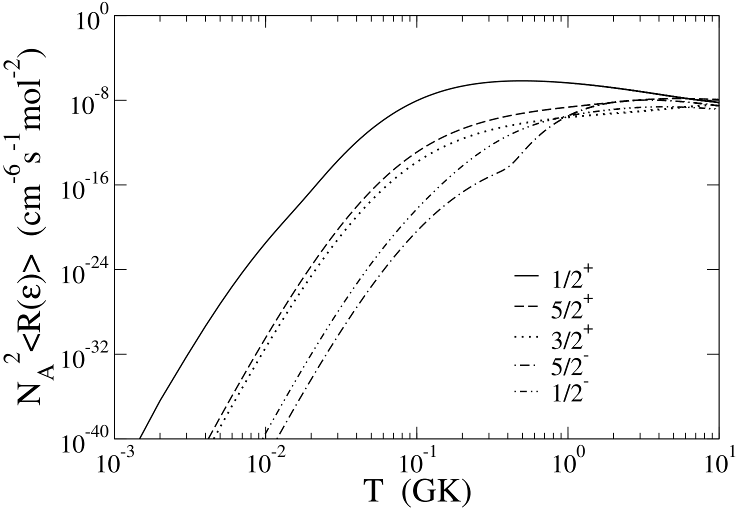


J. Casal *et al.*, Phys. Rev. C 90, 044304 (2014).



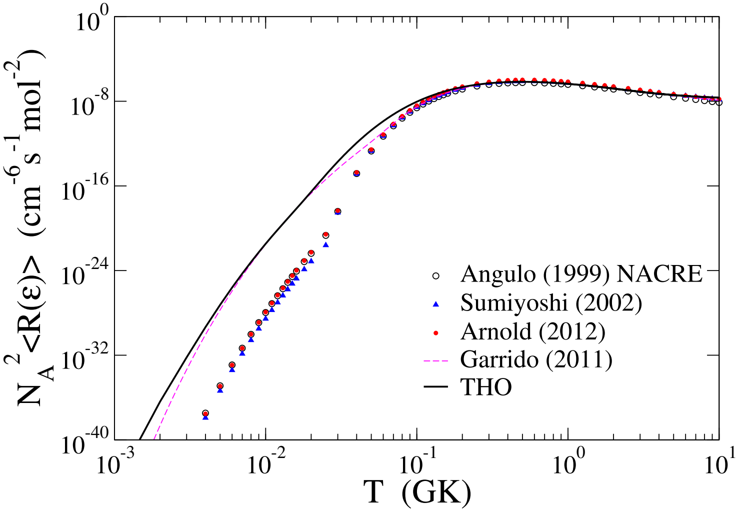
J. Casal *et al.*, Phys. Rev. C 90, 044304 (2014).

$$\alpha + \alpha + n \rightarrow {}^{9}\text{Be} + \gamma$$



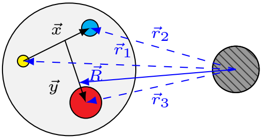
J. Casal *et al.*, Phys. Rev. C 90, 044304 (2014).

$$\alpha + \alpha + n \rightarrow {}^9\text{Be} + \gamma$$



J. Casal *et al.*, Phys. Rev. C 90, 044304 (2014).

Expansion of the 4-b w.f. in internal states of the projectile:



$$\Phi_{JM}(\boldsymbol{R},\boldsymbol{x},\boldsymbol{y}) = \sum_{nj\mu LM_L} \Psi^{\text{THO}}_{nj\mu}(\boldsymbol{x},\boldsymbol{y})\langle LM_L j\mu|JM\rangle i^L \times Y_{LM_L}(\widehat{R})\frac{1}{R} f^J_{Lnj}(R).$$

