Calculating repetitively. III

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We begin with a problem set in stationary 2 + 1 dimensional curved space with the metric

\[ g^{00} = 1 - \frac{\lambda^2}{r^2}, \quad g^{01} = -\frac{\lambda y}{r^2}, \quad g^{02} = \frac{\lambda x}{r^2}, \]

\[ g^{11} = -1, \quad g^{22} = -1, \quad g^{12} = 0, \quad \lambda = 4GJ \]


- J the spin of the massless particle located at the origin
• One gets

\[
g_{00} = 1, \quad g_{01} = -\frac{\lambda y}{r^2}, \quad g_{02} = \frac{\lambda x}{r^2}, \quad g_{12} = -\frac{\lambda^2 xy}{r^4},
\]

\[
g_{11} = -1 + \left(\frac{\lambda y}{r^2}\right)^2, \quad g_{22} = -1 + \left(\frac{\lambda x}{r^2}\right)^2
\]

• Some references:


• K.Bormann and F.Antonsen, The Casimir effect of curved space-time, in Proc. 3rd Alexander Friedmann International Seminar, arXiv:hep-th/9608142v1; these set the framework for the discussion
• The workout relies on the these papers:


• S.G. Kamath, Reworking the Antonsen-Bormann idea, 2012, J. Phys. Conf. Ser. 343, 012051


• S.G. Kamath, “Zeta – function regularization and scale and conformal anomalies for the Landau problem”, Mod. Phys. Lett. A 12, 2631 (1997); they give the impetus for this report.
• Consider the following Lagrangian density for a real massive scalar field:

\[ L = \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \]

• The operator \( B = -\partial_\mu \left( g^{\mu \nu} \partial_\nu \right) - m^2 \)

• can be reworked following Antonsen-Bormann as

\[
B = -\partial_\mu \left( \eta^{ab} e^\mu_a e^\nu_b \partial_\nu \right) - m^2 \\
= -\eta^{ab} \partial_a \partial_b - m^2 - e^\mu_a \partial_a \left( e^\mu_b \right) \partial^b
\]
• with

\[ \eta^{ab} = \text{diag}(1,-1,-1), \eta^{ab} = \text{diag}(1,-1,-1,-1,-1) \]

• in 2+1 or 3+1 dimensions; the vierbeins defined by

\[ g^{\mu \nu} = \eta^{ab} e^\mu_a e^\nu_b \]
Can be chosen in $2 + 1$ dimensions for example as

$$e^0 = \frac{i\lambda}{r}, e^0_1 = 0, e^0_2 = i \quad e^0_0 = 0, e^0_1 = -\frac{iy}{r}, e^0_2 = \frac{ix}{r}$$

$$e^{\mu}_a : \quad e^{1}_0 = \frac{iy}{r}, e^{1}_1 = \frac{x}{r}, e^{1}_2 = 0 \quad e^{a}_0 = 0, e^{1}_0 = \frac{x}{r}, e^{1}_2 = \frac{y}{r}$$

$$e^{2}_0 = -\frac{ix}{r}, e^{2}_1 = \frac{y}{r}, e^{2}_2 = 0 \quad e^{2}_0 = -i, e^{1}_2 = \frac{i\lambda y}{r^2}, e^{2}_2 = -\frac{i\lambda x}{r^2}$$

One has

$$e^\mu_a e^b_\mu = \delta^b_a \quad g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu$$

And a linear expression in $\lambda$ for

$$e^a_\mu \partial_a (e^b_\mu) \partial^b = -\frac{iy}{r^2} \partial^0 - \frac{\lambda}{r^3} (ix \partial^1 + y \partial^0)$$
However the set

\[ e_0^0 = \frac{i\lambda y}{r^2}, e_1^0 = i, e_2^0 = \frac{\lambda x}{r^2} \quad e_0^0 = 0, e_1^0 = -i, e_2^0 = 0 \]

\[ e_\mu^a: e_0^1 = i, e_1^1 = 0, e_2^1 = 0 \quad e_\mu^a: e_0^1 = -i, e_1^1 = \frac{i\lambda y}{r^2}, e_2^1 = -\frac{i\lambda x}{r^2} \]

\[ e_0^2 = 0, e_1^2 = 0, e_2^2 = -1 \quad e_0^2 = 0, e_1^2 = 0, e_2^2 = -1 \]

