Calculating repetitively.III

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We begin with a problem set in stationary 2 + 1 dimensional curved space with the metric

$$g^{00}=1-\frac{\lambda^2}{r^2},\; g^{01}=-\frac{\lambda y}{r^2},\; g^{02}=\frac{\lambda x}{r^2},$$

$$g^{11}=-1,\; g^{22}=-1,\; g^{12}=0,\; \lambda=4GJ$$

S.Deser, R.Jackiw and G. ’tHooft, “Three-dimensional Einstein gravity: Dynamics of Flat space” Ann.Phys.**120**,220(1984)

- G.Clement, “Stationary solutions in three – dimensional general relativity”, Int.J.Theor.Phys. **24**, 267(1985)
- J the spin of the massless particle located at the origin

- One gets

$$g_{00} = 1, g_{01} = -\frac{\lambda y}{r^2}, g_{02} = \frac{\lambda x}{r^2}, g_{12} = -\frac{\lambda^2 xy}{r^4},$$

$$g_{11} = -1 + \left(\frac{\lambda y}{r^2}\right)^2, g_{22} = -1 + \left(\frac{\lambda x}{r^2}\right)^2$$

- Some references:
- F.Antonsen and K.Bormann, Propagators in curved space, arXiv:hep-th/9608141v1.
- K.Bormann and F.Antonsen,The Casimir effect of curved space-time, in Proc. 3rd Alexander Friedmann International Seminar,arXiv:hep-th/9608142v1; these set the framework for the discussion

- The workout relies on the these papers:

- D.G.C.McKeon and T.N.Sherry, “Operator regularization and one loop Green’s functions”, Phys.Rev.D**35**,3854(1987)
- J.Schwinger, “On gauge invariance and vacuum polarization”, Phys.Rev.**82**,664(1951)

- S.G.Kamath, Reworking the Antonsen – Bormann idea.I,presented at FFP11 ,AIP Conf.Proc.**1446**,201(2010)
- S.G.Kamath, Reworking the Antonsen-Bormann idea,2012,J.Phys.Conf.Ser.**343**,012051
- S.G.Kamath, “Operator Regularization ,scale and conformal anomalies for the Landau problem”,Mod.Phys.Lett.A**14**,1391(1999)
- S.G.Kamath, “Zeta – function regularization and scale and conformal anomalies for the Landau problem”, Mod.Phys.Lett.A **12**, 2631(1997);they give the impetus for this report .

- Consider the following Lagrangian density for a real massive scalar field :

$$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2$$

- The operator $B = -\partial_\mu \left(g^{\mu\nu} \partial_\nu \right) - m^2$
- can be reworked following Antonsen-Bormann as

$$\begin{aligned} B &= -\partial_\mu \left(\eta^{ab} e_a^\mu e_b^\nu \partial_\nu \right) - m^2 \\ &= -\eta^{ab} \partial_a \partial_b - m^2 - e_\mu^a \partial_a \left(e_b^\mu \right) \partial^b \end{aligned}$$

- with

$$\eta^{ab} = diag(1,-1,-1), \eta^{ab} = diag(1,-1,-1,-1)$$

- in 2+1 or 3+1 dimensions; the vierbeins defined by

$$g^{\mu\nu} = \eta^{ab} e_a^{\mu} e_b^{\nu}$$

- Can be chosen in 2 + 1 dimensions for example as

$$e_0^0 = \frac{i\lambda}{r}, e_1^0 = 0, e_2^0 = i \qquad e_0^0 = 0, e_1^0 = -\frac{iy}{r}, e_2^0 = \frac{ix}{r}$$

$$e_a^\mu: \quad e_0^1 = \frac{iy}{r}, e_1^1 = \frac{x}{r}, e_2^1 = 0 \qquad e_\mu^a: \; e_0^1 = 0, e_1^1 = \frac{x}{r}, e_2^1 = \frac{y}{r}$$

$$e_0^2 = -\frac{ix}{r}, e_1^2 = \frac{y}{r}, e_2^2 = 0 \qquad e_0^2 = -i, e_1^2 = \frac{i\lambda y}{r^2}, e_2^2 = -\frac{i\lambda x}{r^2}$$

- One has
-

$$e_a^\mu e_\mu^b = \delta_a^b \qquad g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$$

- And a linear expression in λ for

$$e_\mu^a \partial_a \left(e_b^\mu\right) \partial^b = -\frac{iy}{r^2} \partial^0 - \frac{\lambda}{r^3} \left(ix\partial^1 + y\partial^0\right)$$

