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The KTY formalism and nonadiabatic contributions to the neutrino oscillation probability

Osamu Yasuda

Department of Physics, Tokyo Metropolitan University, Minami-Osawa, Hachioji, Tokyo 192-0397, Japan

Abstract

It is shown how to obtain the analytical expression for the effective mixing angle in matter using the formalism which was developed by Kimura, Takamura and Yokomakura. If the baseline of the neutrino path is long enough so that averaging over rapid oscillations is a good approximation, then with the help of Landau's method, the nonadiabatic contribution to the oscillation probability can be expressed analytically by this formalism. We give two examples in which the present method becomes useful.

Keywords: neutrino oscillation, nonadiabatic transition, the KTY formalism

1. Introduction

Neutrino oscillation is a quantum mechanical interference effect which sometimes has complex behaviors, particularly in matter. To discuss the behaviors of neutrino oscillation intuitively, it is important to have analytical formulae for the oscillation probability. Unfortunately, an analytical formula in the three flavor mixing scheme in matter is quite complicated. In 2002 Kimura, Takamura and Yokomakura (KTY) discovered a compact formula [1, 2] for the neutrino oscillation probability in matter with constant density. Subsequently the KTY framework was generalized to more general cases. Ref. [3] discussed the four neutrino mixing scheme in matter with constant density. Ref. [4] discussed the case with unitarity violation. Ref. [5] discussed two cases of neutrino oscillation in the adiabatic approximation, the one with non-standard interactions where the matter potential has non-diagonal elements in the flavor basis, or the other with large neutrino magnetic moments in a magnetic field.

In general, however, adiabatic approximation may not be good, and in this talk I discuss nonadiabatic contributions to the oscillation probability. When there are more than two neutrino mass eigenstates, there can be more than one level crossing. It is believed [1] that the nonadiabatic contributions to the transition phenomena in a problem with three or more eigenstates can be treated approximately well by applying the method for two state problems [8, 9] at each level crossing, if the the two resonances are sufficiently far apart. Throughout this talk I discuss the case in which the baseline of the neutrino path is long enough so that averaging over rapid oscillations is a good approximation, as in the case of the solar neutrino deficit phenomena.

2. The oscillation probability

2.1. The oscillation probability in the adiabatic approximation

The equation of motion for neutrinos propagating in matter with general potential is given by

$$i\frac{d\Psi}{dt} = \left[U\mathcal{E}_0U^{-1} + \mathcal{A}(t)\right]\Psi, \tag{1}$$

[1] See, e.g., Ref. [6] and references therein. See also Ref. [7] for a discussion on the condition to justify such a treatment.

where

$$\mathcal{E}_0 \equiv \mathrm{diag}\,(E_1, E_2, E_3)\,,$$
$$\mathcal{A}(t) \equiv \begin{pmatrix} A_{ee}(t) & A_{e\mu}(t) & A_{e\tau}(t) \\ A_{\mu e}(t) & A_{\mu\mu}(t) & A_{\mu\tau}(t) \\ A_{\tau e}(t) & A_{\tau\mu}(t) & A_{\tau\tau}(t) \end{pmatrix}.$$

Since the matrix which is proportional to identity gives contribution only to the phase of the probability amplitude, instead of $\mathcal{E}_0$ itself, we use the following quantity:

$$\mathcal{E} \equiv \mathcal{E}_0 - E_1\mathbf{1} = \mathrm{diag}(0, \Delta E_{21}, \Delta E_{31}),$$

where $\Delta E_{jk} \equiv E_j - E_k \simeq (m_j^2 - m_k^2)/2|\vec{p}|$. The 3×3 matrix on the right hand side of Eq. (1) can be formally diagonalized as:

$$U\mathcal{E}U^{-1} + \mathcal{A}(t) = \tilde{U}(t)\tilde{\mathcal{E}}(t)\tilde{U}^{-1}(t), \tag{2}$$

where $\tilde{\mathcal{E}}(t) \equiv \mathrm{diag}\left(\tilde{E}_1(t), \tilde{E}_2(t), \tilde{E}_3(t)\right)$ is a diagonal matrix with the energy eigenvalues $\tilde{E}_j(t)$ in the presence of the matter effect. Substituting the diagonalized form (2) of the Hamiltonian into the Dirac equation (1), we have

$$i\frac{d\tilde{\Psi}}{dt} = \left[\tilde{\mathcal{E}} - i\tilde{U}^{-1}\left(\frac{d}{dt}\tilde{U}\right)\right]\tilde{\Psi}, \tag{3}$$

where $\tilde{\Psi}$ is the effective energy eigenstate defined by

$$\tilde{\Psi}(t) \equiv \begin{pmatrix} \tilde{\nu}_1(t) \\ \tilde{\nu}_2(t) \\ \tilde{\nu}_3(t) \end{pmatrix} \equiv \tilde{U}^{-1}(t)\Psi(t).$$

