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Nuclear Physics B Proceedings Supplement 00 (2014) 1–5

Nuclear Physics B Proceedings Supplement

Tau hadronic spectral function moments: perturbative expansion and α_s extractions

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Abstract

In the extraction of α_s from hadronic τ decays different moments of the spectral functions have been used. Furthermore, the two mainstream renormalization group improvement (RGI) frameworks, namely Fixed Order Perturbation Theory (FOPT) and Contour Improved Perturbation Theory (CIPT), lead to conflicting values of α_s. In order to improve the strategy used in α_s determinations, we have performed a systematic study of the perturbative behaviour of these spectral moments in the context of FOPT and CIPT. Higher order coefficients of the perturbative series, yet unknown, were modelled using available knowledge of the renormalon content of the QCD Adler function. We conclude that within these RGI frameworks some of the moments often employed in α_s extractions should be avoided due to their poor perturbative behaviour. Finally, under reasonable assumptions about higher orders, we conclude that FOPT is the preferred method to perform the renormalization group improvement of the perturbative series.

Keywords: α_s, τ decays, Renormalization Group

1. Introduction, framework, and results

The determination of α_s from hadronic τ decays is among of the most precise determinations of the QCD coupling [1, 2]. In spite of the relatively low energy scale set by the τ mass, in the inclusive hadronic width, R_τ, the non-perturbative contribution is subleading and the theoretical description is dominated by perturbative QCD. In detailed analysis of α_s, one exploits the knowledge of the spectral functions, measured by OPAL and ALEPH at LEP [3, 4], in order to construct other observables of interest. Different moments of the spectral functions are used; their theoretical counterpart is evaluated through finite energy sum-rules, as contour integrals in the complex-energy plane. In this context, R_τ can be understood as a particular choice of moment of the spectral functions integrated up to the kinematical limit $s_0 = m_\tau^2$. Other analytic weight functions and upper limits $s_0 \leq m_\tau^2$ (as long as s_0 is large enough to allow a perturbative treatment) also define observables that can be computed theoretically. The use of tailored weight functions can be instrumental to the α_s analysis, e.g., suppressing or enhancing the non-perturbative contributions [3, 4, 5, 6, 7, 8, 9, 10].

Our focus is on the perturbative QCD contribution to the different moments used in α_s analyses. Recently [11, 12], we investigated the convergence of the perturbative series after integration in the complex plane employing two different renormalization group improvement (RGI) prescriptions and discussed how the convergence properties of the series depend on the specific moment used. Here we briefly describe the methods and the results of our analysis and try to summarize the main conclusions.

For the theoretical description, the relevant quantity is the QCD Adler function, which is renormalization group (RG) invariant. The perturbative contribution to the observable defined by the weight function $w_i(s)$, denoted $\delta^{(0)}_{w_i}$, is obtained through an integration on the complex energy plane along the circle of radius s_0. Defining $x = s/s_0$, $W_i(x) = 2\int_x^1 w_i(z)dz$, and $a_\mu \equiv a(\mu^2) \equiv \alpha_s(\mu)/\pi$, the explicit expression reads

$$\delta^{(0)}_{w_i} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} k\, c_{n,k}\, \frac{1}{2\pi i}\oint\limits_{|x|=1}\frac{dx}{x}\, W_i(x)\log^{k-1}\Big(\frac{-s_0 x}{\mu^2}\Big)a_\mu^n. \tag{1}$$

The dynamical input to this series is fully contained in the $c_{n,1}$ coefficients, known at present up to α_s^4 order [13]. The other coefficients can be determined using RG invariance in terms of the $c_{n,1}$ and β-function coefficients.

The scale μ in the last equation can be set in a convenient way due to RG invariance. The two mainstream choices are known as fixed-order perturbation theory [14] (FOPT) obtained by fixing the scale $\mu^2 = s_0$, and contour improved perturbation theory [15, 16] (CIPT) obtained when the running of α_s is resummed along the contour by setting $\mu^2 = xs_0$. (The FOPT series can be reobtained from CIPT via the expansion of the running coupling $a(xs_0)$ in terms of the coupling at a fixed scale $\mu^2 = s_0$.) Both expansions are expected to diverge at large orders due to factorial growth of the perturbative coefficients. Therefore, the two series define *two different asymptotic expansions* (at best) to the value of the $\delta^{(0)}_{w_i}$. In practice, the numerical differences are large at α_s^4 which represents one of the dominant sources of theoretical uncertainty.