is appealing since

\[ e_\mu^a \partial_a (e_\nu^b) \partial^b = -\frac{2\lambda xy}{r^4} \partial^0 - \frac{i\lambda}{r^4} (y^2 - x^2) \partial^2 \]

is a multiple of \( \lambda \); the Schwinger expansion is thus
• an expansion in powers of $\lambda$ in this case.
In 3 + 1 dimensions with the Schwarzschild metric

\[
\begin{align*}
g^{00} &= \frac{1}{A}, & g^{01} &= 0, & g^{02} &= 0, & g^{03} &= 0 \\
g^{11} &= -A \left(1 + L \frac{y^2 + z^2}{r^2}\right), & g^{12} &= AL \frac{xy}{r^2}, & g^{13} &= AL \frac{xz}{r^2} \\
g^{22} &= -A \left(1 + L \frac{x^2 + z^2}{r^2}\right), & g^{23} &= AL \frac{yz}{r^2} \\
g^{33} &= -A \left(1 + L \frac{x^2 + y^2}{r^2}\right), & A &= 1 - \frac{2MG}{r}, & L &= \frac{2MG}{rA} \\
\text{and} & & r &= \sqrt{x^2 + y^2 + z^2}
\end{align*}
\]
• One has

\[ g_{00} = A, \ g_{10} = 0, \ g_{20} = 0, \ g_{30} = 0 \]

\[ g_{11} = -\left(1 + L \frac{x^2}{r^2}\right), \ g_{12} = -L \frac{xy}{r^2}, \ g_{13} = -L \frac{xz}{r^2} \]

\[ g_{22} = -\left(1 + L \frac{y^2}{r^2}\right), \ g_{23} = -L \frac{yz}{r^2}, \]

\[ g_{33} = -\left(1 + L \frac{z^2}{r^2}\right) \]
The vierbein set for example

\[ e_0^0 = \frac{1}{\sqrt{A}}, e_i^0 = 0, e_0^i = 0, i = 1, 2, 3 \]

\[ e_1^1 = \sqrt{A} = e_2^3 = e_3^2, \]

\( e_a^\mu : \)

\[ e_2^1 = \sqrt{AL} \frac{y}{r} = -e_3^3, e_2^2 = -\sqrt{AL} \frac{x}{r} = -e_3^3 \]

\[ e_3^1 = -\sqrt{AL} \frac{z}{r} = -e_1^2 \]

• satisfy

\[ g^{\mu \nu} = \eta^{ab} e_a^\mu e_b^\nu \]
• And their counterparts

\[ e_0^0 = \sqrt{A}, e_0^i = 0 = e_i^0, i = 1,2,3 \]

\[ e_1^1 = \sqrt{A} \left( 1 + L \frac{x^2}{r^2} \right), e_1^2 = \sqrt{AL} \left( \frac{y}{r} + \sqrt{L} \frac{xz}{r^2} \right), e_1^3 = -\sqrt{AL} \left( \frac{z}{r} - \sqrt{L} \frac{xy}{r^2} \right) \]

\[ e_2^a : \]

\[ e_2^1 = \sqrt{AL} \left( \frac{z}{r} + \sqrt{L} \frac{xy}{r^2} \right), e_2^2 = -\sqrt{AL} \left( \frac{x}{r} - \sqrt{L} \frac{yz}{r^2} \right), e_2^3 = \sqrt{A} \left( 1 + L \frac{y^2}{r^2} \right) \]

\[ e_3^1 = -\sqrt{AL} \left( \frac{y}{r} - \sqrt{L} \frac{xz}{r^2} \right), e_3^2 = \sqrt{A} \left( 1 + L \frac{z^2}{r^2} \right), e_3^3 = \sqrt{AL} \left( \frac{x}{r} + \sqrt{L} \frac{yz}{r^2} \right) \]

• Obtained from \( e_a^\mu e_\mu^b = \delta_a^b \) satisfy

\[ g_\mu^\nu = \eta_{ab} e_a^\mu e_b^\nu \]
• But
\[ e^a_\mu \partial_a (e^\mu_b) \partial^b \]

• Involves \( \partial^i, i = 1,2,3 \) only.