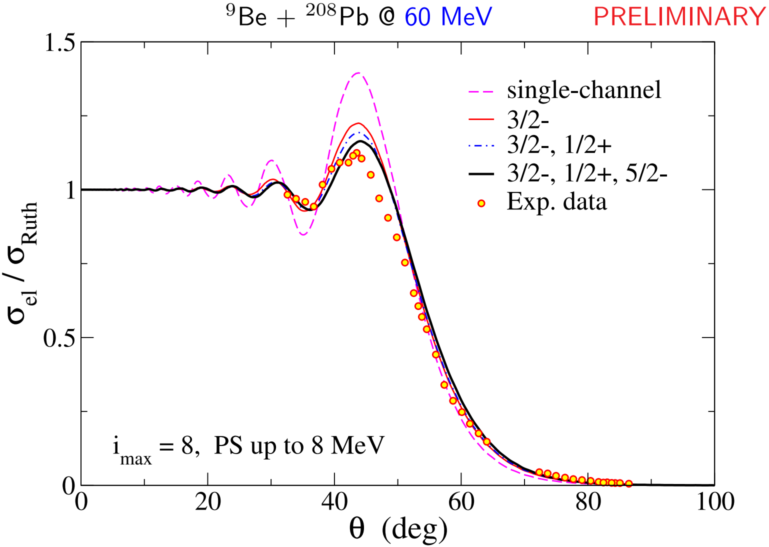
Solve coupled equations:

$$\left[-\frac{\hbar^2}{2m_r}\left(\frac{d^2}{dR^2}-\frac{L(L+1)}{R^2}\right)+\varepsilon_{nj}-E\right] f^J_{Lnj}(R) + \sum_{L'n'j'} i^{L'-L} V^J_{Lnj,L'n'j'}(R) f^J_{L'n'j'}(R) = 0$$

$V^J_{Lnj,L'n'j'}(R)$ coupling potentials projectile-target

$f^J_{Lnj}(R) \Longrightarrow$ **scattering amplitudes**

M. Rodríguez-Gallardo *et al.*, Phys. Rev. C 77, 064609 (2008).



Data from R. J. Wolliscroft *et al.* PRC 69, 044612 (2004)

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4. Conclusions and forthcoming work

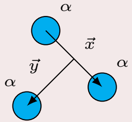
- We have applied the analytical THO method to ^{9}Be $(\alpha + \alpha + n)$.
- The properties of the $3/2^-$ g.s. are well reproduced.
- We have calculated the σ_γ from E1 and M1 contributions. The agreement with experimental data is reasonable.
- The $1/2^+$ resonance is dominat at low ε. Large bases are needed to reach convergence.
- Our reaction rates agree with sequential estimations from experimental data at high T, but it shows an enhancement of several orders of magnitude at $T < 0.1$ GK coming from the direct reaction.
- We have included the 3-body wave functions in a 4-body CDCC formalism in order to reproduce the elastic scattering data of ^{9}Be on lead. This work is in progress.

Reactions induced by ^{9}Be

- ^{9}Be + ^{208}Pb; R. J. Wolliscroft *et al.* PRC 69, 044612 (2004)
- ^{9}Be + ^{120}Sn; Andrés Arazi *et al.*, RIBRAS (São Paulo)
- ^{9}Be + ^{27}Al; P. R. S. Gomes *et al.* PRC 70, 054605 (2004)

