Fits and Related Systematics for the Hadronic Vacuum Polarization on the Lattice

Santi Peris (U.A. Barcelona)

Based on coll. with Christopher Aubin (Fordham U.), Tom Blum (Connecticut U.),
Maarten Golterman (SFSU) & Kim Maltman (York U. & U. Adelaide)

(arXiv:1405.2389 ; arXiv:1309.2153 ; arXiv:1307.4701 ; arXiv:1205.3695)

Outline: How to get $a_\mu^{\rm HVP}$ *w. error less than* 1% *?*

- Low-Q^2 region, where data is scarce, dominates: problem.

- New Strategy: "divide and conquer".

Split $0 \le Q^2 \le Q^2_{min}$ and $Q^2_{min} \le Q^2 \le \infty$.

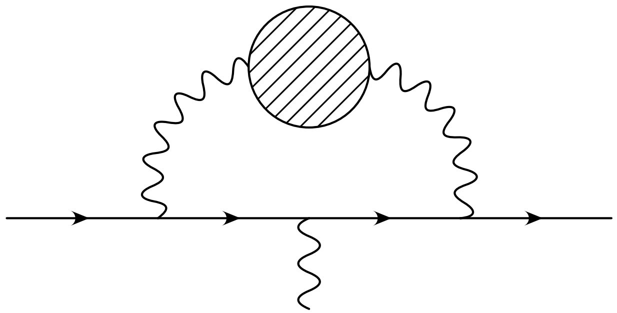
$$Q^2_{min} \sim 0.1 \ \mathrm{GeV}^2$$

- Trapezoid rule approximation for $0.1\mathrm{GeV}^2 \lesssim Q^2 \le \infty$

- Use 3 (independent) methods for $0 \le Q^2 \lesssim 0.1\mathrm{GeV}^2$:
  1. Pades
  2. Polynomial in conformal variable
  3. N^2LO-ChPT supplemented with $O(p^8)$ phenomenological LEC.

Introduction

Lautrup-de Rafael '69
Blum '02

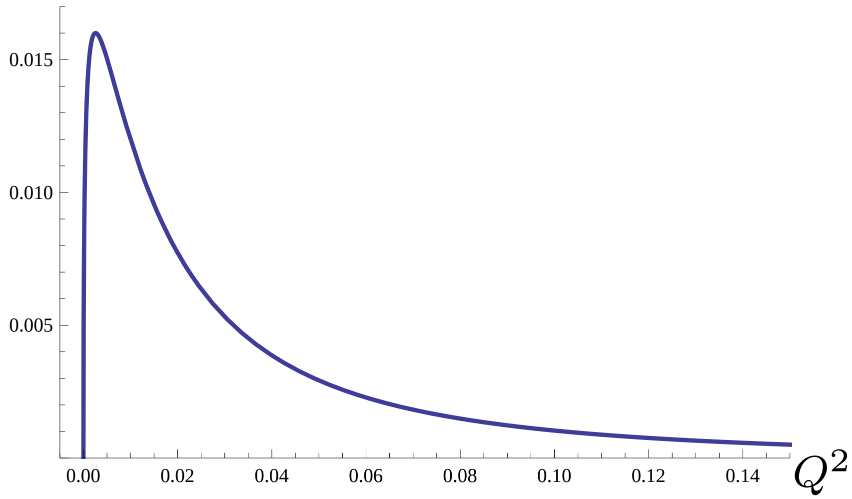


$$(g-2)_\mu^{HVP} \sim \int_0^\infty dQ^2 \underbrace{f(Q^2)}_{\text{known}} \left[\Pi(Q^2) - \Pi(0)\right]$$

$$\widehat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0) \Longrightarrow (g-2)_\mu^{HVP}$$

★ integrand strongly peaked at $Q^2 \sim m_\mu^2/4$ $\sim 0.003\ \mathrm{GeV}^2$

(g-2) integrand



Introduction

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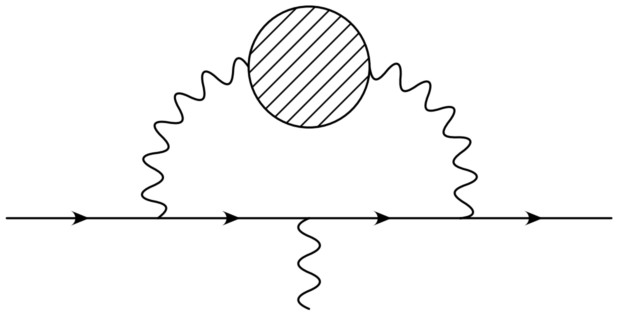

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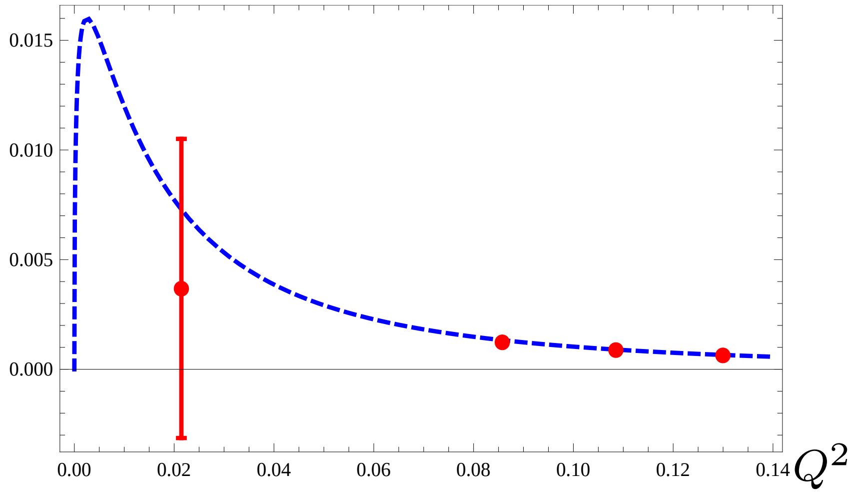
if no good data in region of curvature $\Longrightarrow$ possibly wrong results !