If the term $\tilde{U}^{-1}d\tilde{U}/dt$ in (3) is negligible compared with $\tilde{\mathcal{E}}$, i.e., if adiabatic approximation is good, then the oscillation probability is given by

$$P(\nu_\alpha \to \nu_\beta) = \sum_{j,k} \tilde{U}_{\beta j}(L)\tilde{U}^*_{\beta k}(L)\tilde{U}^*_{\alpha j}(0)\tilde{U}_{\alpha k}(0) \times \exp\left[-i\int_0^L \Delta\tilde{E}_{jk}(t)\,dt\right], \tag{4}$$

where we have defined $\Delta\tilde{E}_{jk}(t) \equiv \tilde{E}_j(t) - \tilde{E}_k(t)$. The bilinear quantity $\tilde{X}_j^{\alpha\beta}(t) \equiv \tilde{U}_{\alpha j}(t)\tilde{U}^*_{\beta j}(t)$ can be expressed analytically as [1, 2, 10, 5]

$$\begin{pmatrix} \tilde{X}_1^{\alpha\beta}(t) \\ \tilde{X}_2^{\alpha\beta}(t) \\ \tilde{X}_3^{\alpha\beta}(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{31}}(\tilde{E}_2\tilde{E}_3, & -(\tilde{E}_2+\tilde{E}_3), & 1) \\ \frac{-1}{\Delta\tilde{E}_{21}\Delta\tilde{E}_{32}}(\tilde{E}_3\tilde{E}_1, & -(\tilde{E}_3+\tilde{E}_1), & 1) \\ \frac{1}{\Delta\tilde{E}_{31}\Delta\tilde{E}_{32}}(\tilde{E}_1\tilde{E}_2, & -(\tilde{E}_1+\tilde{E}_2), & 1) \end{pmatrix} \times \begin{pmatrix} \delta_{\alpha\beta} \\ \left[U\mathcal{E}U^{-1} + \mathcal{A}(t)\right]_{\alpha\beta} \\ \left[\left(U\mathcal{E}U^{-1} + \mathcal{A}(t)\right)^2\right]_{\alpha\beta} \end{pmatrix}, \tag{5}$$

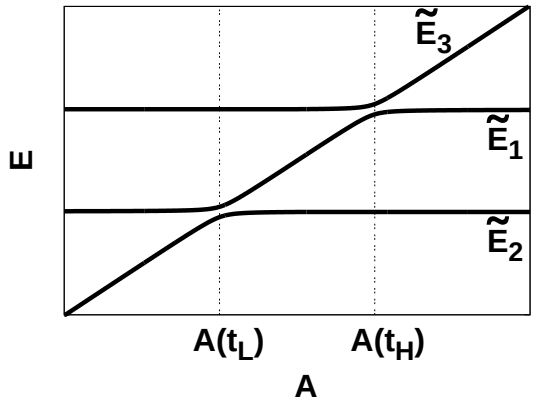

Figure 1: The three energy eigenvalues $\tilde{E}_j$ ($j = 1, 2, 3$). The potential $A_{\alpha\beta}(t)$ is assumed to be of the form diag($A(t)$,0,0) and $A(t)$ is assumed to be linear in t in this figure. We have $\tilde{E}_3 \simeq \tilde{E}_1$ near $t = t_H$ and $\tilde{E}_1 \simeq \tilde{E}_2$ near $t = t_L$.

where the $t-$dependence of the quantities $\tilde{E}_j$, $\Delta\tilde{E}_{jk}$ is suppressed for simplicity in Eq. (5).

2.2. *The nonadiabatic correction to the oscillation probability*

If the adiabatic approximation is not good, on the other hand, then Eq. (4) should be modified by taking non-adiabatic contributions into account.

In the three flavor case, there are at most two level crossings for neutrinos as in the case of a supernova [11]. Let us assume that there are two level crossings. Let $t = t_H$ and $t = t_L$ be the level crossing points, and we assume that the energy levels 1 (1) and 2 (3) cross at $t = t_L$ ($t = t_H$), respectively (See Fig. 1).