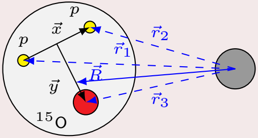
^{12}C

- $(\alpha + \alpha + \alpha)$

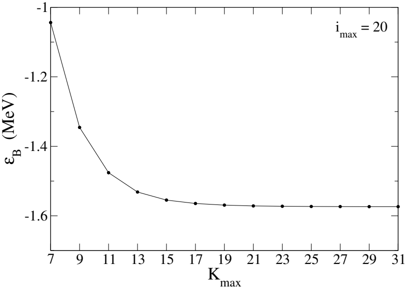


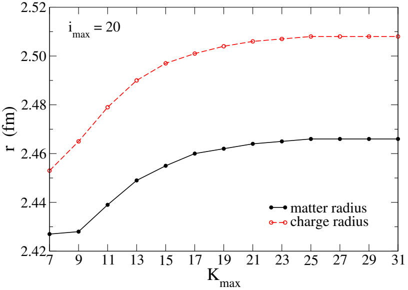
^{17}Ne

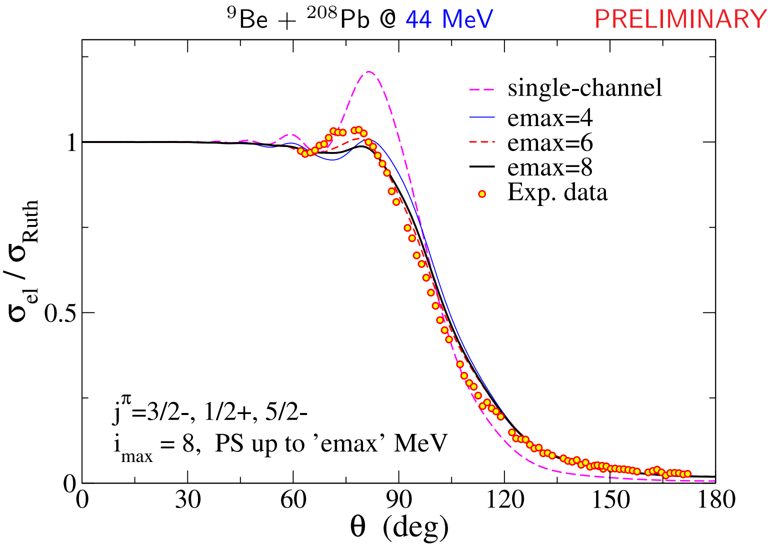
- $(^{15}\text{O} + p + p)$
- 4-b CDCC



APPENDIX

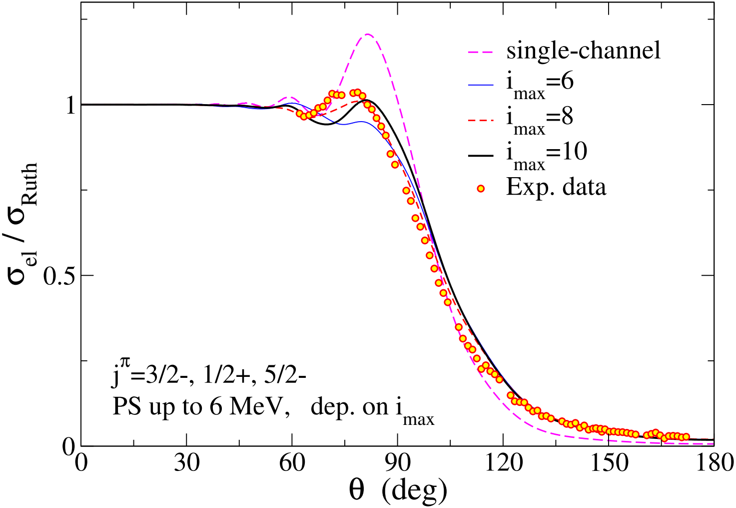

-1
-1.2
-1.4
-1.6
ε_B (MeV)
i_{max} = 20
7
9
11
13
15
17
19
21
23
25
27
29
31
K_{max}


2.52
2.50
2.48
2.46
2.44
2.42
r (fm)
i_{max} = 20
matter radius
charge radius
7
9
11
13
15
17
19
21
23
25
27
29
31
K_{max}



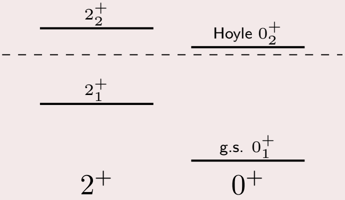
Data from R. J. Wolliscroft *et al.* PRC 69, 044612 (2004)

^{9}Be + ^{208}Pb @ 44 MeV PRELIMINARY



Data from R. J. Wolliscroft *et al.* PRC 69, 044612 (2004)