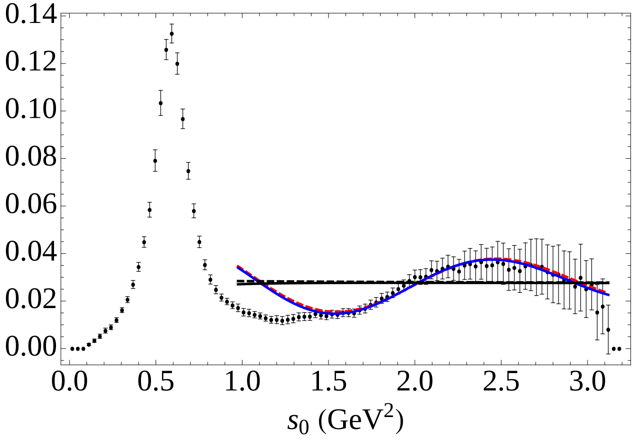
(g-2) integrand



Need reliable function !

how to test this theoretical error?

A τ-based model for $I = 1$ contributions



Boito, Cata, Golterman, Jamin, Mahdavi, Maltman, Osborne, SP '11 + '12

$t \leq t_{min}$ GeV^2 $\rightarrow$ OPAL data.
$t \geq t_{min}$ GeV^2:

$$\mathrm{Im}\Pi(t) = \rho_{\ \mathrm{Pert.Th.}}(t) + e^{-\delta - \gamma\ t} \sin(\alpha + \beta t)$$

$$\Pi(Q^2) = -Q^2 \int_{4m_\pi^2}^{\infty} \frac{dt}{\pi} \frac{\mathrm{Im}\Pi(t)}{t(t+Q^2)}$$

We take $t_{min} = 1.5$ GeV^2.

$$\Pi(0) = 0$$

- Take typical lattice Q^2 values + lattice covariance matrix

  (e.g., Aubin et al. '12, $64^3 \times 144$ lattice, a = 0.06 fm, $m_\pi = 220\mathrm{MeV}$, periodic BCs)

- Generate fake lattice data for $\Pi(Q^2)$ and compare with true answer from model.

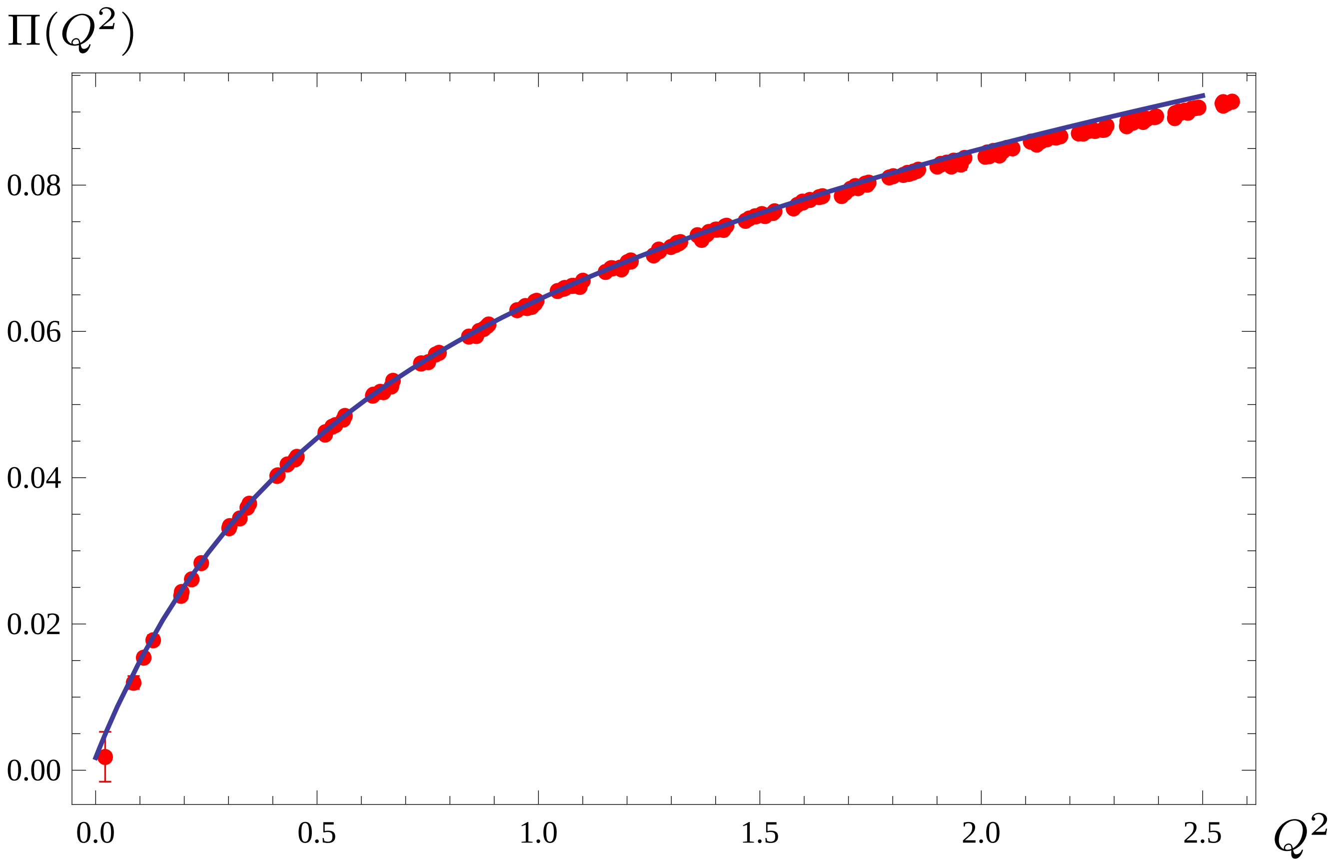
- You should try this model to check your systematics: it's very physically motivated !