• An alternative with
\[ e_0^0 = 0, e_1^0 = \frac{i}{\sqrt{A}} \frac{z}{r}, e_2^0 = \frac{i}{\sqrt{A}} \frac{y}{r}, e_3^0 = -\frac{i}{\sqrt{A}} \frac{x}{r} \]

\[ e_1^1 = i\sqrt{A} \frac{x}{r}, e_1^1 = -\frac{y}{r}, e_1^1 = \frac{z}{r}, e_1^1 = 0 \]

\[ e_\mu^a : \quad e_0^2 = i\sqrt{A} \frac{y}{r}, e_1^2 = -\frac{x}{r}, e_2^2 = 0, e_3^2 = \frac{z}{r} \]

\[ e_0^3 = i\sqrt{A} \frac{z}{r}, e_1^3 = 0, e_2^3 = -\frac{x}{r}, e_3^3 = -\frac{y}{r} \]

- leads to
\[ e_0^0 = 0, e_1^0 = -\frac{ix}{r \sqrt{A}}, e_2^0 = -\frac{i y}{r \sqrt{A}}, e_3^0 = -\frac{i z}{r \sqrt{A}} \]

\[ e^1_0 = -i \sqrt{A} \frac{z}{r}, e^1_1 = -\frac{y}{r}, e^1_2 = \frac{x}{r}, e^1_3 = 0 \]

\[ e^a_\mu : \quad e^2_0 = -i \sqrt{A} \frac{y}{r}, e^2_1 = \frac{z}{r}, e^2_2 = 0, e^2_3 = -\frac{x}{r} \]

\[ e^3_0 = i \sqrt{A} \frac{x}{r}, e^3_1 = 0, e^3_2 = \frac{z}{r}, e^3_3 = -\frac{y}{r} \]
• With \( e^a_\mu e^b_\mu = \delta^b_a \)

• and \( g_{\alpha\beta} = \eta_{ab} e^a_\alpha e^b_\beta \)

• But \( e^a_\mu \partial_a (e^\mu_b) \partial^b \)

• involves all the \( \partial^i, i = 0,1,2,3 \)

• unlike the previous case.
In detail,

\[ e_\mu^a \partial_a (e_\mu^b) \partial^b = \sum_{j=1}^{3} C_j \partial^j + \sum_{j=0}^{3} D_j \partial^j \]

\[ C_1 = -\frac{x}{r^2} - \frac{z(x^2 + z^2)}{r^4}, \quad C_2 = \frac{y}{r^2} - \frac{y(x^2 + z^2)}{r^4}, \quad C_3 = -\frac{z}{r^2} + \frac{x(x^2 + z^2)}{r^4} \]

These are independent of G and
• and

\[ D_0 = -\frac{2iy}{r^2} \sqrt{A}, \quad D_1 = \left(-1 + \frac{2MG}{2r}\right) \frac{y^2z}{r^4A}, \]

\[ D_2 = \left(-1 + \frac{2MG}{2r}\right) \frac{y^3}{r^4A}, \quad D_3 = -\left(-1 + \frac{2MG}{2r}\right) \frac{y^2x}{r^4A} \]

• with \[ A = 1 - \frac{2MG}{r} \]
• The operator

\[ B = -\eta^{ab} \partial_a \partial_b - m^2 - e^a_\mu \partial_a (e^\mu_b) \partial^b = \eta^{ab} p_a p_b - m^2 - e^a_\mu \partial_a (e^\mu_b) \partial^b \]

• becomes

\[ B = p_0^2 - m^2 + \vec{p}^2 - e^a_\mu \partial_a (e^\mu_b) \partial^b \]

• in Euclidean space
With the labels

\[ H_0 = p_0^2 - m^2, \quad H_I = \vec{p}^2 + H_2, \]

\[ H_2 = -e^a_\mu \partial_a (e^b_\mu) \partial^b, \quad B = H_0 + H_I, \]

- We now define after McKeon and Sherry PRD35, 3854 (1987)

\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty du u^{s-1} e^{-Bu} \]
• Since the vierbeins are time-independent one has
\[
e^{-Bu} = e^{-(H_0 + H_I)u} = e^{-H_0u} e^{-H_Iu}
\]
• and the Schwinger expansion is now given to the second order by
\[
e^{-H_Iu} = e^{-(\tilde{p}^2 + H_2)u} = e^{-\tilde{p}^2u} + (-u) \int_0^1 dw e^{-u(1-w)\tilde{p}^2} \langle p \mid H_2 \mid p \rangle e^{-uw\tilde{p}^2}
\]
\[
+ (-u)^2 \int_0^1 w dw \int_0^1 dw_1 e^{-u(1-w)\tilde{p}^2} \left\{ \int_r \langle p \mid H_2 \mid r \rangle e^{-uw(1-w_1)\tilde{r}^2} \langle r \mid H_2 \mid p \rangle e^{-uw_1\tilde{r}^2} \right\}
\]
\[+ \ldots \]
• with \( \tilde{r} \) a momentum vector. The evaluation of these
matrix elements and the Schwinger expansion to the second order will now be taken up for the 2 + 1 dimensional case for which

