

## Classification of lepton mixing patterns from finite flavour symmetries

Renato M. Fonseca<sup>a,1</sup>, Walter Grimus<sup>b</sup>

<sup>a</sup>AHEP Group, Instituto de Física Corpuscular, C.S.I.C./Universitat de València  
Edificio de Institutos de Paterna, Apartado 22085, E-46071, València, Spain

<sup>b</sup>University of Vienna, Faculty of Physics, Boltzmanngasse 5, A-1090 Vienna, Austria

### Abstract

Flavour symmetries have been used to constrain both quark and lepton mixing parameters. In particular, they can be used to completely fix the mixing angles. For the lepton sector, assuming that neutrinos are Majorana particles, we have derived the complete list of mixing patterns achievable in this way, as well as the symmetry groups associated to each case. Partial computer scans done in the past have hinted that such list is limited, and this does indeed turn out to be the case. In addition, most mixing patterns are already 3-sigma excluded by neutrino oscillation data.

**Keywords:** Lepton mixing, Finite flavour symmetries, Roots of unity

### 1. Residual symmetries of the lepton mass matrices

Two lepton mixing angles are known to be large, yet there is no established theory which explains their values. One hypothesis is that the lepton flavour structure is explained by the existence of some finite symmetry which acts on flavour space. Given that symmetry is known to play a critical role in fundamental physics, this is a particularly appealing possibility, which was explored extensively in the literature. However, most research has focused on building models which incorporate particular flavour symmetries, with little attention being given to the problem of systematically cataloging the various possibilities.

References [2, 3] are notable exceptions, where the GAP program [4] was used to make scans over various groups and representations, yielding lists of lepton mixing matrices which can be obtained from finite flavour symmetries. However, even though these are interesting pioneering works in this area, by the very nature of such computational scans, they are necessarily partial since

there are infinitely many groups to be searched. Furthermore, it is useful to have not just a computational understanding of the problem, but an analytical one as well.

In [1] we have done such a systematic study for the cases where the lepton mixing matrix is predicted completely by a finite discrete symmetry, leaving no relation between mixing angles and mass ratios. It is then appropriate to first briefly review the relation between this class of models and the residual symmetries of the lepton mass matrices. Assuming that neutrinos are Majorana particles, the mass terms of leptons are the following:

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2} \nu_L^T C^{-1} M_\nu \nu_L + \text{h.c.} \quad (1)$$

A transformation  $(\ell_L, \nu_L) \rightarrow (T\ell_L, S_i \nu_L)$  with  $T = U^\dagger \cdot \text{diag}(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}) \cdot U$  and  $S_{1,2,3}$  given by  $\text{diag}(-1, +1, +1)$ ,  $\text{diag}(1, -1, +1)$ ,  $\text{diag}(+1, +1, -1)$  will impose  $U$  as the lepton mixing matrix, provided that the eigenvalues of  $T$  are distinct.<sup>2</sup> To be precise, given that this symmetry leaves the lepton masses completely unconstrained, one is able to impose that the

<sup>1</sup>Contribution to the proceedings of the 37th International Conference on High Energy Physics (ICHEP 2014), Valencia, Spain, 2 – 9 July 2014, based on [1].

<sup>2</sup>If this is not the case, two  $T$  matrices are needed.

absolute values of the entries of the mixing matrix are given by  $|U|_{ij} \equiv |U_{ij}|$ , up to row and column permutations of  $U$ .

The  $\{T, S_i\}$  matrices generate a group  $G$ , which is infinite in general. However, it should be noted that the cases where  $G$  is finite are particularly interesting and deserve special attention, as they provide a way to avoid having massless scalar particles. One can ask: when does  $T$  and the  $S_i$  generate a finite group?