The perils of using the wrong function

Take, e.g., the case of VMD+polynomial as fitting function

You may think this is a good fit for an accurate $(g-2)_\mu$:

solid line=VMD+ Fit

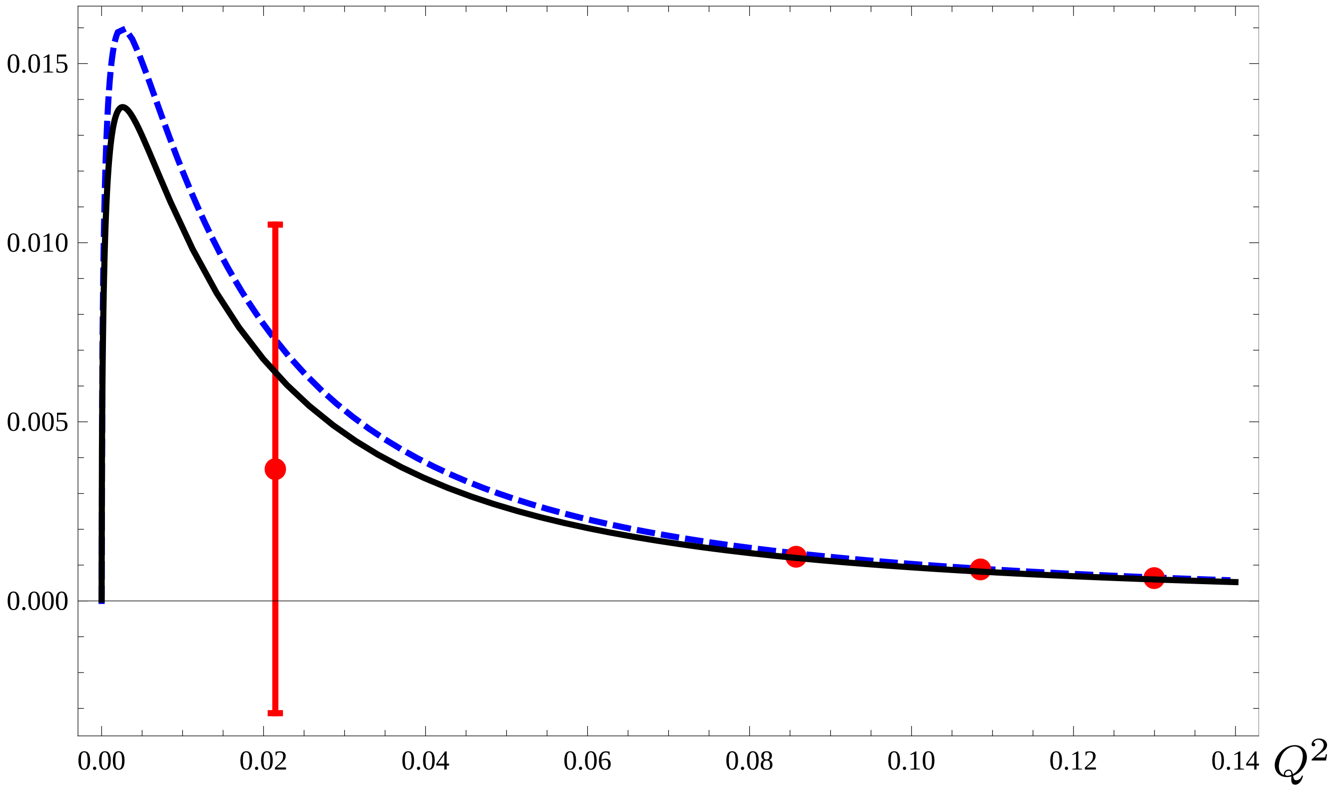


The perils of using the wrong function

while, in fact, this is what you should be looking at:

dashed blue=Truth
solid line=VMD+polynomial Fit

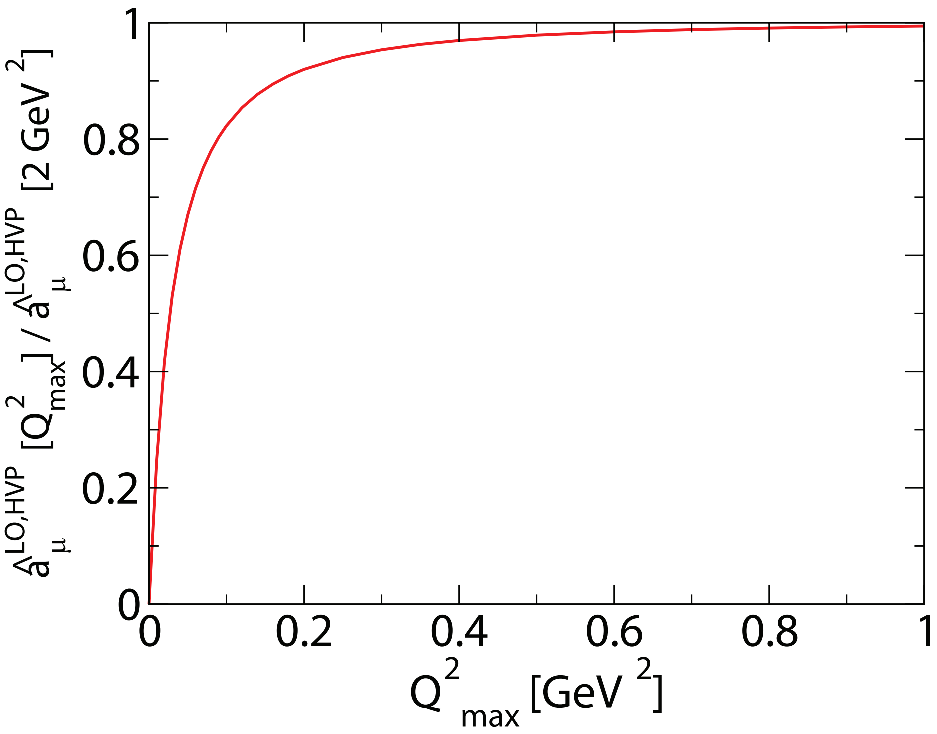
(g-2) Integrand



Systematic error in a_μ^{HVP} corresponds to 18 units of nominal fit error !

Hybrid strategy for a less than 1% *error*

More than 80% of $a_\mu^{\rm HVP}$ accumulates for $Q^2 \lesssim 0.1\ {\rm GeV}^2$.

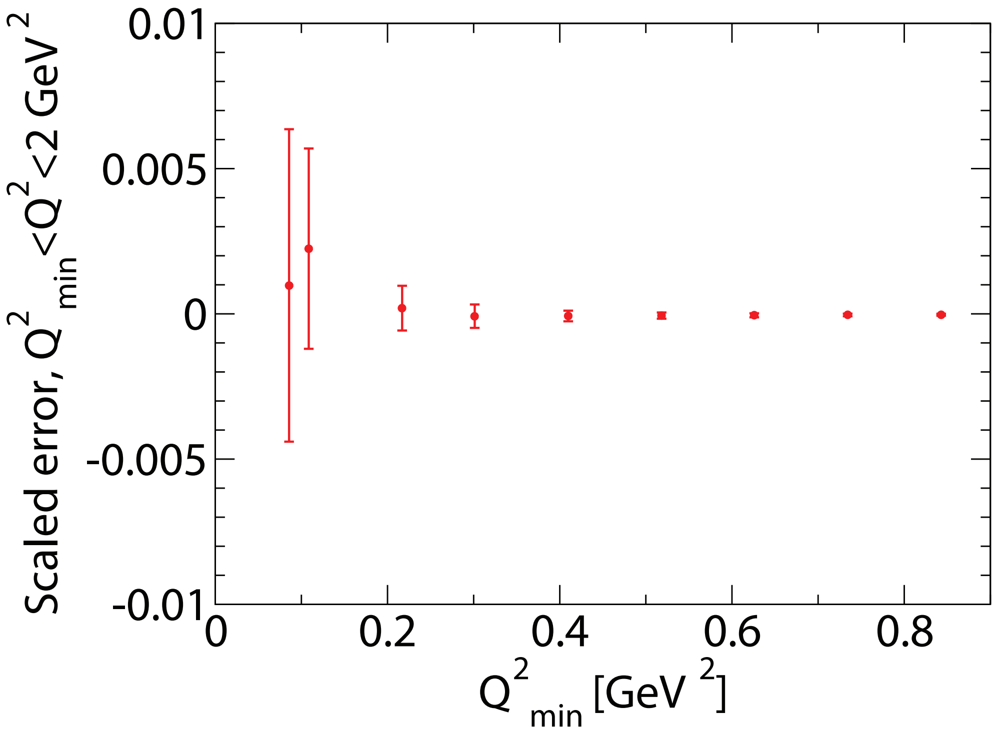


$\Rightarrow$ Split the contributions $0 \leq Q^2 \lesssim 0.1\ {\rm GeV}^2$ and $0.1\ {\rm GeV}^2 \lesssim Q^2 \leq \infty$

The $0.1\ \mathrm{GeV}^2 \lesssim Q^2 \leq \infty$ *interval (I)*

Trapezoid rule approximation (use fake data for vac. pol. from I=1 model with Q^2 from MILC $64^3 \times 144, a = 0.06\ \mathrm{fm}$ lattice data, including covariances, PBC).

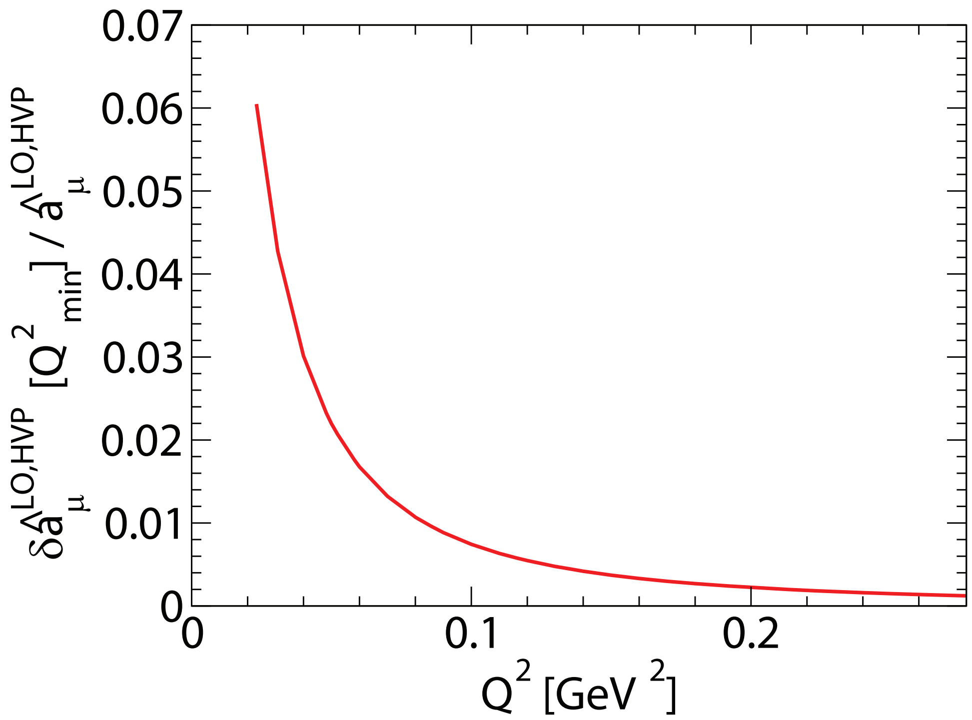
Systematic and statistical error due to trapezoid rule.