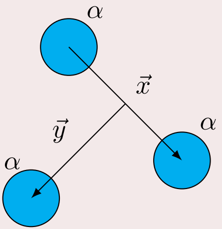
^{12}C $(\alpha + \alpha + \alpha)$



$0^+ \implies 2_1^+$ low T

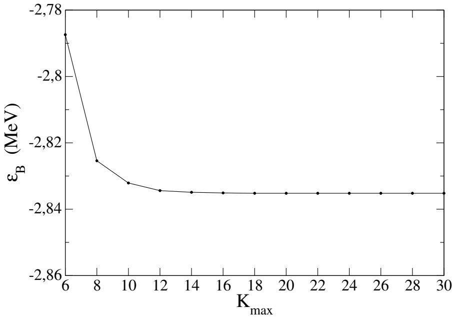
$2^+ \implies 0_1^+$

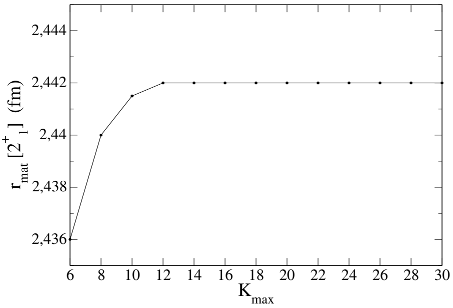
$2^+ \implies 2_1^+$



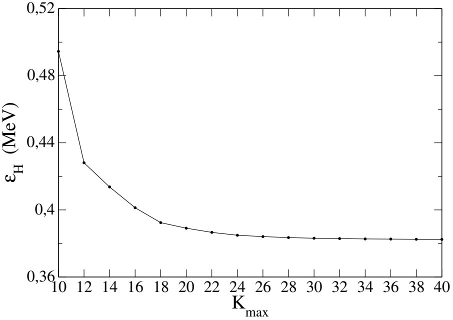
α-α: Ali-Bodmer "a"
(modified by Fedorov *et al.* (1996))
repulsive core (**PC**) s-,d-waves

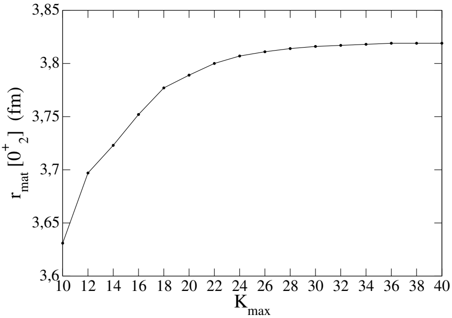
^{12}C, 2_1^+ bound state convergence



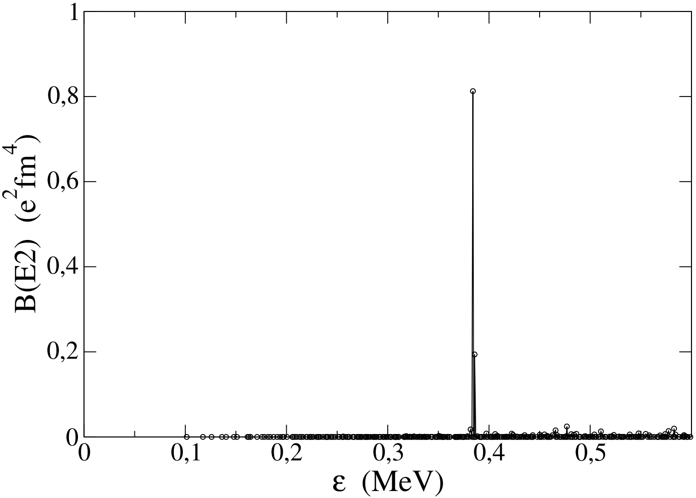


^{12}C, 0_2^+ Hoyle state convergence



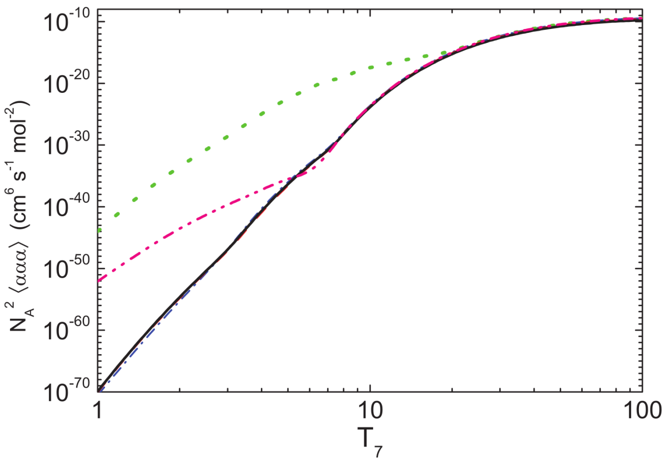


^{12}C, $2_1^+ \Rightarrow 0^+$ $B(E2)$ discrete values

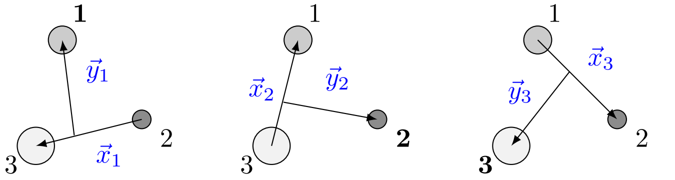


^{12}C $(\alpha + \alpha + \alpha)$ rate

S. Ishikawa, Phys. Rev. C 87, 055804 (2013)



Jacobi and hyperspherical coordinates (6D)



$$\{\vec{x}_k, \vec{y}_k\} \Rightarrow \{\rho, \Omega_k\} \; (\Omega_k = \{\alpha_k, \widehat{x}_k, \widehat{y}_k\})$$