The answer to this question depends on both  $U$  and the eigenvalues of the  $T$  matrix (the  $\lambda_i^{(0)}$ ). For example, in [5] it was proven that for tribimaximal mixing (TBM) — which was an acceptable ansatz until the reactor angle was measured to be appreciable — the resulting group is finite if and only if the eigenvalues of  $T$  are  $(1, \omega, \omega^2)$  up to permutations of these values and multiplication by a common root of unity ( $\omega \equiv \exp(2\pi i/3)$ ). Depending on this multiplicative root of unity,  $G$  might be  $S_4$  or some other group which contains it as a subgroup. It is therefore natural to associate tribimaximal mixing with the effective symmetry group  $S_4$ .

The technique used in [5] to derive this connection relied on scanning all finite subgroups of  $SU(3)$  (as listed in [6, 7]; see also [8]) and on analyzing their 3-dimensional irreducible representations. However, there is a simpler way to derive the same result.

## 2. Roots of unity, and finite groups

The finiteness of a group  $G$  requires that the eigenvalues of its elements  $g \in G$ , in any representation, must be roots of unity. This is a direct consequence of the fact that there is always some positive integer  $n$  such that  $g^n = e$  (the identity element). As such, the eigenvalues of  $T$  (the  $\lambda_i^{(0)}$ ) as well as those of  $TS_j$  (which we may call  $\lambda_i^{(j)}$ ) must all be roots of unity. Computing the eigenvalues of the  $TS_j$  matrices as a function of those of  $T$  can be done, but the expressions are in general complicated so, instead of looking at the  $\lambda_i^{(j)}$  individually, we may consider the combinations  $\lambda_1^{(j)} + \lambda_2^{(j)} + \lambda_3^{(j)}$  which are just the traces of the matrices  $TS_j$  (no additional useful information is obtainable from  $\lambda_1^{(j)}\lambda_2^{(j)} + \lambda_1^{(j)}\lambda_3^{(j)} + \lambda_2^{(j)}\lambda_3^{(j)}$  and  $\lambda_1^{(j)}\lambda_2^{(j)}\lambda_3^{(j)}$ ). For  $U = U_{TBM}$ , it is easy to check that

$$\lambda_1^{(0)} + \lambda_2^{(0)} + \lambda_3^{(0)} + 3(\lambda_1^{(2)} + \lambda_2^{(2)} + \lambda_3^{(2)}) = 0. \quad (2)$$

This is a vanishing sum of roots of unity with integer coefficients, whose solutions can be found using theorem 6 of [9]. There are two: either (a)  $\lambda_1^{(0)} = \lambda_2^{(0)} = \lambda_3^{(0)} = -\lambda_i^{(2)}$  and  $\lambda_j^{(2)} = -\lambda_k^{(2)}$  ( $i \neq j \neq k \neq i$ ), or

(b)  $(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}) = \xi(1, \omega, \omega^2)$  and  $(\lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_3^{(2)}) = \xi'(1, \omega, \omega^2)$  for some roots of unity  $\xi, \xi'$  (up to permutations amongst the  $\lambda_i^{(0)}/\lambda_i^{(2)}$ ). Solution (a) implies that the eigenvalues of  $T$  are completely degenerate ( $T \propto \mathbb{1}$ ) which is unacceptable from the physical point of view (see section 1). Solution (b) is the one reported in [5].

## 3. The lepton mixing matrices compatible with finite symmetries

Instead of just scanning the allowed  $\lambda_i^{(0)}$  for a particular value of  $U$ , in [1] we considered the problem in full generality, which resulted in the list of all the values of  $(\lambda_i^{(0)}, U)$  which lead to a finite group. To do so, we applied known results concerning roots of unity (see also [10]), without having to rely on the classification of the finite subgroups of  $SU(3)$ , nor did we have to use group or representation theory. This is particularly relevant given that finding the finite subgroups of  $SU(3)$  and their representations over the last century was not a straightforward task (see [11]).