- However the set

$$e_0^0 = \frac{i\lambda y}{r^2}, e_1^0 = i, e_2^0 = \frac{\lambda x}{r^2} \qquad e_0^0 = 0, e_1^0 = -i, e_2^0 = 0$$

$$e_a^\mu: \; e_0^1 = i, e_1^1 = 0, e_2^1 = 0 \qquad e_\mu^a: \; e_0^1 = -i, e_1^1 = \frac{i\lambda y}{r^2}, e_2^1 = -\frac{i\lambda x}{r^2}$$

$$e_0^2 = 0, e_1^2 = 0, e_2^2 = -1 \qquad e_0^2 = 0, e_1^2 = 0, e_2^2 = -1$$

- is appealing since

$$e_\mu^a \partial_a \left(e_b^\mu\right) \partial^b = -\frac{2\lambda xy}{r^4} \partial^0 - \frac{i\lambda}{r^4}\left(y^2 - x^2\right)\partial^2$$

- is a multiple of λ;the Schwinger expansion is thus

- an expansion in powers of λ in this case.

- In 3 + 1 dimensions with the Schwarzschild metric

$$g^{00} = \frac{1}{A}, g^{01} = 0, g^{02} = 0, g^{03} = 0$$

$$g^{11} = -A\left(1 + L\frac{y^2 + z^2}{r^2}\right), g^{12} = AL\frac{xy}{r^2}, g^{13} = AL\frac{xz}{r^2}$$

$$g^{22} = -A\left(1 + L\frac{x^2 + z^2}{r^2}\right), g^{23} = AL\frac{yz}{r^2}$$

$$g^{33} = -A\left(1 + L\frac{x^2 + y^2}{r^2}\right), A = 1 - \frac{2MG}{r}, L = \frac{2MG}{rA}$$

- and

$$r = \sqrt{x^2 + y^2 + z^2}$$

- One has

$$g_{00} = A, g_{10} = 0, g_{20} = 0, g_{30} = 0$$

$$g_{11} = -\left(1 + L\frac{x^2}{r^2}\right), g_{12} = -L\frac{xy}{r^2}, g_{13} = -L\frac{xz}{r^2}$$

$$g_{22} = -\left(1 + L\frac{y^2}{r^2}\right), g_{23} = -L\frac{yz}{r^2},$$

$$g_{33} = -\left(1 + L\frac{z^2}{r^2}\right)$$

- The vierbein set for example

$$e_0^0 = \frac{1}{\sqrt{A}}, e_i^0 = 0, e_0^i = 0, i = 1,2,3$$

$$e_1^1 = \sqrt{A} = e_2^3 = e_3^2,$$

e_a^μ :

$$e_2^1 = \sqrt{AL}\,\frac{y}{r} = -e_1^3, e_2^2 = -\sqrt{AL}\,\frac{x}{r} = -e_3^3$$

$$e_3^1 = -\sqrt{AL}\,\frac{z}{r} = -e_1^2$$

- satisfy $$g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu$$

- And their counterparts

$$e^a_\mu : \quad \begin{aligned} & e^0_0 = \sqrt{A}, e^i_0 = 0 = e^0_i, i = 1,2,3 \\ & e^1_1 = \sqrt{A}\left(1 + L\frac{x^2}{r^2}\right), e^2_1 = \sqrt{AL}\left(\frac{y}{r} + \sqrt{L}\frac{xz}{r^2}\right), e^3_1 = -\sqrt{AL}\left(\frac{z}{r} - \sqrt{L}\frac{xy}{r^2}\right) \\ & e^1_2 = \sqrt{AL}\left(\frac{z}{r} + \sqrt{L}\frac{xy}{r^2}\right), e^2_2 = -\sqrt{AL}\left(\frac{x}{r} - \sqrt{L}\frac{yz}{r^2}\right), e^3_2 = \sqrt{A}\left(1 + L\frac{y^2}{r^2}\right) \\ & e^1_3 = -\sqrt{AL}\left(\frac{y}{r} - \sqrt{L}\frac{xz}{r^2}\right), e^2_3 = \sqrt{A}\left(1 + L\frac{z^2}{r^2}\right), e^3_3 = \sqrt{AL}\left(\frac{x}{r} + \sqrt{L}\frac{yz}{r^2}\right) \end{aligned}$$

- Obtained from $e^\mu_a e^b_\mu = \delta^b_a$ satisfy
- $$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

- But

$$e_\mu^a \partial_a \left(e_b^\mu \right) \partial^b$$

- Involves $\partial^i \;, i = 1,2,3$ only.