Pseudo-State (PS) method

$$\psi_{i\beta j\mu}(\rho, \Omega_k) = \rho^{-5/2} U_{i\beta}(\rho) \mathcal{Y}_{\beta j\mu}(\Omega_k); \quad \beta = \{K, l_x, l_y, l, S_x, j_{ab}\}$$

$$|nj\mu\rangle = \Psi_{nj\mu}(\rho, \Omega) = \sum_{\beta} \sum_{i=0}^{i_{max}} C_n^{i\beta j} \psi_{i\beta j\mu}(\rho, \Omega)$$

Basis states: $\psi_{i\beta j\mu}(\rho,\Omega) = R_{i\beta}(\rho)\ \mathcal{Y}_{\beta j\mu}(\Omega)$

- $\beta \equiv \{K, l_x, l_y, l, S_x, j_{ab}\}$ defines a channel
- i denotes the hyperradial excitation
- $R_{i\beta}(\rho)$ is the hyperradial function
- $\mathcal{Y}_{\beta j\mu}(\Omega)$ are states of good total angular momentum j, expanded in hyperspherical harmonics (HH) $\Upsilon^{l_x l_y}_{Klm}(\Omega)$ as

$$\mathcal{Y}_{\beta j\mu}(\Omega) = \sum_{\nu\iota}\langle j_{ab}\nu I\iota|j\mu\rangle\kappa_I^\iota \sum_{m\sigma}\langle lmS_x\sigma|j_{ab}\nu\rangle\Upsilon^{l_x l_y}_{Klm}(\Omega)\chi^\sigma_{S_x}$$

where:

- K is the hypermomentum, $\widehat{K}^2\Upsilon^{l_x l_y}_{Klm} = K(K+4)\Upsilon^{l_x l_y}_{Klm}$

- $\vec{l} = \vec{l}_x + \vec{l}_y$ is the total orbital angular momentum
- S_x the spin of the particles in subsystem $\vec{x}$
- $\vec{j}_{ab} = \vec{l} + \vec{S}_x$
- I the spin of the third particle
- $\vec{j} = \vec{j}_{ab} + \vec{I}$ the total angular momentum
- $\chi^{\sigma}_{S_x}$ the spin wave function of the particles related by $\vec{x}$
- κ^{ι}_{I} the spin wave function of the third particle

- $\mu, \nu, \iota, m_l, \sigma$ are the third components of $\vec{j}, \vec{j}_{ab}, \vec{I}, \vec{l}, \vec{S}_x$

The HH are eigenfunctions of the hypermomentum operator $\widehat{K}^2$, and can be expressed in terms of the spherical harmonics as:

$$\begin{aligned}
\Upsilon_{Klm_l}^{l_x l_y}(\Omega) &= \sum_{m_x m_y} \langle l_x m_x l_y m_y | l m_l \rangle \Upsilon_K^{l_x l_y m_x m_y}(\Omega), \\
\Upsilon_K^{l_x l_y m_x m_y}(\Omega) &= \varphi_K^{l_x l_y}(\alpha) Y_{l_x m_x}(\widehat{x}) Y_{l_y m_y}(\widehat{y}), \\
\varphi_K^{l_x l_y}(\alpha) &= N_K^{l_x l_y} (\sin\alpha)^{l_x} (\cos\alpha)^{l_y} P_n^{l_x+\frac{1}{2}, l_y+\frac{1}{2}}(\cos 2\alpha)
\end{aligned}$$

where $P_n^{a,b}$ is a Jacobi polynomial with order $n = (K - l_x - l_y)/2$ and $N_K^{l_x l_y}$ is the normalization constant.