\[
H_2 = -e^a_\mu \partial_\mu (e^\mu_b) \partial^b = \frac{2 \lambda xy}{r^4} \partial^0 + \frac{i \lambda}{r^4} (y^2 - x^2) \partial^2
\]

\[
= \frac{2 \lambda xy}{r^4} ip_0 + \frac{\lambda}{r^4} (y^2 - x^2) p_2
\]

in Euclidean space. One gets

\[
\langle p | H_2 | r \rangle = \frac{\lambda}{4 \pi (\vec{p} - \vec{r})^2} \left\{ -2ip_0 (p_1 - r_1)(p_2 - r_2) + r_2 \left[ (p_1 - r_1)^2 - (p_2 - r_2)^2 \right] \right\}
\]

\[
\langle r | H_2 | p \rangle = \frac{\lambda}{4 \pi (\vec{p} - \vec{r})^2} \left\{ -2ip_0 (p_1 - r_1)(p_2 - r_2) + p_2 \left[ (p_1 - r_1)^2 - (p_2 - r_2)^2 \right] \right\}
\]
The integral
\[ \int \langle p \mid H_2 \mid r \rangle e^{-uw(1-w_1)\vec{r}^2} \langle r \mid H_2 \mid p \rangle e^{-uw_1\vec{p}^2} \]

with \( y = uw(1-w_1), a = z + y, b = y^2p^2 \) involves 4 terms

\[ \int_0^\infty dz \int_0^\infty (-2ip_0)^2 (p_1 - r_1)^2 (p_2 - r_2)^2 ze^{-z(\vec{p} - \vec{r})^2 - yr^2} \]

\[ = \pi e^{-yp^2} (-2ip_0)^2 \int_0^\infty dz \frac{y^2p^2}{a^3} \left( \frac{1}{4} \frac{y^2p^2}{2a} + \frac{y^4p_1^2p_2^2}{a^2} \right) \]

A.

\[ = \pi e^{-yp^2} (-2ip_0)^2 \frac{3y^4}{b^4} \left( 6p_1^2p_2^2 - p_1^2 - p_2^2 \right) + \frac{1}{b} e^{yp^2} \left\{ \frac{1}{y} \left( \frac{1}{2} + yp_1^2 \right) \left( \frac{1}{2} + yp_2^2 \right) - \frac{1}{b} \left( \frac{1}{4} + yp^2 + 3y^2p_1^2p_2^2 \right) + \frac{y^2}{b^3} \left( p^2 + \frac{6yzp_1^2p_2^2}{b} \right) \right\} \]
• Similarly,

\[ -2ip_0 p_2 \int_0^\infty dz \int r (p_1 - r_1)(p_2 - r_2) \left[ (p_1 - r_1)^2 - (p_2 - r_2)^2 \right] z e^{-(\bar{p} - \bar{r})^2 - y\bar{r}^2} \]

B.

\[ = -2ip_0 p_2 e^{-y\bar{p}^2} \int_0^\infty dz \frac{\pi y^4 p_1 p_2 e^a}{a^5} (p_1^2 - p_2^2) \]

\[ -2ip_0 \int_0^\infty dz \int r_2 (p_1 - r_1)(p_2 - r_2) \left[ (p_1 - r_1)^2 - (p_2 - r_2)^2 \right] z e^{-(\bar{p} - \bar{r})^2 - y\bar{r}^2} \]

C.

\[ = -2ip_0 e^{-y\bar{p}^2} \int_0^\infty dz \frac{\pi e^a}{a^5} \left[ \frac{z}{a} y^4 p_1 p_2^2 (p_1^2 - p_2^2) + \frac{y^3 p_1}{2} (3p_2^2 - p_1^2) \right] \]

whose sum is

\[ -2ip_0 e^{-y\bar{p}^2} \int_0^\infty dz \frac{\pi e^a}{a^5} \left[ \left( 1 + \frac{z}{a} \right) y^4 p_1 p_2^2 (p_1^2 - p_2^2) + \frac{y^3 p_1}{2} (3p_2^2 - p_1^2) \right] \]
Lastly,
\[
\int_0^\infty dz \int_r \frac{y^2 p^2}{r_2 p_2} \left[(p_1 - r_1)^2 - (p_2 - r_2)^2\right]^2 z e^{-z(p_1 - r_2)^2 - y r^2}
\]
is the sum of