- An alternative with

•

$$e_a^\mu: \quad \begin{aligned} & e_0^0 = 0, e_1^0 = \frac{i}{\sqrt{A}}\frac{z}{r}, e_2^0 = \frac{i}{\sqrt{A}}\frac{y}{r}, e_3^0 = -\frac{i}{\sqrt{A}}\frac{x}{r} \\ & e_0^1 = i\sqrt{A}\frac{x}{r}, e_1^1 = -\frac{y}{r}, e_2^1 = \frac{z}{r}, e_3^1 = 0 \\ & e_0^2 = i\sqrt{A}\frac{y}{r}, e_1^2 = -\frac{x}{r}, e_2^2 = 0, e_3^2 = \frac{z}{r} \\ & e_0^3 = i\sqrt{A}\frac{z}{r}, e_1^3 = 0, e_2^3 = -\frac{x}{r}, e_3^3 = -\frac{y}{r} \end{aligned}$$

• leads to

$$e_0^0 = 0, e_1^0 = -\frac{ix}{r\sqrt{A}}, e_2^0 = -\frac{iy}{r\sqrt{A}}, e_3^0 = -\frac{iz}{r\sqrt{A}}$$

$$e_0^1 = -i\sqrt{A}\frac{z}{r}, e_1^1 = -\frac{y}{r}, e_2^1 = \frac{x}{r}, e_3^1 = 0$$

$$e_\mu^a: \quad e_0^2 = -i\sqrt{A}\frac{y}{r}, e_1^2 = \frac{z}{r}, e_2^2 = 0, e_3^2 = -\frac{x}{r}$$

$$e_0^3 = i\sqrt{A}\frac{x}{r}, e_1^3 = 0, e_2^3 = \frac{z}{r}, e_3^3 = -\frac{y}{r}$$

- With $e_a^{\mu} e_{\mu}^{b} = \delta_a^{b}$
- and $g_{\alpha\beta} = \eta_{ab} e_{\alpha}^{a} e_{\beta}^{b}$
- But $e_{\mu}^{a} \partial_{a} \left(e_{b}^{\mu} \right) \partial^{b}$
- involves all the $\partial^{i}, i = 0,1,2,3$
- unlike the previous case.

.In detail,

$$e_{\mu}^{a}\partial_{a}\left(e_{b}^{\mu}\right)\partial^{b}=\sum_{j=1}^{3}C_{j}\partial^{j}+\sum_{j=0}^{3}D_{j}\partial^{j}$$

$$C_{1}=-\frac{x}{r^{2}}-\frac{z\left(x^{2}+z^{2}\right)}{r^{4}},C_{2}=\frac{y}{r^{2}}-\frac{y\left(x^{2}+z^{2}\right)}{r^{4}},$$

$$C_{3}=-\frac{z}{r^{2}}+\frac{x\left(x^{2}+z^{2}\right)}{r^{4}}$$

These are independent of G and

- and

$$D_0 = -\frac{2iy}{r^2}\sqrt{A}, \qquad D_1 = \left(-1+\frac{2MG}{2r}\right)\frac{y^2 z}{r^4 A},$$

$$D_2 = \left(-1+\frac{2MG}{2r}\right)\frac{y^3}{r^4 A}, \qquad D_3 = -\left(-1+\frac{2MG}{2r}\right)\frac{y^2 x}{r^4 A}$$

- with $A = 1 - \frac{2MG}{r}$

- The operator

$$B = -\eta^{ab}\partial_a\partial_b - m^2 - e^a_\mu\partial_a\left(e^\mu_b\right)\partial^b$$
$$= \eta^{ab}p_a p_b - m^2 - e^a_\mu\partial_a\left(e^\mu_b\right)\partial^b$$

- becomes

$$B = p_0^2 - m^2 + \vec{p}^{\,2} - e^a_\mu\partial_a\left(e^\mu_b\right)\partial^b$$

- in Euclidean space

With the labels

$$H_0 = p_0^2 - m^2, \quad H_I = \vec{p}^{\,2} + H_2,$$

$$H_2 = -e_\mu^a \partial_a \left(e_b^\mu\right) \partial^b, \quad B = H_0 + H_I,$$

