Fits and Related Systematics for the Hadronic Vacuum Polarization on the Lattice

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Outline: How to get $\alpha_H^{\mu V}$ w. error less than 1%?

- Low-$Q^2$ region, where data is scarce, dominates: problem.

- New Strategy: "divide and conquer".

\[
\text{Split } 0 \leq Q^2 \leq Q^2_{\text{min}} \text{ and } Q^2_{\text{min}} \leq Q^2 \leq \infty.
\]

\[
Q^2_{\text{min}} \sim 0.1 \text{ GeV}^2
\]

- Trapezoid rule approximation for $0.1 \text{GeV}^2 \lesssim Q^2 \lesssim \infty$

- Use 3 (independent) methods for $0 \leq Q^2 \lesssim 0.1 \text{GeV}^2$
  1. Pades
  2. Polynomial in conformal variable
  3. $N^2\text{LO-ChPT}$ supplemented with $O(p^8)$ phenomenological LEC.
\[(g - 2)_{\mu}^{HV P} \sim \int_0^\infty dQ^2 \frac{f(Q^2)}{\text{known}} \left[ \Pi(Q^2) - \Pi(0) \right] \]

\[\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0) \implies (g - 2)_{\mu}^{HV P} \]

- \[\star \text{ integrand strongly peaked at } Q^2 \sim \frac{m^2_{\mu}}{4} \sim 0.003 \text{ GeV}^2\]
\[ (g - 2)^{HV P}_\mu \sim \int_0^\infty dQ^2 \ f(Q^2) \left[ \Pi(Q^2) - \Pi(0) \right] \]

\[ \hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0) \implies (g - 2)^{HV P}_\mu \]

\[ \star \text{ integrand strongly peaked at } Q^2 \sim m^2_{\mu}/4 \sim 0.003 \text{ GeV}^2 \]

if no good data in region of curvature \implies possibly wrong results!

Need reliable function!

how to test this theoretical error?
**A $\tau$-based model for $I = 1$ contributions**

Boito, Cata, Golterman, Jamin, Mahdavi, Maltman, Osborne, SP ’11 + ’12

$t \leq t_{min}$ GeV$^2$ $\rightarrow$ OPAL data.

$t \geq t_{min}$ GeV$^2$:

\[ \text{Im}\Pi(t) = \rho \text{Pert. Th.}(t) + e^{-\delta - \gamma t} \sin(\alpha + \beta t) \]

\[ \Pi(Q^2) = -Q^2 \int_{4m_{\pi}^2}^\infty \frac{dt}{\pi} \frac{\text{Im}\Pi(t)}{t(t+Q^2)} \]

We take $t_{min} = 1.5$ GeV$^2$.

$\Pi(0) = 0$

- Take typical lattice $Q^2$ values + lattice covariance matrix

  (e.g., Aubin et al. ’12, $64^3 \times 144$ lattice, $a = 0.06$ fm, $m_\pi = 220$ MeV, periodic BCs)

- Generate fake lattice data for $\Pi(Q^2)$ and compare with true answer from model.

- You should try this model to check your systematics: it’s very physically motivated!
The perils of using the wrong function

Take, e.g., the case of VMD+polynomial as fitting function

You may think this is a good fit for an accurate \((g - 2)_\mu\):

\[
\Pi(Q^2)
\]

solid line=VMD+ Fit
The perils of using the wrong function

while, in fact, this is what you should be looking at:

\( \text{dashed blue=Truth} \)
\( \text{solid line=VMD+polynomial Fit} \)

(g-2) Integrand

Systematic error in \( a^\text{HVP}_\mu \) corresponds to 18 units of nominal fit error!
Hybrid strategy for a less than 1% error

More than 80% of $a_{\mu}^{\text{HVP}}$ accumulates for $Q^2 \lesssim 0.1$ GeV$^2$.