In a first step, it can be shown that, up to row and column permutations, the representation matrix  $X$  of any element of a finite group which contains the  $S_i$  is of the form

$$|X| = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix},$$

or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, \quad (3)$$

where  $\theta$  is some rational angle ( $\theta = \pi q$ , with  $q \in \mathbb{Q}$ ). This applies to  $X = T$  but also to  $X = T^2$ , for example. It is then possible, in a second step, to enumerate all the possible  $T$ 's by knowing the allowed values of  $|T|$  and  $|T^2|$ .

Given that the rows of  $U^*$  are the eigenvectors of  $T$ , we end up with a list of the lepton mixing matrices which are compatible with a finite flavor group  $G$ . This list contains 17 sporadic  $U$ 's involving three-flavour mixing, as well as the infinite series of mixing matrices

$$|U|^2 = \frac{1}{3} \begin{pmatrix} 1 & 1 + \operatorname{Re} \rho & 1 - \operatorname{Re} \rho \\ 1 & 1 + \operatorname{Re}(\omega \rho) & 1 - \operatorname{Re}(\omega \rho) \\ 1 & 1 + \operatorname{Re}(\omega^2 \rho) & 1 - \operatorname{Re}(\omega^2 \rho) \end{pmatrix} \quad (4)$$

where  $\rho = \exp(2\pi i p/n)$  can be any root of unity. The sporadic cases are associated with the minimal groups

$A_4$ ,  $S_4$ ,  $A_5$ ,  $PSL(2, 7)$  and  $\Sigma(360 \times 3)$  and they are excluded by neutrino oscillation data at more than  $3\sigma$  since they all predict either  $\sin^2 \theta_{13} = 0$  or  $\sin^2 \theta_{13}$  equal or larger than  $(5 - \sqrt{21})/12 \approx 0.035$ .

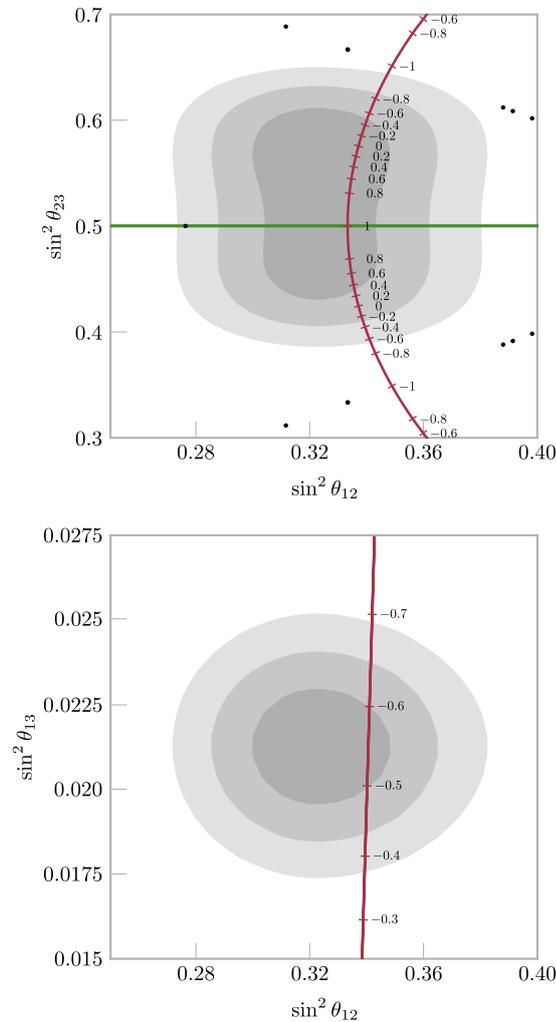


Figure 1: Plots showing the angles predicted by some of the mixing matrices which can be obtained with finite flavour symmetries (figure taken from [1]). The sporadic mixing matrices appear as dots, while the infinite series of  $U$ 's displayed in equation (4) is represented as a red ( $\cos^2 \theta_{13} \sin^2 \theta_{12} = 1/3$ ) and a green ( $\sin^2 \theta_{13} = 1/3$ ) line, depending on the column permutation which is considered. The numbers along the red curves indicate the values of  $\text{Re}(\rho^6)$ . The gray areas indicate the 1, 2 and  $3\sigma$  regions allowed by neutrino oscillation experiments, as computed by Forero et al. [12].