Should expect clear improvement w. AMA, twisting or analytic continuation (Feng et al. '13).

The $0.1\ \mathrm{GeV}^2 \lesssim Q^2 \leq \infty$ *interval (II)*

Systematic error due to $\delta\Pi(0) = 0.001$ (obtained from $[1,1]$ Pade fit to $Q^2 \leq 1\ \mathrm{GeV}^2$).



The $0 \leq Q^2 \lesssim 0.1\ \mathrm{GeV}^2$ *interval*

Three independent strategies:

- Pades .
- Polynomial in a conformal variable.
- N^2LO- ChPT + C Q^4

Pades (I)

(Aubin, Blum, Golterman, S.P. '12; Chakraborty et al. '14)

Rational function matched to derivatives of $\Pi(Q^2)$ at $Q^2 = 0$, or matched/fitted to several Q^2 values.

Derivatives of $\Pi(Q^2)$ at $Q^2 = 0 \Leftrightarrow$ time moments

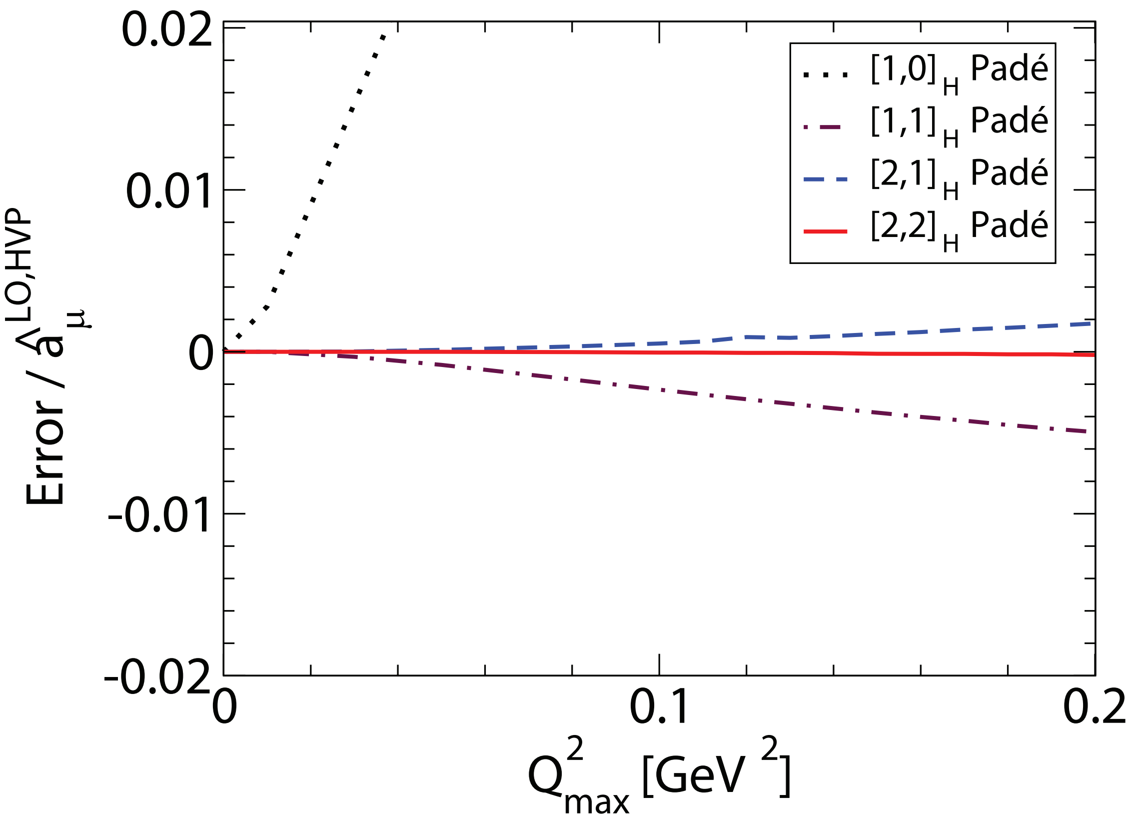
$$\widehat{\Pi}(Q^2) = Q^2 \left(\Pi_1 + \Pi_2 Q^2 + \Pi_3 Q^4 + \Pi_4 Q^6 + ...\right)$$

with $\Pi_j = \frac{(-1)^{j+1}}{(2j+2)!} \int d^4x \; t^{2j+2} \langle J_1(x) J_1(0) \rangle$

Interesting property (convergence+ alternating bounds):

$$[1,0]_H \leq [2,1]_H \leq \cdots \leq [N+1,N]_H \leq \; \widehat{\Pi}(Q^2) \; \leq [N,N]_H \leq \cdots \leq [2,2]_H \leq [1,1]_H$$

Pades (II)



$[1,1]_H$ is good enough: 2 parameters i.e. up to t^6 moment.