$\Rightarrow$ Split the contributions $0 \leq Q^2 \lesssim 0.1$ GeV$^2$ and $0.1$ GeV$^2 \lesssim Q^2 \leq \infty$
The $0.1 \, \text{GeV}^2 \lesssim Q^2 \leq \infty$ interval (I)

Trapezoid rule approximation (use fake data for vac. pol. from l=1 model with $Q^2$ from MILC $64^3 \times 144$, $a = 0.06 \, \text{fm}$ lattice data, including covariances, PBC).

Systematic and statistical error due to trapezoid rule.

![Graph showing systematic and statistical error due to trapezoid rule.]  

Should expect clear improvement w. AMA, twisting or analytic continuation (Feng et al. '13).
Systematic error due to $\delta \Pi(0) = 0.001$ (obtained from [1, 1] Padé fit to $Q^2 \leq 1$ GeV$^2$).
The $0 \leq Q^2 \leq 0.1 \text{ GeV}^2$ interval

Three independent strategies:

- Pades .
- Polynomial in a conformal variable.
- $N^2\text{LO- ChPT} + C Q^4$
Rational function matched to derivatives of $\Pi(Q^2)$ at $Q^2 = 0$, or matched/fitted to several $Q^2$ values.

Derivatives of $\Pi(Q^2)$ at $Q^2 = 0 \Leftrightarrow$ time moments

$$\hat{\Pi}(Q^2) = Q^2 (\Pi_1 + \Pi_2 Q^2 + \Pi_3 Q^4 + \Pi_4 Q^6 + \ldots)$$

with $\Pi_j = \frac{(-1)^{j+1}}{(2j+2)!} \int d^4x \; t^{2j+2} \langle J_1(x) J_1(0) \rangle$

Interesting property (convergence $+$ alternating bounds):

$$[1, 0]_H \leq [2, 1]_H \leq \ldots \leq [N + 1, N]_H \leq \hat{\Pi}(Q^2) \leq [N, N]_H \leq \ldots \leq [2, 2]_H \leq [1, 1]_H$$
[1, 1]_H is good enough: 2 parameters i.e. up to \( t^6 \) moment.

If, e.g., we fit Pade to data in interval \( 0.1 \text{GeV}^2 \leq Q^2 \leq 0.2 \text{GeV}^2 \) then need \([2, 1]_H\), typically.
Polynomials in conformal variable

\[ w(Q^2) = \frac{1 - \sqrt{1 + z}}{1 + \sqrt{1 + z}} , \quad z = \frac{Q^2}{4m^2_\pi} , \quad \Pi(Q^2) = \sum_{n=0}^{\infty} p_n w^n \]

Also convergent (like Pades) but no alternating bounds.

Sub-1% systematic error \(\Rightarrow\) quadratic polynomial in \(w\) (2 parameters).
Similar to \([1, 1]_H\) Pade.
Again, if fit in an interval, then \(\Rightarrow\) cubic polynomial.
$N^2LO - ChPT + C Q^4$

(Golterman, Maltman, S.P. ’14)

Phenomenologically motivated, but not fully model independent (no chiral log at $O(p^8)$).

Best strategy is get LEC from time moments (as did with Pades and conf. polynomial).

However, even with the term $CQ^4$, this ChPT functional form performs worse than Pades and/or conformal polynomial.

$\Rightarrow$ Mostly useful as a cross check.
Very Preliminary Results

Used time moments to construct Pades. \( u, d, s \) quarks.
RBC/UKQCD DWF data (Hudspith, Lewis, Maltman, Portelli '14)

(BDKZ11: Boyle, Del Debbio, Kerrane, Zanotti '11)
Conclusions

- Fitting on a large $Q^2$ window very dangerous.