A comparison between the two approaches regarding their success in approximating $\delta^{(0)}_{w_i}$ depends on assumptions about the higher order terms. A strategy to deal with this problem based on the available knowledge of the renormalon singularities of the Borel transformed Adler function was put forward by Beneke and Jamin [17]. They were able to show that under reasonable assumptions — to be discussed below — FOPT is to be preferred for the inclusive τ hadronic width. Later we extended this analysis [11, 12] in order to ascertain how the behavior of the perturbative series depends on the moment $w_i(x)$ as well as on the value of s_0.

We work with the Adler function $\widehat{D}$, which contains only the corrections to the parton model result,

$$\widehat{D}(s) \equiv \sum_{n=0}^{\infty} r_n\, \alpha_s(\sqrt{s})^{n+1}, \tag{2}$$

and define its Borel transform $B[\widehat{D}](t)$ as

$$B[\widehat{D}](t) \equiv \sum_{n=0}^{\infty} r_n \frac{t^n}{n!}, \tag{3}$$

with $c_{n,1} = \pi^n r_{n-1}$. The original series $\widehat{D}$ can be understood as an asymptotic expansion of the inverse of transform $B[\widehat{D}](t)$:

$$\widehat{D}(\alpha) \equiv \int_0^\infty e^{-t/\alpha} B[\widehat{D}](t), \tag{4}$$

when the integral exists. This equation defines the Borel sum of the series.

The Borel transformed Adler function has singularities along the real axis both for negative and positive values of t, known as renormalon singularities (for a review see [18]). General RG arguments and the structure of the OPE allows one to determine the position and, in principle, the strength of these singularities — the residues are unknown. Singularities on the positive real axis, infrared (IR) renormalons, give rise to fixed sign divergent series. These singularities obstruct the integration in Eq. (3) and produce an ambiguity in the Borel sum related to the prescription used to define the integral. Singularities for $t < 0$, ultraviolet (UV) renormalons, give rise to sign alternating divergent series. The fixed sign nature of the exactly known coefficients of the Adler function suggests that the series is dominated by IR singularities at low and intermediate orders.

The strategy then consists in constructing a model for the Borel transformed Adler function containing a small number of dominant renormalon singularities whose residues are unknown. The residues are then fixed in order to reproduce the known coefficients and an estimate of the α_s^5 term. The Adler function can be reconstructed to all orders and the RG improved result can be compared with a "true" result for $\delta^{(0)}_{w_i}$, obtained using Eq. (4).

The main assumptions behind this strategy is that the series exhibits some regularity, and that sufficiently many terms are known in order to fix the contribution of the leading renormalons. This has been tested in detail using the large-β_0 limit of QCD and the plausibility of these assumptions has been confirmed [11]. In this limit, the Adler function is exactly known to all orders and contains all leading terms in the number of flavours, N_f. The procedure outlined above, employed using as input the first few coefficients of the series in the large-β_0 limit, leads to a robust post-diction of the higher order coefficients of the Adler function. The success of the procedure in the context of the large-β_0 limit provides strong evidence against the notion that "there are no visible signs of renormalonic behaviour" [19] in the known terms of the Adler function.

General RG arguments and the structure of the OPE allow one to determine the position and strength of the renormalon singularities in the t plane, though not their residues [18]. The fixed-sign nature of the exactly known coefficients of the Adler function suggest that at low and intermediate orders the series is dominated by IR singularities. Two models were then constructed [11, 17]. The first, and more realistic one in our opinion, assumes a logical hierarchy between the IR renormalon contributions. The *reference model* (RM) of [17] contains the first two IR and the leading UV singularities. The Borel transform Adler function in this model can be cast as

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\rm UV}](u) + B[\widehat{D}_2^{\rm IR}](u) + B[\widehat{D}_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO}\, u. \tag{5}$$