Except the two regions near $t = t_H$ and $t = t_L$, we can integrate (3) in the adiabatic approximation. Near the two level crossings $t = t_H$ and $t = t_L$, the energy eigenstates at $t = t_H \pm \epsilon$ and $t = t_L \pm \epsilon$ are related by the matrices W_H and W_L as in the case of the two level problem [12]:

$$\tilde{\Psi}(t_H + \epsilon) = W_H\tilde{\Psi}(t_H - \epsilon)$$
$$\tilde{\Psi}(t_L + \epsilon) = W_L\tilde{\Psi}(t_L - \epsilon)$$

In this case the flavor eigenstate is given by

$$\Psi(L) = \tilde{U}(L)\exp\left[-i\int_{t_L}^{L}\tilde{\mathcal{E}}(t)\,dt\right]W_L\exp\left[-i\int_{t_H}^{t_L}\tilde{\mathcal{E}}(t)\,dt\right] \times W_H\exp\left[-i\int_0^{t_H}\tilde{\mathcal{E}}(t)\,dt\right]\tilde{U}(0)^{-1}\Psi(0).$$

Taking average over rapid oscillations, we have the probability

$$
\begin{aligned}
&P(\nu_\alpha \to \nu_\beta) \\
&= \sum_{j,k,\ell} \left|\tilde{U}(L)_{\beta j}\right|^2 \left|(W_L)_{jk}\right|^2 \left|(W_H)_{k\ell}\right|^2 \left|\tilde{U}(0)_{\alpha\ell}\right|^2 \\
&= \left(\begin{array}{ccc} \left|U_{\beta 1}\right|^2 & \left|U_{\beta 2}\right|^2 & \left|U_{\beta 3}\right|^2 \end{array} \right) \left(\begin{array}{ccc} 1-P_L & P_L & 0 \\ P_L & 1-P_L & 0 \\ 0 & 0 & 1 \end{array} \right) \\
&\times \left(\begin{array}{ccc} 1-P_H & 0 & P_H \\ 0 & 1 & 0 \\ P_H & 0 & 1-P_H \end{array} \right) \left(\begin{array}{c} \left|\tilde{U}_{\alpha 1}(0)\right|^2 \\ \left|\tilde{U}_{\alpha 2}(0)\right|^2 \\ \left|\tilde{U}_{\alpha 3}(0)\right|^2 \end{array} \right). \qquad (6)
\end{aligned}
$$

In (6) it was assumed that the nonadiabatic contributions to the transition phenomena in a problem with three eigenstates can be treated approximately well by applying the method for two state problems at each level-crossing, when the the two resonances are sufficiently far apart. Using the WKB approximation [8], the jumping factors in (6) are given by

$$P_H = \exp\left[-\,\mathrm{Im} \int_C \Delta\tilde{E}_{31}(t)\, dt \right], \qquad (7)$$

$$P_L = \exp\left[-\,\mathrm{Im} \int_C \Delta\tilde{E}_{21}(t)\, dt \right]. \qquad (8)$$

2.3. The effective mixing angles in matter

To evaluate the jumping factors P_H and P_L in the previous subsection, we assume here as in the two flavor case that the exponents in (7) and (8) are related by the ratio of the difference of the energy eigenvalues of the two levels to the derivative of the effective mixing angle at the level-crossing. For that purpose, it is necessary to know the effective mixing angles at the level crossings. In this subsection I show how to derive the expression for the effective mixing angle in the presence of the matter using the KTY formalism. Our strategy here is to start with effective matrix elements $\tilde{X}_j^{\alpha\beta}$ which are obtained by the KTY formalism and to determine the phase of each element by demanding that it be consistent with the standard parametrization of the mixing matrix element:

$$\tilde{U} = e^{i\tilde{\theta}_{23}\lambda_7}\, \Gamma_{\tilde{\delta}}^{(13)}\, e^{i\tilde{\theta}_{13}\lambda_5}\, (\Gamma_{\tilde{\delta}}^{(13)})^{-1}\, e^{i\tilde{\theta}_{12}\lambda_2}, \qquad (9)$$

where λ_j ($j = 2, 5, 7$) are the Gell-Mann matrices defined by $\lambda_2 \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda_5 \equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$, $\lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, and $\Gamma_\delta^{(13)} \equiv \mathrm{diag}(e^{-i\delta/2}, 1, e^{i\delta/2})$.