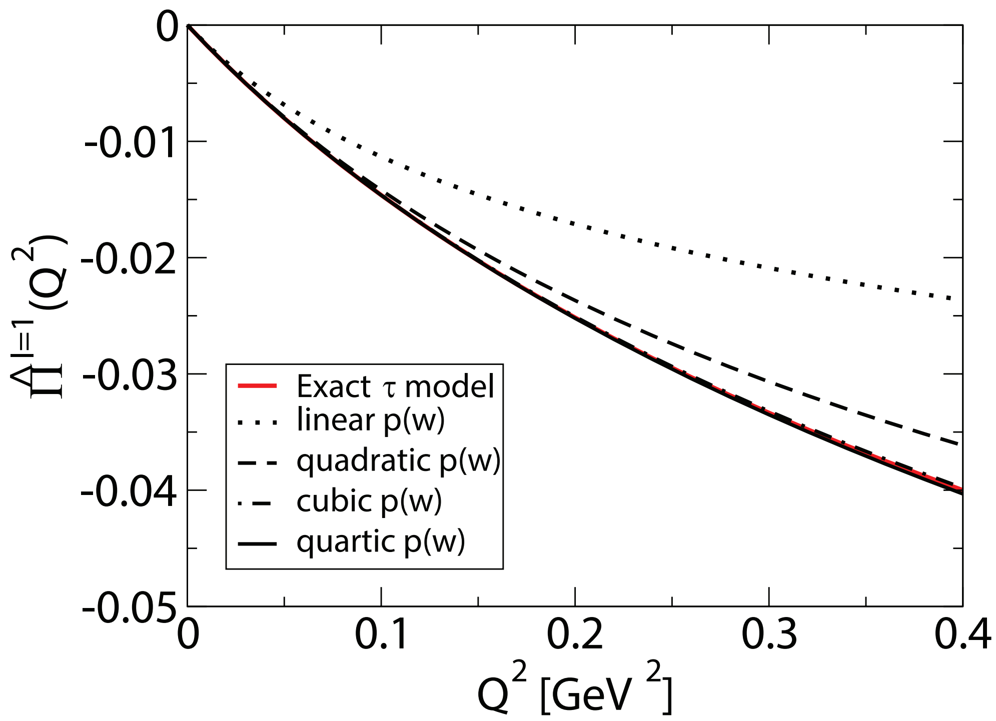
If,e.g., we fit Pade to data in interval $0.1\mathrm{GeV}^2 \leq Q^2 \leq 0.2\mathrm{GeV}^2$ then need $[2,1]_H$, typically.

Polynomials in conformal variable

(Golterman, Maltman, S.P. '14)

$$w(Q^2) = \frac{1-\sqrt{1+z}}{1+\sqrt{1+z}} \quad , \quad z = \frac{Q^2}{4m_\pi^2} \quad , \quad \Pi(Q^2) = \sum_{n=0}^{\infty} p_n w^n$$

Also convergent (like Pades) but no alternating bounds.

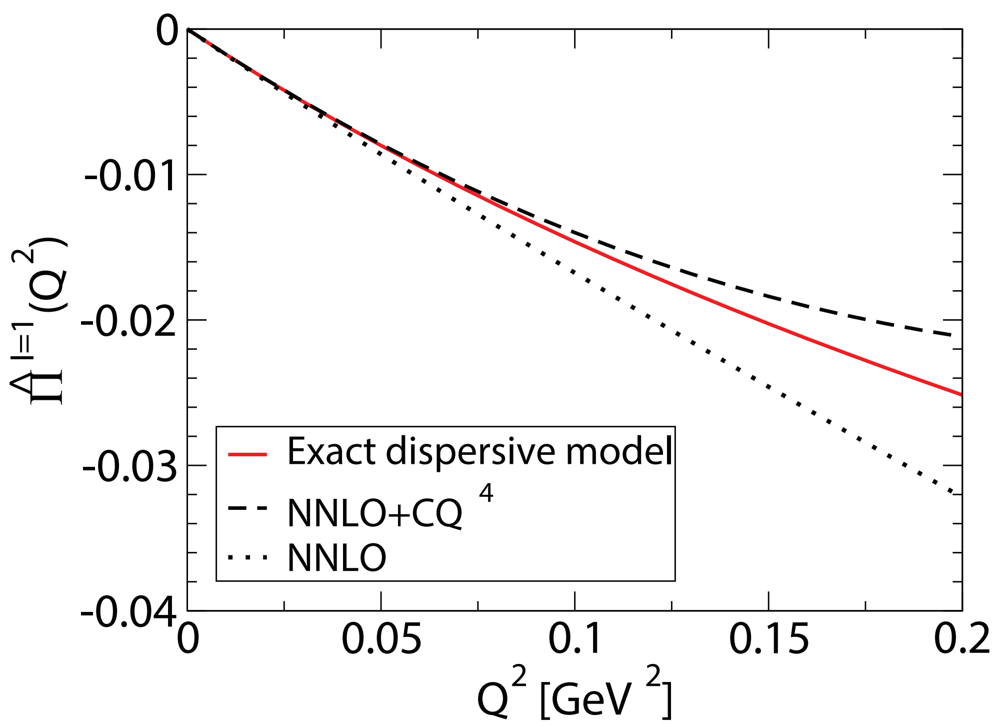


Sub-1% systematic error $\Rightarrow$ quadratic polynomial in w (2 parameters).
Similar to $[1,1]_H$ Pade.
Again, if fit in an interval, then $\Rightarrow$ cubic polynomial.