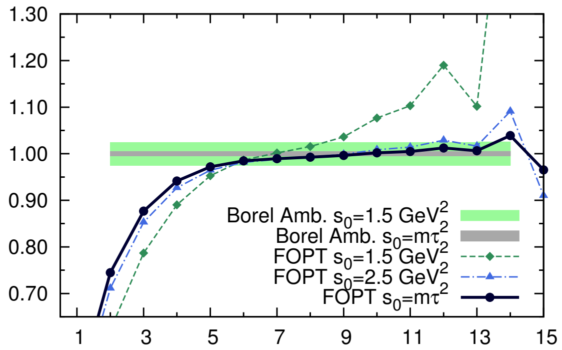
The general structure of the individual branch-cut singularities can be found in [17]. The residues and the coefficients $d_{0,1}^{\rm PO}$ are fixed by matching to the exactly known $c_{1,1}$ to $c_{4,1}$ augmented by the estimate $c_{5,1} = 283$ [17].

Within this model, the conclusion of Ref. [17] in favour of FOPT has been corroborated and extended in our recent work [11]. All moments that display a good perturbative behaviour favour the FOPT prescription within the RM. This conclusion can be traced back to the contribution of the leading IR singularity, related to the $D = 4$ corrections in the OPE. If this singularity is arbitrarily suppressed, one generates a model — less realistic, in our opinion — in which CIPT is the preferred prescription. To realize this scenario in practice, and assess possible model dependencies in our conclusions, we introduced the following *alternative model* (AM) where the leading singularity is absent whereas the sub-leading one at $u = 4$ is explicitly taken into account:

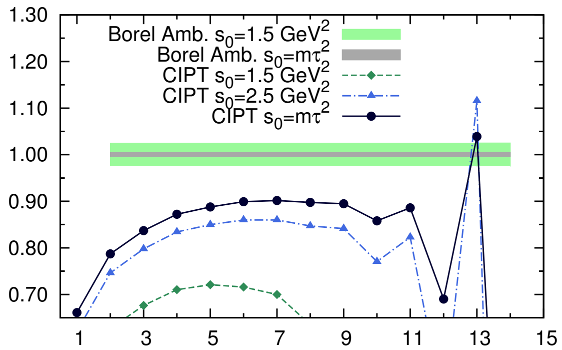
$$B[\widehat{D}](u) = B[\widehat{D}_1^{\rm UV}](u) + B[\widehat{D}_3^{\rm IR}](u) + B[\widehat{D}_4^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO}\, u\,. \tag{6}$$

Within the AM, moments with good perturbative behaviour favour CIPT.

The models represent, therefore, two quite different situations regarding the interplay of the Adler function coefficients and the running coupling effects. In the RM, there are cancellations between the contribution from the higher-order coefficients $c_{n,1}$ and the running coupling effects, at a given order in α_s. In this case FOPT is superior since it treats these contributions on an equal footing, while CIPT misses the cancellations due to the resummation of the running effects to all orders. On the other hand, the AM represents a situation where the running effects are dominant and should be resummed. In this case, the high-order coefficients can be neglected and CIPT is a better prescription.



(a) $w_\tau(x) = (1-x)^2(1+2x)$, FOPT.



(b) $w_\tau(x) = (1-x)^2(1+2x)$, CIPT.

Figure 1: Reference model. $\delta^{(0)}_{w_\tau}(s_0)$ order by order in α_s normalised to the Borel sum for FOPT (left) and CIPT (right) with three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. Bands give the Borel ambiguities.

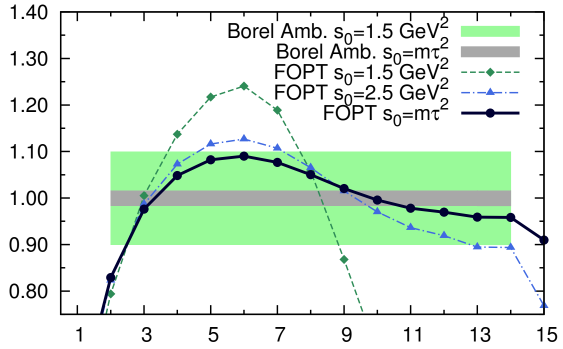
Since there is no known mechanism that would naturally suppress the leading IR singularity in QCD, we believe the scenario of Eq. (5) to be more realistic.