- We now define after McKeon and Sherry PRD**35**, 3854 (1987)
-

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty du\, u^{s-1} e^{-Bu}$$

- Since the vierbeins are time-independent one has

$$e^{-Bu} = e^{-(H_0+H_I)u} = e^{-H_0 u} e^{-H_I u}$$

- and the Schwinger expansion is now given to the second order by

$$e^{-H_I u} = e^{-\left(\vec{p}^2+H_2\right)u} = e^{-\vec{p}^2 u} + (-u)\int_0^1 dw\, e^{-u(1-w)\vec{p}^2} \langle p|H_2|p\rangle e^{-uw\vec{p}^2}$$

$$+(-u)^2 \int_0^1 w\,dw \int_0^1 dw_1 e^{-u(1-w)\vec{p}^2} \left\{ \int_r \langle p|H_2|r\rangle e^{-uw(1-w_1)\vec{r}^2} \langle r|H_2|p\rangle e^{-uww_1\vec{p}^2} \right\}$$

$$+\ldots$$

- with $\vec{r}$ a momentum vector. The evaluation of these

- matrix elements and the Schwinger expansion to the second order will now be taken up for the 2 + 1 dimensional case for which

$$H_2 = -e_\mu^a \partial_a \left(e_b^\mu\right) \partial^b = \frac{2\lambda xy}{r^4} \partial^0 + \frac{i\lambda}{r^4}\left(y^2 - x^2\right) \partial^2$$

$$= \frac{2\lambda xy}{r^4} i p_0 + \frac{\lambda}{r^4}\left(y^2 - x^2\right) p_2$$

- in Euclidean space. One gets

$$\langle p | H_2 | r \rangle = \frac{\lambda}{4\pi\left(\vec{p} - \vec{r}\right)^2} \left\{ -2ip_0 \left(p_1 - r_1\right)\left(p_2 - r_2\right) + r_2 \left[\left(p_1 - r_1\right)^2 - \left(p_2 - r_2\right)^2\right] \right\}$$

$$\langle r | H_2 | p \rangle = \frac{\lambda}{4\pi\left(\vec{p} - \vec{r}\right)^2} \left\{ -2ip_0 \left(p_1 - r_1\right)\left(p_2 - r_2\right) + p_2 \left[\left(p_1 - r_1\right)^2 - \left(p_2 - r_2\right)^2\right] \right\}$$

The integral $\int_r \langle p|H_2|r\rangle e^{-uw(1-w_1)\vec{r}^2} \langle r|H_2|p\rangle e^{-uww_1\vec{p}^2}$

with $y = uw(1-w_1), a = z+y, b = y^2p^2$ involves 4 terms

$$\int_0^\infty dz \int_r (-2ip_0)^2 (p_1 - r_1)^2 (p_2 - r_2)^2 z e^{-z(\vec{p}-\vec{r})^2 - y\vec{r}^2}$$

$$= \pi e^{-y\vec{p}^2} (-2ip_0)^2 \int_0^\infty dz \frac{e^{\frac{y^2p^2}{a}}}{a^3} \left(\frac{1}{4} + \frac{y^2p^2}{2a} + \frac{y^4 p_1^2 p_2^2}{a^2} \right)$$

A.

$$= \pi e^{-y\vec{p}^2} (-2ip_0)^2 \left[\frac{3y^4}{b^4} \left(6p_1^2 p_2^2 - p_1^2 - p_2^2 \right) + \frac{1}{b} e^{y\vec{p}^2} \begin{Bmatrix} \frac{1}{y}\left(\frac{1}{2} + yp_1^2\right)\left(\frac{1}{2} + yp_2^2\right) \\ -\frac{1}{b}\left(\frac{1}{4} + yp^2 + 3y^2 p_1^2 p_2^2\right) \\ +\frac{y^2}{b^3}\left(p^2 + \frac{6yzp_1^2 p_2^2}{b}\right) \end{Bmatrix} \right]$$

- Similarly,

$$-2ip_0p_2\int_0^\infty dz\int_r (p_1-r_1)(p_2-r_2)\left[(p_1-r_1)^2-(p_2-r_2)^2\right]ze^{-z(\vec{p}-\vec{r})^2-y\vec{r}^2}$$

B.

$$=-2ip_0p_2e^{-y\vec{p}^2}\int_0^\infty dz\,\frac{\pi y^4p_1p_2e^{\frac{y^2p^2}{a}}}{a^5}\left(p_1^2-p_2^2\right)$$

$$-2ip_0\int_0^\infty dz\int_r r_2(p_1-r_1)(p_2-r_2)\left[(p_1-r_1)^2-(p_2-r_2)^2\right]ze^{-z(\vec{p}-\vec{r})^2-y\vec{r}^2}$$

C.

$$=-2ip_0e^{-y\vec{p}^2}\int_0^\infty dz\,\frac{\pi e^{\frac{y^2p^2}{a}}}{a^5}\left[\frac{z}{a}y^4p_1p_2{}^2\left(p_1^2-p_2^2\right)+\frac{y^3p_1}{2}\left(3p_2^2-p_1^2\right)\right]$$

whose sum is

$$-2ip_0e^{-y\vec{p}^2}\int_0^\infty dz\,\frac{\pi e^{\frac{y^2p^2}{a}}}{a^5}\left[\left(1+\frac{z}{a}\right)y^4p_1p_2{}^2\left(p_1^2-p_2^2\right)+\frac{y^3p_1}{2}\left(3p_2^2-p_1^2\right)\right]$$