A. \[
p_2^2 e^{-y p^2} \int_0^\infty \frac{\pi z^2 e^{a}}{a^6} \left(y^4 p_1^4 + 3 a y^2 p_1^2 + \frac{3}{4} a^2 \right)
\]

B. \[
p_2^2 e^{-y p^2} \int_0^\infty \frac{\pi z e^{a}}{a^6} \left(z \left(y^4 p_2^4 + a p_2^2 \left(5 z^2 - 12 a z + 9 a^2\right)\right)\right] + a^2 \left(\frac{15}{4} z - 3 a - 2 a^2 p_2^2\right)
\]

and

C. \[
-2 p_2^2 e^{-y p^2} \int_0^\infty dz \frac{\pi z e^{a}}{a^6} \left[zy^4 p_1^2 p_2^2 + a z \frac{y^2}{2} \left(p_2^2 + 3 p_1^2\right)\right] + \frac{3 z}{4} a^2 - a^2 \left(\frac{a}{2} + y^2 p_1^2\right)
\]
• And it works to

\[
p_2^2 e^{-y^2 p^2} \int_0^\infty dz \frac{\pi z e^{\frac{y^2 p^2}{a}}}{a^6} \left[ z y^4 \left( p_1^2 - p_2^2 \right)^2 + 3za^2 + 2a^2 \left( y^2 p_1^2 - a^2 p_2^2 \right) \right] \\
+ azp_2^2 \left( 5z^2 - 12az + 9a^2 - y^2 \right) - 2a^3
\]

• Each of the integrals over ‘z’ can be done and the
• \(O(G^2)\) term can then be calculated.
• Consider the set

\[ e_0^0 = 1, \quad e_1^0 = \frac{\lambda x}{r^2}, \quad e_2^0 = \frac{\lambda y}{r^2} \]

\[ e_a^\mu : \quad e_0^1 = 0, \quad e_1^1 = 0, \quad e_2^1 = 1 \]

\[ e_0^2 = 0, e_1^2 = -1, \quad e_2^2 = 0 \]

\[ e_0^0 = 1, e_1^0 = -\frac{\lambda y}{r^2}, e_2^0 = \frac{\lambda x}{r^2} \]

\[ e_a^\mu : \quad e_0^1 = 0, \quad e_1^1 = 0, \quad e_2^1 = -1 \]

\[ e_0^2 = 0, \quad e_1^2 = 1, e_2^2 = 0 \]

• in 2+1 dimensions; they satisfy

\[ g_{\mu \nu} = \eta_{ab} e_{\mu}^a e_{\nu}^b, \quad e_{\mu}^a e_{\mu}^b = \delta_{ab}^b, \quad g^{\mu \nu} = \eta^{ab} e_{\mu}^a e_{\nu}^b \]

• but lead to \[ e_{\mu}^a \partial_a \left( e_{\nu}^\mu \right) \partial_b^b = 0 \]; equally,
• the set

\[ e_0^0 = 1, e_1^0 = -\frac{\lambda}{r}, e_2^0 = 0 \quad \text{and} \quad e_0^1 = 1, e_1^0 = -\frac{\lambda y}{r^2}, e_2^0 = \frac{\lambda x}{r^2} \]

\[ e_a^\mu : \quad e_0^1 = 0, e_1^1 = -\frac{y}{r}, e_2^1 = \frac{x}{r} \quad \text{and} \quad e_a^\mu : \quad e_0^1 = 0, e_1^1 = -\frac{y}{r}, \quad e_2^1 = \frac{x}{r} \]

\[ e_2^0 = 0, e_2^2 = \frac{x}{r}, e_2^2 = \frac{y}{r} \quad \text{and} \quad e_2^0 = 0, e_2^2 = \frac{x}{r}, e_2^2 = \frac{y}{r} \]

• leads to

\[ e_a^\mu \partial_a (e_b^\mu) \partial^b = -\frac{x}{r^2} \partial^1 \]

• implying a Schwinger expansion that is independent of G in 2 + 1 dimensions. For the extension to 4+1 dimensions one has in mind

• G. Clement, Stationary solutions in five dimensional general relativity, Gen. Rel. Grav. 18, 137 (1986)