Using these two models for the higher orders, we performed a systematic analysis of a collection of different moments, using different s_0 values, and comparing the performance of FOPT and CIPT. As an example, Figs. 1 and 2 show results for the kinematic moment (which gives R_τ for $s_0 = m_\tau^2$) within the two models [11, 12]. They clearly show the preference for FOPT within the more realistic RM. CIPT gives the better approximation when the leading IR renormalon is artificially suppressed.

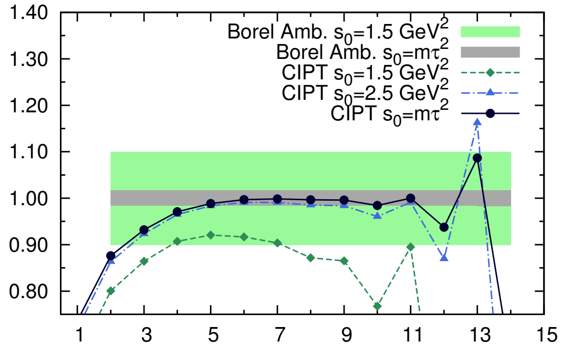
2. Conclusions

Here we try to summarize our main conclusions [11, 12, 17]

- The finiteness of the radius of convergence of the expansion of the running coupling in terms of $\alpha_s(\sqrt{s_0})$ [16] cannot be used as an argument



(a) $w_\tau(x) = (1-x)^2(1+2x)$, FOPT.



(b) $w_\tau(x) = (1-x)^2(1+2x)$, CIPT.

Figure 2: Alternative model. $\delta^{(0)}_{w_\tau}(s_0)$ order by order in α_s normalised to the Borel sum for FOPT (left) and CIPT (right) with three values of s_0: 1.5 GeV^2, 2.5 GeV^2, and m_τ^2. Bands give the Borel ambiguities.

against FOPT, for the perturbative series of $\widehat{D}$ is expected to be divergent. The models corroborate this conclusion: FOPT provides a better asymptotic expansion than CIPT in the case of the RM.

- CIPT and FOPT define two different asymptotic series. FOPT treats the running of α_s and the $c_{n,1}$ coefficients on an equal footing and only includes at a given order n terms up to order $\alpha_s(\sqrt{s_0})^n$. In CIPT the running of α_s is always resummed to all orders although only a finite number of $c_{n,1}$ coefficients contribute at a given order. Contrary to what is often stated, there is no reason to believe that the differences in the α_s values from FOPT and CIPT can be attributed to missing higher orders.

- The preference for FOPT or CIPT can be mapped into an assumption about the renormalon content of the QCD Adler function. FOPT is superior whenever a sizable contribution from the leading ($D = 4$) IR renormalon is present. The naturalness of this scenario is a strong argument in favor of FOPT. The (artificial) suppression of this contribution realizes a scenario where CIPT is superior.

- In the context of the RGI frameworks discussed here, some of the moments that are commonly employed in determinations of α_s should be avoided due to their poor perturbative behavior. In particular, the moments that emphasise higher OPE condensates ($D \geq 8$) used in [3, 4, 5, 6] should be avoided. In contrast, the moments used in [7, 8, 9, 10] have a better convergence, and at least one of the series (FOPT or CIPT) approaches the "true" value at relatively low orders.

- Several detailed tests of our procedure have been performed with and without the use of the large-β_0 limit [11]. They support the notion that the method is rather robust, and that the model dependence — although unavoidable — is smaller than stated in the literature [19].

- Ideally, the goodness of the RGI framework should be moment independent. Preferentially, it should also be independent on the assumptions about the renormalon singularities of the Adler function (in the context of our work this can be rephrased as being *model independent*). To our knowledge, the most promising strategy in this direction is the use of conformal mapping techniques based on the partial knowledge of the Borel transformed Adler function [20].

Acknowledgements

I would like to thank Martin Beneke and Matthias Jamin for the collaboration in this subject and for the reading of the manuscript. This work was supported in part by the Gottfried Wilhelm Leibniz programme of the Deutsche Forschungsgemeinschaft (DFG), the Alexander von Humboldt Foundation, and the São Paulo Research Foundation (FAPESP) grant 14/50683-0.

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