- Pointed out that a hybrid strategy with
  - trapezoid rule approximation for $0.1 \text{ GeV}^2 \lesssim Q^2 \lesssim \infty$
  - Pades, conformal polynomials and $N^2\text{LO- ChPT} + C Q^4$ for $0 \leq Q^2 \lesssim 0.1 \text{ GeV}^2$

is capable of reaching the desired below 1% error in $\alpha_{\mu}^{\text{HVP}}$ provided time moments can be determined accurately enough (rule of thumb: $\delta \Pi_j / \Pi_j \lesssim j \%$)

- First attempt at time moments by Chakraborty et al. ’14 for $s, c$ quarks.

  What about $u, d$?

  (encouraging results from Hudspith et al. RBC/UKQCD DWF data, still very preliminary...)

- Alternatively (and a good check!): get good data in the low $Q^2 \sim 0.1 \text{GeV}^2$ interval (twisting, analytic continuation, etc...) and fit (Pades, Conf. polynomial,...)
BACK-UP SLIDES
Duality Violations

- OPE valid in euclidean, but not in minkowski. We know that spectrum $\neq$ OPE

\[ \text{Im} \Pi(t) \]

- We expect (at large $t$):

\[ \text{Im}\Pi_{DV} \sim \text{Im}(\Pi - \Pi_{OPE}) \sim \kappa e^{-\gamma t} \sin(\alpha + \beta t) \]

\[ \text{OPE asympt.} \quad \text{Regge} \]

- $\Pi_{DV}(s) \to 0$ as $|s| \to \infty$. Then:

\[ -\frac{1}{2\pi i} \oint_{|z|=s_0} dz \ w(z) \ \Pi_{DV}(z) = - \int_{s_0}^{\infty} ds \ w(s) \ \frac{1}{\pi} \text{Im}\Pi_{DV}(s) \]

\[ \text{extrapolation!} \]
Fitting functions

Padés, model independent, they enjoy a convergence theorem for \( N \to \infty \):

\[
\Pi(Q^2) = \Pi(0) + Q^2 \left( a_0 + \sum_{r=1}^{N} \frac{a_r}{Q^2 + b_r} \right)
\]

\( \Pi(0) \), \( a' \)'s and \( b' \)'s are fitting parameters.

VMD is not a Pade, since you fix \( b_1 = M_p^2 \). (true \( \Pi(Q^2) \) has cut starting at \( 4m^2 \pi^2 \) ...)

We have: \( a_0 \neq 0 \implies [N, N] \) Pade; \( a_0 = 0 \implies [N - 1, N] \) Pade.

For instance:

- \( \frac{a_1}{Q^2 + b_1} \) is a [0,1] Pade \( \implies \Pi(Q^2) = \Pi(0) + Q^2 \left( \frac{a_1}{Q^2 + b_1} \right) \)

- \( a_0 + \frac{a_1}{Q^2 + b_1} \) is a [1,1] Pade \( \implies \Pi(Q^2) = \Pi(0) + Q^2 \left( a_0 + \frac{a_1}{Q^2 + b_1} \right) \)

etc...
**Example: Stieltjes. No errors.**

Toy Vac. Pol. function \((\text{Im}\Pi \geq 0)\), with a cut, \(-\infty \leq Q^2 \leq -1\):

\[
\frac{\Pi(0) - \Pi(Q^2)}{Q^2} = \int_1^{\infty} \frac{dt}{(t + Q^2)} \frac{1}{t}
\]

\[
= \frac{1}{Q^2} \log (1 + Q^2)
\]

**Theorem:** As \(N \to \infty\), with \(a_i, b_i\) determined from the function (and derivatives) at \(Q^2 = 0\), or at multiple points,

\[
\frac{a_0 + a_1 Q^2 + a_2 Q^4 + \ldots + a_N Q^{2N}}{1 + b_1 Q^2 + \ldots + b_N Q^{2N}} \to \frac{\Pi(0) - \Pi(Q^2)}{Q^2}
\]

everywhere in a compact region in complex \(Q^2\), except on the cut.

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Graph showing the relation between \(Q^2\) and Relat. Error, with markers indicating poles.

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