$\widehat{\mathcal{H}}(\rho,\Omega)$ between states which separate the hyperradial and hyper-angular parts. $\psi_{i\beta j\mu}(\rho,\Omega) = R_{i\beta}(\rho)\mathcal{Y}_{\beta j\mu}(\Omega)$

$$\int_0^\infty d\rho\ \rho^5 R_{i\beta}(\rho)R_{i'\beta}(\rho) = \delta_{ii'}$$

Defining the convinient functions $U(\rho) = \rho^{5/2}R(\rho)$:

$$\int_0^\infty d\rho\ U_{i\beta}(\rho)U_{i'\beta}(\rho) = \delta_{ii'},$$

$$\widehat{T}(\rho,\Omega)\ \Rightarrow\ \widehat{T}_U(\rho) = -\frac{\hbar^2}{2m}\left[\frac{d^2}{d\rho^2} - \frac{15/4 + K(K+4)}{\rho^2}\right]$$

Kinetic energy matrix elements

$$\langle i\beta j|\widehat{T}(\rho,\Omega)|i'\beta' j\rangle = \langle i\beta j|\widehat{T}_U(\rho)|i'\beta j\rangle \ \delta_{\beta\beta'}$$
$$= \frac{\hbar^2}{2m} \left[\int_0^\infty d\rho \ \frac{dU_{i\beta}}{d\rho} \ \frac{dU_{i'\beta}}{d\rho} \right.$$
$$\left. + \left(\frac{15}{4} + K(K+4)\right) \int_0^\infty d\rho \ U_{i\beta}(\rho)\frac{1}{\rho^2}U_{i'\beta}(\rho)\right] \delta_{\beta\beta'}$$

- Analytical form
- Not connecting different channels β.

$$\widehat{V}^{(2)} = V_c(r) + V_{so}(r)\ \widehat{SO} + V_q(r)\ \widehat{Q} + V_t(r)\ \widehat{Ts} + V_{ss}(r)\ \widehat{SS}\ ,$$

where

- $V_c(r)$ central interaction,
- $V_{so}(r)\ \widehat{SO}$ spin-orbit,
- $V_q(r)\ \widehat{Q}$ deformation potential multipoles,
- $V_t(r)\ \widehat{Ts}$ *NN* tensor,
- $V_{ss}(r)\ \widehat{SS}$ spin-spin.

$$V_j^{(3)}(\rho) = \frac{s_{3bj}}{\left[1 + \left(\frac{\rho}{r_{3bj}}\right)^{a_{3bj}}\right]}$$

Matrix elements

$$\langle i\beta j|\widehat{\mathcal{H}}(\rho,\Omega)|i\beta j\rangle = \langle i\beta j|\widehat{T}(\rho,\Omega)|i\beta j\rangle + \langle i\beta j|\widehat{V}(\rho,\Omega)|i\beta j\rangle$$

Kinetic energy

$$\langle i\beta j|\widehat{T}(\rho,\Omega)|i\beta j\rangle$$

- Analytical form
- Not connecting different channels β

Potential

I. J. Thompson *et al.*, Comp. Phys. Commun. 161 (2004)

FaCE code ⇒ hyperangular elements $V^{j}_{\beta\beta'}(\rho)$

$$\langle i\beta j|\widehat{V}(\rho,\Omega)|i'\beta' j\rangle = \int_0^\infty d\rho\; U_{i\beta}(\rho) V^{j}_{\beta\beta'}(\rho)\; U_{i'\beta'}(\rho)$$

We need the transition probabilities $(\mathcal{O} = E, M)$

$$\boldsymbol{B(\mathcal{O}\lambda)(\varepsilon_n)} = B(\mathcal{O}\lambda; n_0 j_0 \rightarrow nj) = \left(\frac{2\lambda+1}{4\pi}\right) |\langle n_0 j_0 || \widehat{\mathcal{O}}_\lambda || nj \rangle|^2$$

Energy distribution

$$\boldsymbol{\frac{dB(\mathcal{O}\lambda)}{d\varepsilon}(\varepsilon, w)} = \sum_n D(\varepsilon, \varepsilon_n, w) \boldsymbol{B(\mathcal{O}\lambda)(\varepsilon_n)}$$

$D(\varepsilon, \varepsilon_n, w)$ are Poisson distributions with w width parameter

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^{6}He: PRC 88 014327 (2013)

^{9}Be: PRC 90 044304 (2014)