From the identity $\tilde{U}_{\alpha j} = e^{i \arg \tilde{U}_{ej}} \tilde{X}_j^{\alpha e} / \sqrt{\tilde{X}_j^{ee}}$, we can postulate the form for $\tilde{U}$

$$\tilde{U} \equiv e^{i\varphi_0}\, e^{i\varphi_3\lambda_3}\, e^{i\varphi_9\lambda_9}\, \tilde{U}_0\, e^{i\varphi_9'\lambda_9}\, e^{i\varphi_3'\lambda_3}, \qquad (10)$$

where $(\tilde{U}_0)_{\alpha j} \equiv \tilde{X}_j^{\alpha e} / \sqrt{\tilde{X}_j^{ee}}$, $\lambda_3 \equiv \mathrm{diag}(1, -1, 0)$, $\lambda_9 \equiv \mathrm{diag}(1, 0, -1)$. and choose the phases φ_0, φ_3, φ_9, φ_3' and φ_9' so that the form (10) is consistent with (9) (See Ref. [13] for details). Comparing Eqs. (9) and (10), we find

$$\cos 2\tilde{\theta}_{12} = \frac{\tilde{X}_1^{ee} - \tilde{X}_2^{ee}}{\tilde{X}_1^{ee} + \tilde{X}_2^{ee}}, \qquad (11)$$

$$\cos 2\tilde{\theta}_{13} = 1 - 2\tilde{X}_3^{ee}, \qquad (12)$$

$$\cos 2\tilde{\theta}_{23} = \frac{|\tilde{X}_3^{\tau e}|^2 - |\tilde{X}_3^{\mu e}|^2}{|\tilde{X}_3^{\tau e}|^2 + |\tilde{X}_3^{\mu e}|^2}, \qquad (13)$$

$$\tilde{\delta} = -\arg\det \tilde{U}_0 + \arg \tilde{X}_1^{\mu e} + \arg \tilde{X}_3^{\mu e} + \arg \tilde{X}_3^{\tau e}. \qquad (14)$$

The quantities $\tilde{X}_j^{\alpha\beta}$ and $\det \tilde{U}_0$ in Eqs. (11)–(14) are expressed in closed form by the known variables as is seen in Eq. (5) on the assumption that analytical expressions for all the eigenvalues are known.[2].

The standard parametrization (9) is not the only one for 3×3 unitary matrices, and other parametrizations are possible as is described in Appendix B of Ref. [13]. In the three flavor case, there can be at most two level crossings. Depending on which pair of the energy eigenvalues gets close at each level crossing, the relevant effective mixing angle varies. The appropriate parametrization is the one in which the orthogonal matrix, which mixes the two energy eigenstates, is located on the most right-hand side of the unitary matrix U, because in such a parametrization the diagonalized matrix looks like $\cdots O(\tilde{\theta}_{jk})\mathrm{diag}(\cdots, \tilde{E}_j, \cdots, \tilde{E}_k, \cdots) O(\tilde{\theta}_{jk})^T \cdots$, and it becomes clear that $\tilde{\theta}_{jk}$ in the orthogonal matrix $O(\tilde{\theta}_{jk})$ plays a role of the effective mixing angle which mixes the energy eigenstates with the energy $\tilde{E}_j$ and $\tilde{E}_k$. Furthermore in order for the effective mixing angle $\tilde{\theta}_{jk}$ to be consistent with the two flavor description, $\tilde{\theta}_{jk}$ should become maximal at the level crossing. Thus, while the effective mixing angle at $t = t_L$ is $\tilde{\theta}_{12}$ defined by (11) for the standard parametrization (9), the one at $t = t_H$ should be $\tilde{\varphi}_{13}$ defined by

$$\cos 2\tilde{\varphi}_{13} = \frac{\tilde{X}_1^{ee} - \tilde{X}_3^{ee}}{\tilde{X}_1^{ee} + \tilde{X}_3^{ee}}, \qquad (15)$$

[2]Here we note that the effective mixing angles were given in the standard three flavor case in Ref. [14], whereas Eqs. (11)–(14) are the results for the case of the general potential.

for the parametrization

$$\tilde{U} = e^{i\tilde{\varphi}_{23}\lambda_7}\,\Gamma_\delta^{(12)}\,e^{i\tilde{\varphi}_{12}\lambda_2}\,(\Gamma_\delta^{(12)})^{-1}\,e^{i\tilde{\varphi}_{13}\lambda_5},$$

where $\Gamma_\delta^{(12)} \equiv \mathrm{diag}(e^{-i\delta/2}, e^{i\delta/2}, 1)$. From the analogy with the two flavor case, the jumping factors P_H and P_L are given by

$$P_H = \exp\left(-\frac{\pi}{2}\,F\,\frac{\Delta\tilde{E}_{31}}{2\,|d\tilde{\varphi}_{13}/dt|_{t=t_H}}\right), \tag{16}$$

$$P_L = \exp\left(-\frac{\pi}{2}\,F\,\frac{\Delta\tilde{E}_{21}}{2\left|d\tilde{\theta}_{12}/dt\right|_{t=t_L}}\right).$$

$\tilde{\varphi}_{13}$ and $\tilde{\theta}_{12}$ are defined by (15) and (11), and F is the factor which depends on the form of the potential $A_{\alpha\beta}(t)$, and $F = 1$ in the case of a linear potential.