That leaves the infinite series of equation (4) — for  $-0.69 \lesssim \text{Re}(\rho^6) \lesssim -0.37$  — as the only phenomenologically viable case (see figure 1). Note that this type of mixing implies a  $\theta_{23}$  reasonably far from maximal ( $\sin^2 \theta_{23} \sim 0.4$  or  $0.6$ ), and a trivial Dirac-type phase  $\delta$ . If  $n$  is not divisible by 9, it can be

shown that the minimal group  $G$  is  $\Delta(6m^2)$ , with  $m = \text{lcm}(6, n)/3$ ; otherwise  $G$  is a semi-direct product of the form  $(\mathbb{Z}_m \times \mathbb{Z}_{m/3}) \rtimes S_3$ , with  $m = \text{lcm}(2, n)$ .

## Acknowledgments

The work of R.F. was supported by the Spanish grants FPA2011-22975 and Multidark CSD2009-00064 (MINECO), PROMETEOII/2014/084 (Generalitat Valenciana), and by the Portuguese grants CERN/FP/123580/2011 and EXPL/FIS-NUC/0460/2013 (Fundação para a Ciência e a Tecnologia).

## References

- [1] R. M. Fonseca, W. Grimus, Classification of lepton mixing matrices from finite residual symmetries, JHEP 1409 (2014) 033. arXiv:1405.3678, doi:10.1007/JHEP09(2014)033.
- [2] M. Holthausen, K. S. Lim, M. Lindner, Lepton mixing patterns from a scan of finite discrete groups, Phys. Lett. B721 (2013) 61–67. arXiv:1212.2411, doi:10.1016/j.physletb.2013.02.047.
- [3] K. M. Parattu, A. Wingerter, Tribimaximal mixing from small groups, Phys. Rev. D84 (2011) 013011. arXiv:1012.2842, doi:10.1103/PhysRevD.84.013011.
- [4] The GAP Group, GAP – Groups, Algorithms, and Programming (2014). URL <http://www.gap-system.org>
- [5] C. Lam, The unique horizontal symmetry of leptons, Phys. Rev. D78 (2008) 073015. arXiv:0809.1185, doi:10.1103/PhysRevD.78.073015.
- [6] W. Fairbairn, T. Fulton, W. Klink, Finite and disconnected subgroups of SU3 and their application to the elementary-particle spectrum, J. Math. Phys. 5 (8) (1964) 1038. doi:10.1063/1.1704204.
- [7] A. Bovier, M. Lüling, D. Wyler, Finite subgroups of SU(3), J. Math. Phys. 22 (1981) 1543. doi:10.1063/1.525096.
- [8] W. Grimus, P. O. Ludl, On the characterization of the SU(3)-subgroups of type C and D, J. Phys. A47 (2014) 075202. arXiv:1310.3746, doi:10.1088/1751-8113/47/7/075202.
- [9] J. H. Conway, A. J. Jones, Trigonometric diophantine equations (On vanishing sums of roots of unity), Acta Arith. 30 (1976) 229240.
- [10] W. Grimus, Discrete symmetries, roots of unity, and lepton mixing, J. Phys. G40 (2013) 075008. arXiv:1301.0495, doi:10.1088/0954-3899/40/7/075008.
- [11] P. O. Ludl, Comments on the classification of the finite subgroups of SU(3), J. Phys. A44 (2011) 255204. arXiv:1101.2308, doi:10.1088/1751-8113/45/6/069502, 10.1088/1751-8113/44/25/255204.
- [12] D. Forero, M. Tórtola, J. Valle, Neutrino oscillations refitted. arXiv:1405.7540.