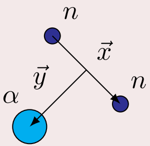
Electric multipole operator (λ)

$$\widehat{Q}_{\lambda M_\lambda}(\boldsymbol{x}_k, \boldsymbol{y}_k) = \sum_{q=1}^{3} Z_q \; e \; r_q^\lambda Y_{\lambda M_\lambda}(\widehat{r}_q), \quad \vec{r}_q = \sqrt{\frac{m}{m_q}\frac{(m_k + m_p)}{M_T}}\vec{y}_q$$

$$y_q^\lambda Y_{\lambda M_\lambda}\left(\widehat{y}_q\right) = \sum_{l=0}^{\lambda} (-1)^\lambda \, x_k^{\lambda - l} \left(\sin\varphi_{qk}\right)^{\lambda - l} y_k^l \left(\cos\varphi_{qk}\right)^l \times \sqrt{\frac{4\pi\,(2\lambda+1)!}{(2l+1)!\,(2\lambda - 2l + 1)!}} \times \left[Y_{\lambda - l, m_1}\left(\widehat{x}_k\right) \otimes Y_{l, m_2}\left(\widehat{y}_k\right)\right]^{\lambda M_\lambda}$$

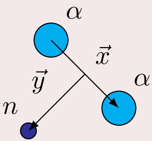
E1 operator

Only 1 charged particle: m_c, Z



$$\widehat{Q}_{1M_1}(y) = Ze\frac{\sqrt{ma_y}}{m_c} y Y_{1M_1}(\widehat{y})$$

2 identical charged particles: m_2, Z_2

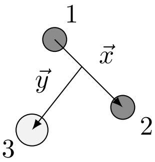


$$\widehat{Q}_{1M_1}(y) = 2\left(\cos\varphi_{23}\right) Z_2 e\frac{\sqrt{ma_{y_2}}}{m_2} y Y_{1M_1}(\widehat{y})$$

1 charged particle

$$
\begin{aligned}
\langle nj \| \widehat{Q}_\lambda \| n'j' \rangle \; = \; & (-1)^{j+2j'} \; \hat{j}' Z \; e \left(\frac{\sqrt{m a_y}}{m_c} \right)^\lambda \\
\times \; & \sum_{\beta\beta'} \delta_{l_x l'_x} \delta_{S_x S'_x} \delta_{j j_{ab}} \delta_{j' j'_{ab}} (-1)^{l_x + S_x} \\
\times \; & \hat{l}_y \hat{l}'_y \hat{l} \hat{l}' W(ll' l_y l'_y; \lambda l_x) W(jj' ll'; \lambda S_x) \\
\times \; & \begin{pmatrix} l_y & \lambda & l'_y \\ 0 & 0 & 0 \end{pmatrix} \sum_{ii'} C_n^{i\beta j} C_{n'}^{i'\beta' j'} \\
\times \; & \int \int d\alpha \; d\rho \; (\sin\alpha)^2 (\cos\alpha)^2 \\
\times \; & U_{i\beta}^{\mathsf{THO}}(\rho) \varphi_K^{l_x l_y}(\alpha) y^\lambda \varphi_{K'}^{l_x l'_y}(\alpha) U_{i'\beta'}^{\mathsf{THO}}(\rho)
\end{aligned}
$$

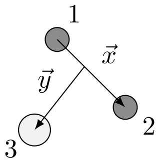
Jacobi-T system



Particles 1 and 2 are identical

$$|\beta; \mathsf{T}\rangle^j = \left|\left\{\left[Y_{l_x}(\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_1, s_2) S_x\right\} j_{ab}; I\right\rangle^j$$

Jacobi-T system



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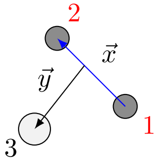
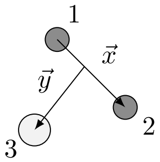


Jacobi-T system



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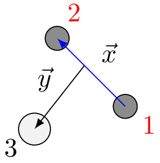
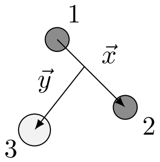


$$\widehat{P}_{12}|\beta; \mathsf{T}\rangle^j = \left|\left\{\left[Y_{l_x}(-\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_2, s_1) S_x\right\} j_{ab}; I\right\rangle^j$$