3. Examples

Now I discuss two examples to demonstrate how the general discussions in the previous section are applied.

3.1. The case with non-standard interactions

The first example is the oscillation probability in the presence of new physics in propagation [15, 16, 17]. In this case the mass matrix is given by

$$U\mathcal{E}U^{-1} + \mathcal{A}_{NP} \tag{17}$$

where $\mathcal{A}_{NP} \equiv \sqrt{2}G_F N_e \begin{pmatrix} 1+\epsilon_{ee} & \epsilon_{e\mu} & \epsilon_{e\tau} \\ \epsilon_{e\mu}^* & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\ \epsilon_{e\tau}^* & \epsilon_{\mu\tau}^* & \epsilon_{\tau\tau} \end{pmatrix}$. The dimensionless quantities $\epsilon_{\alpha\beta}$ stand for possible deviation from the standard matter effect. It is known [18] that the constraints on the parameters $\epsilon_{e\mu}$, $\epsilon_{\mu\mu}$, $\epsilon_{\mu\tau}$ are strong ($|\epsilon_{\alpha\mu}| \simeq O(10^{-2})$ ($\alpha = e, \mu, \tau$) while those on the parameters $\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}$ are weak ($|\epsilon_{ee}|$, $|\epsilon_{e\tau}|, |\epsilon_{\tau\tau}| \simeq O(1)$. In Ref. [19] it was found that large values ($\sim O(1)$) of the parameters $\epsilon_{ee}, \epsilon_{e\tau}, \epsilon_{\tau\tau}$ are consistent with all the experimental data including those of the atmospheric neutrino data, provided that one of the eigenvalues of the matrix (17) at high energy limit becomes zero, and that such a constraint implies the relation $\epsilon_{\tau\tau} \simeq |\epsilon_{e\tau}|^2/(1+\epsilon_{ee})$. For simplicity, therefore, we consider the potential matrix

$$\mathcal{A}_{NP} = A \begin{pmatrix} 1+\epsilon_{ee} & 0 & \epsilon_{e\tau} \\ 0 & 0 & 0 \\ \epsilon_{e\tau}^* & 0 & |\epsilon_{e\tau}|^2/(1+\epsilon_{ee}) \end{pmatrix}. \tag{18}$$

Then $\mathcal{A}_{NP}$ can be diagonalized as

$$\mathcal{A}_{NP} = e^{i\gamma'\lambda_9} e^{-i\beta\lambda_5}\,\mathrm{diag}\,(\lambda_{e'}, 0, 0)\,e^{i\beta\lambda_5} e^{-i\gamma'\lambda_9}, \tag{19}$$

where $\tan\beta = |\epsilon_{e\tau}|/(1+\epsilon_{ee})$, $\gamma' \equiv \arg(\epsilon_{e\tau})/2$, $\lambda_{e'} = A(1+\epsilon_{ee})/\cos^2\beta$. The mass matrix (17) can be written as

$$\begin{aligned} &U\mathcal{E}U^{-1} + \mathcal{A}_{NP} \\ &= e^{i\gamma'\lambda_9} e^{-i\beta\lambda_5} e^{-i\phi_9\lambda_9} e^{-i\phi_3\lambda_3} \left[U''\mathcal{E}U''^{-1} + \mathrm{diag}\,(\lambda_{e'}, 0, 0)\right] \\ &\times e^{-i\omega_3\lambda_3} e^{-i\omega_9\lambda_9} e^{i\beta\lambda_5} e^{-i\gamma'\lambda_9}, \end{aligned} \tag{20}$$

where the phases ϕ_3, ϕ_9, ω_3 and ω_9, which are defined in Appendix C of Ref. [13], are introduced to make U'' consistent with the standard parametrization (9). The expressions for th mixing angles θ''_{jk} and the CP phase δ'' in the standard parametrization of U'' are given in Ref. [13]. Among others the mixing angles θ''_{12} and θ''_{13} are given by

$$\theta''_{12} = \tan^{-1}\frac{|c_\beta e^{-i\gamma'} U_{e2} + s_\beta e^{i\gamma'} U_{\tau 2}|}{|c_\beta e^{-i\gamma'} U_{e1} + s_\beta e^{i\gamma'} U_{\tau 1}|},$$

$$\theta''_{13} = \sin^{-1}|c_\beta e^{-i\gamma'} U_{e3} + s_\beta e^{i\gamma'} U_{\tau 3}|,$$