N^2LO - $ChPT + C\,Q^4$ (Golterman, Maltman, S.P. '14)

Phenomenologically motivated, but not fully model independent (no chiral log at $\mathcal{O}(p^8)$).

Best strategy is get LEC from time moments (as did with Pades and conf. polynomial).

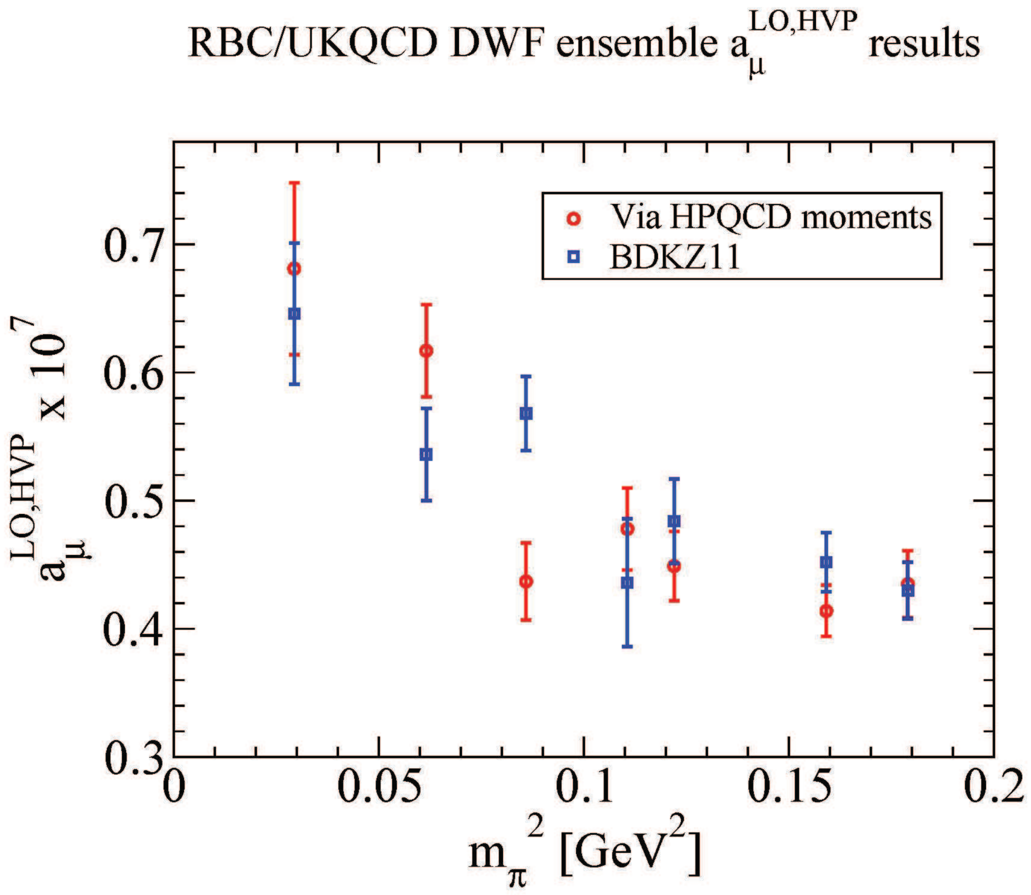


However, even with the term CQ^4, this ChPT functional form performes worse than Pades and/or conformal polynomial.

$\Rightarrow$ Mostly useful as a cross check.

Very Preliminary Results

Used time moments to construct Pades. u, d, s quarks.
RBC/UKQCD DWF data (Hudspith, Lewis, Maltman, Portelli ’14)



(BDKZ11: Boyle, Del Debbio, Kerrane, Zanotti ’11)

Conclusions

- Fitting on a large Q^2 window very dangerous.
- Pointed out that a hybrid strategy with
  - trapezoid rule approximation for $0.1\ \mathrm{GeV}^2 \lesssim Q^2 \leq \infty$
  - Pades, conformal polynomials and N^2LO- ChPT + C Q^4 for $0 \leq Q^2 \lesssim 0.1\ \mathrm{GeV}^2$

  is capable of reaching the desired below 1% error in $a_{\mu^{\mathrm{HVP}}}$ provided time moments can be determined accurately enough (rule of thumb: $\delta\Pi_j/\Pi_j \lesssim j\ \%$)

  $\exists$ First attempt at time moments by Chakraborty et al. '14 for s, c quarks.

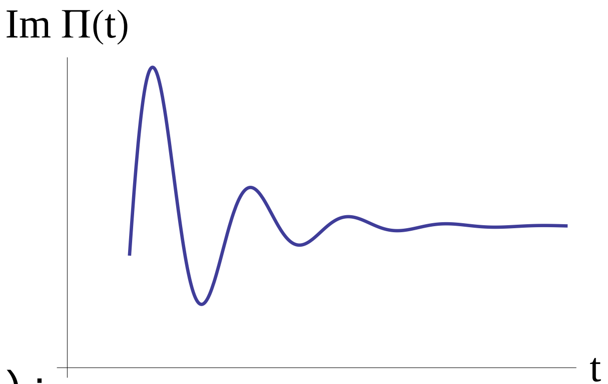
  What about u, d?

  (encouraging results from Hudspith et al. RBC/UKQCD DWF data, still very preliminary...)
- Alternatively (and a good check!): get good data in the low $Q^2 \sim 0.1\mathrm{GeV}^2$ interval (twisting, analytic continuation, etc...) and fit (Pades, Conf. polynomial,...)