Jacobi-T system



Particles 1 and 2 are identical

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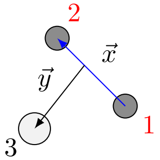
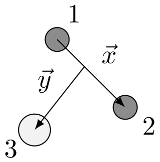


$$\begin{aligned} \widehat{P}_{12}|\beta; \mathsf{T}\rangle^j &= \left|\left\{\left[Y_{l_x}(-\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_2, s_1) S_x\right\} j_{ab}; I\right\rangle^j \\ &= (-)^{l_x - S_x + s_1 + s_2} |\beta; \mathsf{T}\rangle^j \end{aligned}$$

Jacobi-T system



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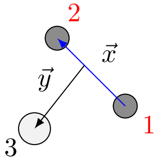
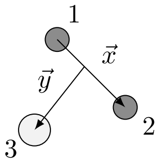


$$\begin{aligned}\widehat{P}_{12}|\beta;\mathsf{T}\rangle^j &= \left|\left\{\left[Y_{l_x}(-\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_2, s_1) S_x\right\} j_{ab}; I\right\rangle^j \\ &= (-)^{l_x - S_x + s_1 + s_2}|\beta;\mathsf{T}\rangle^j \\ &= \pm|\beta;\mathsf{T}\rangle^j\end{aligned}$$

Jacobi-T system



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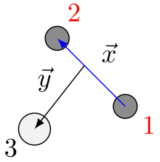
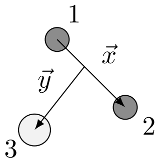


$$\begin{aligned}\widehat{P}_{12}|\beta;\mathsf{T}\rangle^j &= \left|\left\{\left[Y_{l_x}(-\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_2, s_1) S_x\right\} j_{ab}; I\right\rangle^j \\ &= (-)^{l_x - S_x + s_1 + s_2}|\beta;\mathsf{T}\rangle^j \\ &= \pm|\beta;\mathsf{T}\rangle^j\end{aligned}$$

$$Y_l(-\widehat{r}) = (-)^l Y_l(\widehat{r}); \quad (l_1, l_2)l = (-)^{l_1+l_2-l}(l_2, l_1)l$$

Jacobi-T system



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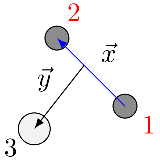
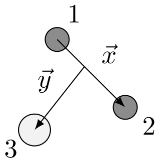


$$\begin{aligned}\widehat{P}_{12}|\beta;\mathsf{T}\rangle^j &= \left|\left\{\left[Y_{l_x}(-\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_2, s_1)S_x\right\} j_{ab}; I\right\rangle^j \\ &= (-)^{l_x - S_x + s_1 + s_2}|\beta;\mathsf{T}\rangle^j \\ &= \pm|\beta;\mathsf{T}\rangle^j\end{aligned}$$

$$Y_l(-\widehat{r}) = (-)^l Y_l(\widehat{r}); \quad (l_1, l_2)l = (-)^{l_1+l_2-l}(l_2, l_1)l$$

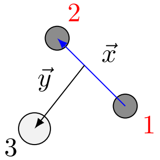
- 2 id. bosons $(s = 0) \quad \Rightarrow l_x$ even

Jacobi-T system



Particles 1 and 2 are identical

$$|\beta;\mathsf{T}\rangle^j = \left|\left\{\left[Y_{l_x}(\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_1, s_2)S_x\right\} j_{ab}; I\right\rangle^j$$



$$\begin{aligned}\widehat{P}_{12}|\beta;\mathsf{T}\rangle^j &= \left|\left\{\left[Y_{l_x}(-\widehat{x}), Y_{l_y}(\widehat{y})\right] l, (s_2, s_1)S_x\right\} j_{ab}; I\right\rangle^j \\ &= (-)^{l_x - S_x + s_1 + s_2}|\beta;\mathsf{T}\rangle^j \\ &= \pm|\beta;\mathsf{T}\rangle^j\end{aligned}$$

$$Y_l(-\widehat{r}) = (-)^l Y_l(\widehat{r}); \quad (l_1, l_2)l = (-)^{l_1+l_2-l}(l_2, l_1)l$$

- 2 id. bosons $(s = 0) \Rightarrow l_x$ even
- 2 id. fermions $(s = 1/2) \Rightarrow l_x + S_x$ even