where $c_\beta \equiv \cos\beta$, $s_\beta \equiv \sin\beta$. The inside of the square bracket in the mass matrix (20) has exactly the same form as that of the standard case with replacement $\theta_{jk} \to \theta''_{jk}$, $\delta \to \delta''$ and $A \to \lambda_{e'}$. Furthermore, at the two level-crossings specified by $\Delta E_{31}\cos 2\theta''_{13} = \lambda_{e'}$ and $\Delta E_{21}\cos 2\theta''_{12} = (c''_{13})^2\lambda_{e'}$, $\tilde{\theta}''_{13}$ and $\tilde{\theta}''_{12}$ become $\pi/4$, respectively. Therefore, $\tilde{\theta}''_{13}$ and $\tilde{\theta}''_{12}$ can be regarded as the appropriate mixing angles to describe the nonadiabatic transition at the two level-crossings. Hence we can deduce the jumping factors at the two level-crossings [3]:

$$P_H = \exp\left(-\frac{\pi}{2}\cdot\frac{\Delta E_{31}\sin^2 2\theta''_{13}}{\cos 2\theta''_{13}|d\log A/dt|_{\mathrm{resonance}}}\right) \tag{21}$$

$$P_L = \exp\left(-\frac{\pi}{2}\cdot\frac{\Delta E_{21}\sin^2 2\theta''_{12}}{\cos 2\theta''_{12}|d\log A/dt|_{\mathrm{resonance}}}\right) \tag{22}$$

To estimate the effective mixing matrix elements at the origin $L = 0$, we assume that the matter effect A is much larger than the energy difference $|\Delta E_{jk}|$. In this case we can ignore the term $\mathcal{E}$ in Eq. (25), and Eq. (19) indicates that the mixing matrix $\tilde{U}$ is given by $e^{i\gamma'\lambda_9} e^{-i\beta\lambda_5}$, and we get

$$|\tilde{U}_{\alpha j}(0)|^2 = \begin{pmatrix} c_\beta^2 & 0 & s_\beta^2 \\ 0 & 1 & 0 \\ s_\beta^2 & 0 & c_\beta^2 \end{pmatrix}. \tag{23}$$

From Eqs. (6), (21), (22) and (23), we can obtain the transition probability $P(\nu_\alpha \to \nu_\beta)$ in the case with the nonstandard neutrino interaction in propagation.

[3] The quantity P_L was given first in Ref. [20] whose result agrees with ours.

3.2. The case with large magnetic moments and a magnetic field

The second example is the case where there are three active neutrinos with magnetic moments and a large magnetic field [4]. This is an example where the energy eigenvalues cannot be expressed as roots of a quadratic equation, and this case demonstrates the usefulness of the KTY formalism. Here we assume the magnetic interaction of Majorana type $\mu_{\alpha\beta}\bar{\nu}_\alpha F_{\lambda\kappa}\sigma^{\lambda\kappa}\nu^c_\beta + h.c.$, and in this case the magnetic moments $\mu_{\alpha\beta}$ are real and antisymmetric in flavor indices: $\mu_{\alpha\beta} = -\mu_{\beta\alpha}$.

$$\mathcal{M} \equiv \begin{pmatrix} U\mathcal{E}U^{-1} & \mathcal{B} \\ \mathcal{B}^\dagger & U^*\mathcal{E}(U^*)^{-1} \end{pmatrix} \tag{24}$$

with $\mathcal{B}_{\alpha\beta} \equiv B\mu_{\alpha\beta}$ is the hermitian mass matrix for neutrinos and anti-neutrinos without the matter effect where neutrinos have the magnetic moments $\mu_{\alpha\beta}$ in the magnetic field B.

For simplicity we consider the limit $\theta_{13} \to 0$ and $\Delta m^2_{21} \to 0$, and we assume that all the CP phases vanish. Then the matrix (24) can be rewritten as

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ i\mathbf{1} & \mathbf{1} \end{pmatrix}\begin{pmatrix} U\mathcal{E}U^{-1} + i\mathcal{B} & 0 \\ 0 & U\mathcal{E}U^{-1} - i\mathcal{B} \end{pmatrix} \times\begin{pmatrix} \mathbf{1} & -i\mathbf{1} \\ -i\mathbf{1} & \mathbf{1} \end{pmatrix},$$

so the problem of diagonalizing the 6×6 matrix (24) is reduced to diagonalizing the 3×3 matrices $U\mathcal{E}U^{-1} \pm i\mathcal{B}$. Since we are assuming that all the CP phases vanish, all the matrix elements $U_{\alpha j}$ and $\mathcal{B}_{\alpha\beta} = -\mathcal{B}_{\beta\alpha}$ are real, $U\mathcal{E}U^{-1} \pm i\mathcal{B}$ can be diagonalized by a unitary matrix and its complex conjugate:

$$\begin{aligned} U\mathcal{E}U^{-1} + i\mathcal{B} &= \tilde{U}\tilde{\mathcal{E}}\tilde{U}^{-1} \\ U\mathcal{E}U^{-1} - i\mathcal{B} &= \tilde{U}^*\tilde{\mathcal{E}}(\tilde{U}^*)^{-1}, \end{aligned} \tag{25}$$

and the equation for motion is given by

$$\begin{aligned} & i\frac{d}{dt}\begin{pmatrix} \Psi(t) + i\Psi^c(t) \\ \Psi(t) - i\Psi^c(t) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{U}(t)\tilde{\mathcal{E}}(t)\tilde{U}^{-1}(t)\{\Psi(t) + i\Psi^c(t)\} \\ \tilde{U}^*(t)\tilde{\mathcal{E}}(t)(\tilde{U}^*)^{-1}(t)\{\Psi(t) - i\Psi^c(t)\} \end{pmatrix}. \end{aligned}$$

Introducing the notations

$$\mathcal{B}_{\alpha\beta} = B\mu_{\alpha\beta} \equiv \begin{pmatrix} 0 & -p_0 & -q_0 \\ p_0 & 0 & -r_0 \\ q_0 & r_0 & 0 \end{pmatrix},$$

$$\begin{aligned} & e^{-i\theta_{23}\lambda_7}\,\mathcal{B}\,e^{i\theta_{23}\lambda_7} \\ &= \begin{pmatrix} 0 & -p_0c_{23} + q_0s_{23} & -p_0s_{23} - q_0c_{23} \\ p_0c_{23} - q_0s_{23} & 0 & -r_0 \\ p_0s_{23} + q_0c_{23} & r_0 & 0 \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{pmatrix}, \end{aligned}$$

it can be shown [13] that Eq. (25) can be rewritten as

$$\begin{aligned} & e^{i\theta_{23}\lambda_7}\,e^{i\omega\lambda_2}\left[\mathrm{diag}(0,0,\Delta E_{31}) + \Lambda e^{i\chi\lambda_5}\,\lambda_2\,e^{-i\chi\lambda_5}\right] \\ &\times e^{-i\omega\lambda_2}\,e^{-i\theta_{23}\lambda_7}, \end{aligned} \tag{26}$$

where Λ, ω and χ are defined by $\Lambda \equiv \sqrt{p^2+q^2+r^2}$, $\omega \equiv \tan^{-1}(r/q)$, $\chi \equiv \tan^{-1}(\sqrt{q^2+r^2}/p)$. Since the we are mainly interested in the effective mixing angle which mixes the two energy eigenstates, the matrices $e^{i\theta_{23}\lambda_7}e^{i\omega\lambda_2}$ on the left-hand side and $e^{-i\omega\lambda_2}e^{-i\theta_{23}\lambda_7}$ on the right-hand side of the square bracket in Eq. (26) are irrelevant, so we discuss the following matrix:

$$\mathcal{M} \equiv \mathrm{diag}(0,0,\Delta E_{31}) - \frac{\Delta E_{31}}{3}\mathbf{1} + \Lambda e^{i\chi\lambda_5}\,\lambda_2\,e^{-i\chi\lambda_5}, \tag{27}$$

where a matrix which is proportional to identity was subtracted for convenience in later calculations so that the trace of $\mathcal{M}$ vanishes. The eigenvalues $\tilde{E}_j$ of the matrix $\mathcal{M}$ are given by $\tilde{E}_j = 2\sqrt{\Delta E^2_{31}/9 + \Lambda^2/3}\,\cos(\varphi + 2j\pi/3)$ $(j = 1,2,3)$, where $\cos 3\varphi \equiv \left\{(\Delta E_{31}/3)^3 - (1+3\cos 2\chi)\,\Lambda^2\Delta E_{31}/12\right\} / \left(\Delta E^2_{31}/9 + \Lambda^2/3\right)^{3/2}$. In Fig. 2 the three eigenvalues t_j $(j = 1,2,3)$ which are normalized by $2\sqrt{\Delta E^2_{31}/9 + \Lambda^2/3}$ are depicted as a function of $u \equiv 3\Lambda^2/\Delta E^2_{31}$. If χ is small, then the two of the three eigenvalues get close to each other, and χ can be regarded as the vacuum mixing angle near the level-crossing in the present case. In this example, for a large value of $\Lambda \gg \Delta E_{31}$, the energy eigenvalues are 0 and $\pm\Lambda$, and we found that there is only one level-crossing for $|\Delta E_{31}| \sim \Lambda$, unlike in the standard three flavor case. So in the following we discuss the contribution from one level-crossing only.