BACK-UP SLIDES

Duality Violations

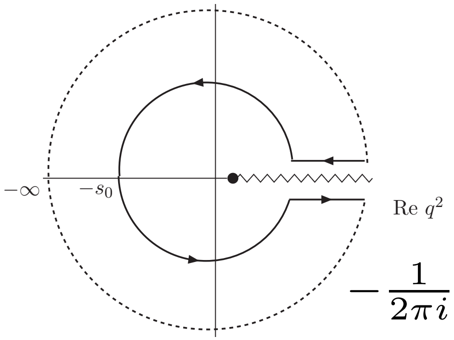
- OPE valid in euclidean, but not in minkowski. We know that spectrum $\neq$ OPE



- We expect (@ large t) :

$$\mathrm{Im}\Pi_{DV} \sim \mathrm{Im}(\Pi - \Pi_{OPE}) \sim \kappa \underbrace{e^{-\gamma t}}_{\text{OPE asympt.}} \underbrace{\sin(\alpha + \beta t)}_{\text{Regge}}$$

- $\Pi_{DV}(s) \to 0$ as $|s| \to \infty$. Then:



(Cata-Golterman-S.P. '05)

$$-\frac{1}{2\pi i}\oint_{|z|=s_0} dz\, w(z)\, \Pi_{DV}(z) = -\underbrace{\int_{s_0}^{\infty} ds}_{\text{extrapolation!}} \; w(s)\, \frac{1}{\pi}\mathrm{Im}\Pi_{DV}(s)$$

Fitting functions

Aubin, Blum, Golterman, SP '12

★ Padés, model independent, they enjoy a convergence theorem for $N \to \infty$:

$$\Pi(Q^2) = \Pi(0) + Q^2 \underbrace{\left(a_0 + \sum_{r=1}^{N} \frac{a_r}{Q^2 + b_r} \right)}_{\text{Pade}}$$

$\Pi(0), a's$ and $b's$ are fitting parameters.

★ VMD is not a Pade, since you fix $b_1 = M_\rho^2$. (true $\Pi(Q^2)$ has cut starting at $4m_\pi^2$...)

We have: $a_0 \neq 0 \Longrightarrow [N, N]$ Pade; $a_0 = 0 \Longrightarrow [N-1, N]$ Pade.

For instance:

- $\frac{a_1}{Q^2+b_1}$ is a [0,1] Pade $\Longrightarrow \Pi(Q^2) = \Pi(0) + Q^2 \left(\frac{a_1}{Q^2+b_1} \right)$
- $a_0 + \frac{a_1}{Q^2+b_1}$ is a [1,1] Pade $\Longrightarrow \Pi(Q^2) = \Pi(0) + Q^2 \left(a_0 + \frac{a_1}{Q^2+b_1} \right)$

etc...

Example: Stieltjes. No errors.

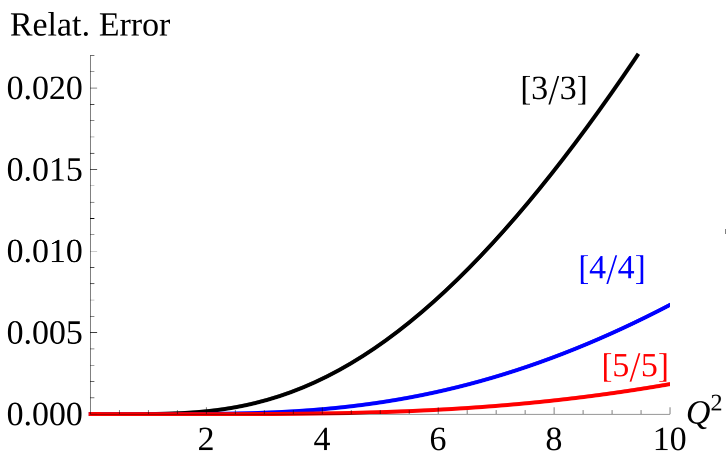
Toy Vac. Pol. function ($\mathrm{Im}\Pi \geq 0$), with a cut, $-\infty \leq Q^2 \leq -1$:

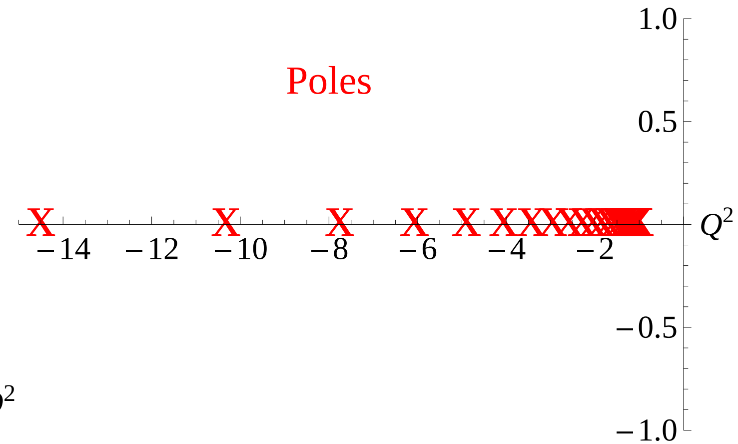
$$\frac{\Pi(0) - \Pi(Q^2)}{Q^2} = \int_1^\infty \frac{dt}{(t+Q^2)} \frac{1}{t} = \frac{1}{Q^2} \log\left(1+Q^2\right)$$

Theorem: As $N \to \infty$, with a_i, b_i determined from the function (and derivatives) at $Q^2 = 0$, or at multiple points,

$$\frac{a_0 + a_1 Q^2 + a_2 Q^4 + \ldots + a_N Q^{2N}}{1 + b_1 Q^2 + \ldots + b_N Q^{2N}} \longrightarrow \frac{\Pi(0) - \Pi(Q^2)}{Q^2}$$

everywhere in a compact region in complex Q^2, except on the cut.





Poles $\neq$ Physics