- Lastly,

$$\int_0^\infty dz \int_r r_2 p_2 \left[(p_1 - r_1)^2 - (p_2 - r_2)^2 \right]^2 z e^{-z(\vec{p}-\vec{r})^2 - y\vec{r}^2}$$

- is the sum of
- A.

$$p_2^2 e^{-y\vec{p}^2} \int_0^\infty dz \frac{\pi z^2 e^{\frac{y^2 p^2}{a}}}{a^6} \left(y^4 p_1^4 + 3ay^2 p_1^2 + \frac{3}{4} a^2 \right)$$

- B. $$p_2^2 e^{-y\vec{p}^2} \int_0^\infty dz \frac{\pi z e^{\frac{y^2 p^2}{a}}}{a^6} \left[z\left(y^4 p_2^4 + a p_2^2 \left(5z^2 - 12az + 9a^2 \right) \right) + a^2 \left(\frac{15}{4} z - 3a - 2a^2 p_2^2 \right) \right]$$

- and
- C.

$$-2 p_2^2 e^{-y\vec{p}^2} \int_0^\infty dz \frac{\pi z e^{\frac{y^2 p^2}{a}}}{a^6} \left[z y^4 p_1^2 p_2^2 + az \frac{y^2}{2} \left(p_2^2 + 3p_1^2 \right) + \frac{3z}{4} a^2 - a^2 \left(\frac{a}{2} + y^2 p_1^2 \right) \right]$$

- And it works to

$$p_2^2 e^{-y\vec{p}^2} \int_0^\infty dz \, \frac{\pi z e^{\frac{y^2 p^2}{a}}}{a^6} \left[\begin{array}{l} zy^4\left(p_1^2 - p_2^2\right)^2 + 3za^2 + 2a^2\left(y^2 p_1^2 - a^2 p_2^2\right) \\ + azp_2^2\left(5z^2 - 12az + 9a^2 - y^2\right) - 2a^3 \end{array} \right]$$

- Each of the integrals over ‘z’ can be done and the
- $O\left(G^2\right)$ term can then be calculated.

- Consider the set

$$e_0^0 = 1, e_1^0 = \frac{\lambda x}{r^2}, e_2^0 = \frac{\lambda y}{r^2}$$

$$e_a^\mu : \quad e_0^1 = 0, e_1^1 = 0, \quad e_2^1 = 1$$

$$e_0^2 = 0, e_1^2 = -1, \; e_2^2 = 0$$

$$e_0^0 = 1, e_1^0 = -\frac{\lambda y}{r^2}, e_2^0 = \frac{\lambda x}{r^2}$$

$$e_\mu^a : \quad e_0^1 = 0, \quad e_1^1 = 0, \quad e_2^1 = -1$$

$$e_0^2 = 0, \quad e_1^2 = 1, e_2^2 = 0$$

- in 2+ 1 dimensions; they satisfy

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \; e_a^\mu e_\mu^b = \delta_a^b, \; g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu$$

- but lead to $e_\mu^a \partial_a \left(e_b^\mu \right) \partial^b = 0$;equally,

- the set

$$e_a^\mu : \quad e_0^0 = 1, e_1^0 = -\frac{\lambda}{r}, e_2^0 = 0 \qquad e_0^1 = 0, e_1^1 = -\frac{y}{r}, e_2^1 = \frac{x}{r} \qquad e_0^2 = 0, e_1^2 = \frac{x}{r},\ e_2^2 = \frac{y}{r}$$

$$e_\mu^a : \quad e_0^0 = 1, e_1^0 = -\frac{\lambda y}{r^2}, e_2^0 = \frac{\lambda x}{r^2} \qquad e_0^1 = 0, e_1^1 = -\frac{y}{r},\quad e_2^1 = \frac{x}{r} \qquad e_0^2 = 0, e_1^2 = \frac{x}{r},\ e_2^2 = \frac{y}{r}$$

- leads to $$e_\mu^a \partial_a \left(e_b^\mu\right)\partial^b = -\frac{x}{r^2}\partial^1$$

- implying a Schwinger expansion that is independent of G in 2 + 1 dimensions. For the extension to 4+1 dimensions one has in mind
- G.Clement, Stationary solutions in five dimensional general relativity,Gen.Rel.Grav.**18**,137(1986)