Furthermore, it can be shown [13] that the following relation holds:

$$\begin{aligned} & P(\nu_\alpha \to \nu_\beta) + P(\nu_\alpha \to \bar{\nu}_\beta) \\ &= \sum_{j,k} |\tilde{U}_{\beta j}(L)|^2\,|(W_H)_{jk}|^2\,|\tilde{U}^*_{\alpha k}(0)|^2 \\ &= \begin{pmatrix} |U_{\beta 1}|^2 & |U_{\beta 2}|^2 & |U_{\beta 3}|^2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-P_H & P_H \\ 0 & P_H & 1-P_H \end{pmatrix} \end{aligned}$$

[4] The possibility that the magnetic moments of neutrinos in a large magnetic field affect the neutrino flavor transition caught a lot of attention after this idea was applied to the solar neutrino deficit in Refs. [21, 22, 23, 24].

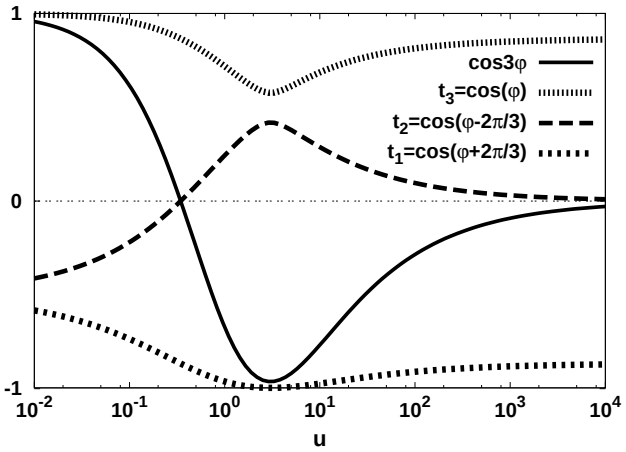

Figure 2: The behaviors of the normalized eigenvalues $t_j \equiv \tilde{E}_j/\left(2\sqrt{\Delta E^2_{31}/9+\Lambda^2/3}\right) = \cos(\varphi + 2j\pi/3)\,(j = 1,2,3)$ and $\cos 3\varphi$ as functions of $u \equiv 3\Lambda^2/\Delta E^2_{31}$. See Ref. [13] for details.

$$\times \begin{pmatrix} \left|\tilde{U}_{\alpha 1}(0)\right|^2 \\ \left|\tilde{U}_{\alpha 2}(0)\right|^2 \\ \left|\tilde{U}_{\alpha 3}(0)\right|^2 \end{pmatrix}, \tag{28}$$

where we have assumed that the level-crossing occurs between the energy eigenstates 2 and 3, and we have assumed that there is no magnetic field at the endpoint $t = L$, and $|\tilde{U}_{\alpha j}(0)|^2$ in the transition probability can be obtained from the KTY formula (5).

In the approximation of the small mixing angle χ, the jumping factor P_H can be calculated as [13]

$$P_H \simeq \exp\left(-\frac{\pi}{|d\Lambda/dt|_{u=u_0}}\Delta E^2_{31}\chi^2\right). \tag{29}$$

It can be also shown that the exponent of the jumping factor P_H coincides with $-\pi/2$ times the γ factor in the case of a linear potential ($F = 1$):

$$\gamma = \left.\frac{\Delta\tilde{E}_{32}}{2|d\tilde{\psi}_{23}/dt|}\right|_{u=u_0} \simeq \frac{2\Delta E^2_{31}\chi^2}{|d\Lambda/dt|_{u=u_0}} = -\frac{\log P_H}{\pi/2}.$$

4. Conclusions

Using the formalism which was developed by Kimura, Takamura and Yokomakura to express analytically the combination $\tilde{X}^{\alpha\beta}_j \equiv \tilde{U}_{\alpha j}\tilde{U}^*_{\beta j}$ of the mixing matrix elements in matter with constant density, we have shown that the effective mixing angle can be analytically expressed in terms of the mixing matrix elements in vacuum and the energy eigenvalues. The analytical expression for the effective mixing angle enables us to evaluate the nonadiabatic contribution to the transition probability based on the two assumptions: (i) The nonadiabatic transitions in the case with more than two energy eigenstates can be separately treated as a two state problem at each level crossing. (ii) The exponent of the probability obtained by the WKB method is proportional to the factor γ which is the ratio of the energy difference of the two eigenstates to the derivative of the effective mixing angle at the level crossing. We have given two examples: one with flavor dependent non-standard interactions in neutrino propagation and the other with magnetic moments in a large magnetic field.

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