Lifshitz hyperscaling violating holography

Ioannis Papadimitriou

Institúto de Física Teórica UAM/CSIC, Madrid

ICHEP 2014
Valencia

5 July 2014

Based on: 1407.xxxx [W. Chemissany, I. P.]
1405.3965 [W. Chemissany, I. P.]
1106.4826 [I. P.]
1007.4592 [I. P.]

Symplectic space of asymptotic data & holography

- Does any theory of (quantum) gravity admit a dual holographic description?
- To answer this we need a complete theory of quantum gravity, but there is considerable evidence for an affirmative answer, e.g. Bekenstein-Hawking formula for black hole entropy, Brown-Henneaux asymptotic symmetries in AdS_3, black hole microstate counting, *AdS/CFT correspondence*.
- Given our limited understanding of quantum gravity and of strongly coupled QFTs, a pragmatic approach to gauge gravity dualities has become popular. Namely, one is trying to *model* a given strongly coupled QFT of interest using a gravitational theory, based e.g. on symmetries or the spectrum of observables, and the hope is that the gravity description captures certain universal features of the dual strongly coupled QFT.
- Typical examples are holographic models of QCD, high temperature superconductors or other strongly correlated condensed matter systems exhibiting quantum critical transitions.
- Within this approach to gauge/gravity duality, the holographic dictionary can be derived systematically starting from the gravity side by i) providing a theory that supports the desired backgrounds, ii) identifying the radial coordinate emanating from the boundary of these geometries with the RG scale of the dual QFT iii) constructing the symplectic space of asymptotic data.

Lifshitz & hyperscaling violating Lifshitz holography

- Holographic description of quantum critical points and QFTs exhibiting hyperscaling violation
- Geometries suffer from IR pathologies – not relevant here
- These backgrounds can emerge in the IR or some intermediate energy scale starting with some other UV completion – e.g. AdS in the same or higher dimensions
- Here we will focus on the case where these geometries are considered as the UV. Otherwise we can develop the holographic dictionary in whatever UV completion these geometries emerge from

Related work on Lifshitz holography

- [Ross & Saremi '09] (Einstein-Proca)
- [Ross '11] (Einstein-Proca, vielbein formalism, counteterms derived using dilatation operator method [I.P. & Skenderis '04])
- [Griffin, Hořava, Melby-Thompson '11] (Einstein-Proca)
- [Mann & McNees '11] (Einstein-Proca)
- [Baggio, de Boer & Holsheimer '11] (Einstein-Proca)
- [Chemissany, Geissbühler, Hartong & Rollier '12] ($D = 4$ $z = 2$, Scherk-Schwarz reduction from 5d axion-dilaton model [I.P. '11])
- [Korovin, Skenderis & Taylor '13] ($z = 1 + \epsilon$)
- [Christensen, Hartong, Obers & Rollier '13] ($D = 4$ $z = 2$, Scherk-Schwarz reduction from 5d axion-dilaton model [I.P. '11])
- [Andrade & Ross '13] (Einstein-Proca, linear metric fluctuations)

Outline

- The Lifshitz (Lif) metric is

$$ds^2_{d+2} = \ell^2 u^{-2}\left(du^2 - u^{-2(z-1)}dt^2 + dx^a dx^a\right)$$

with dynamical exponent $z \neq 1$

- This metric is invariant under the scaling transformation

$$x^a \to \lambda x^a\ ,\ t \to \lambda^z t\ ,\ u \to \lambda u$$

- The null energy condition

$$T_{\mu\nu}k^\mu k^\nu \geq 0, \quad k^\mu k_\mu = 0$$

requires

$$z \geq 1$$

- Hyperscaling refers to the property that the free energy and other thermodynamic quantities scale with temperature by their naive dimension. e.g. $S \sim T^{(2-\theta)/z}$, where θ is the hyperscaling violating parameter
- The Hyperscaling violating Lifshitz (hvLf) metric is [Huijse, Sachdev & Swingle '11]

$$ds^2_{d+2} = \ell^2 u^{-2(d-\theta)/d} \left(du^2 - u^{-2(z-1)} dt^2 + dx^a dx^a \right)$$

  with dynamical exponent $z \neq 1$ and hyperscaling violation exponent $\theta \neq 0$
- This metric has the scaling property that under

$$x^a \to \lambda x^a \ , \ t \to \lambda^z t \ , \ u \to \lambda u$$

  the metric transforms as

$$ds^2_{d+2} \to \lambda^{2\theta/d} ds^2_{d+2}$$

- The null energy condition requires

$$(d-\theta)(d(z-1)-\theta) \geq 0, \quad (d-\theta+z)(z-1) \geq 0$$

- The solutions of the null energy condition are:

I	$z \leq 0$	$\theta \geq d$
II	$0 < z \leq 1$	$\theta \geq d + z$
IIIa IIIb	$1 \leq z \leq 2$	$\theta \leq d(z-1)$ $d \leq \theta \leq d + z$
IVa IVb	$2 < z \leq \frac{2d}{d-1}$	$\theta \leq d$ $d(z-1) \leq \theta \leq d + z$
V	$z > \frac{2d}{d-1}$	$\theta \leq d$

- For $\theta \geq d + z$ the on-shell action does not diverge and hence there is no well defined asymptotic expansion/holographic dictionary (cf. D6 branes)
- We therefore exclude cases I and II and consider all cases with $z > 1$

The model

- We consider a generic “bottom up” model of the form

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}\left(R[g] - \alpha\partial_\mu\phi\partial^\mu\phi - Z(\phi)F^2 - W(\phi)A^2 - V(\phi)\right)$$

- Preserve $U(1)$ gauge symmetry via the Stückelberg mechanism:

$$A_\mu \to B_\mu = A_\mu - \partial_\mu\omega$$

  such that under a $U(1)$ transformation

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \quad \omega \to \omega + \Lambda$$

- Go to generic Weyl frame in order to accommodate both Lifshitz and hyperscaling violating backgrounds:

$$g \to e^{2\xi\phi}g$$

$$S_\xi = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}e^{d\xi\phi}\left(R[g] - \alpha_\xi\partial_\mu\phi\partial^\mu\phi - Z_\xi(\phi)F^2 - W_\xi(\phi)B^2 - V_\xi(\phi)\right)$$

$$\alpha_\xi = \alpha - d(d+1)\xi^2, \quad Z_\xi(\phi) = e^{-2\xi\phi}Z(\phi), \quad W_\xi(\phi) = W(\phi), \quad V_\xi(\phi) = e^{2\xi\phi}V(\phi)$$

- This model admit Lif or hvLf solutions at least asymptotically provided the potentials are of the form

$$V_\xi = V_o e^{2(\rho+\xi)\phi}, \qquad Z_\xi = Z_o e^{-2(\xi+\nu)\phi}, \qquad W_\xi = W_o e^{2\sigma\phi}$$

- The parameters are related to the parameters of the Lif solutions

$$ds^2 = dr^2 - e^{2zr}dt^2 + e^{2r}d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o}e^{\epsilon r}dt, \quad \phi = \mu r, \quad \omega = const.$$

as

$$\rho = -\xi, \quad \nu = -\xi + \frac{\epsilon - z}{\mu}, \quad \sigma = \frac{z-\epsilon}{\mu},$$
$$\epsilon = \frac{(\alpha_\xi + d^2\xi^2)\mu^2 - d\mu\xi + z(z-1)}{z-1}, \quad \mathcal{Q}^2 = \frac{1}{2}Z_o(z-1)\epsilon,$$
$$W_o = 2Z_o\epsilon(d + z + d\mu\xi - \epsilon), \quad V_o = -d(1+\mu\xi)(d+z+d\mu\xi) - (z-1)\epsilon.$$

- HvLf solutions take the form

$$ds^2 = dr^2 - r^{2\nu_z} dt^2 + r^{2\nu_1} d\vec{x}^2, \quad A = \frac{\mathcal{Q}}{\epsilon Z_o} r^{\epsilon} dt, \quad \phi = \mu \log r, \quad \omega = const.$$

where

$$\nu_z = 1 - \frac{dz}{\theta}, \quad \nu_1 = 1 - \frac{d}{\theta}, \quad u = \frac{|\theta|}{d} r^{\frac{d}{\theta}}, \quad \theta \neq 0$$

with

$$\mu(\xi + \rho) = -1, \quad \nu = -\xi - \frac{\nu_z - \epsilon}{\mu}, \quad \sigma = \frac{\nu_z - \epsilon - 1}{\mu}, \quad \mathcal{Q}^2 = \frac{1}{2} Z_o (\nu_z - \nu_1)\epsilon,$$

$$\epsilon = \frac{\left(\alpha_\xi + d^2\xi^2\right)\mu^2 - d\xi(\nu_1 + 1)\mu - \nu_1(d + \nu_z - 1) + \nu_z(\nu_z - 1)}{\nu_z - \nu_1},$$

$$W_o = 2\epsilon Z_o (d(\nu_1 + \mu\xi) + \nu_z - 1 - \epsilon),$$

$$V_o = \epsilon(\nu_1 - \nu_z) - d(\nu_1 + \mu\xi)(d(\nu_1 + \mu\xi) + \nu_z - 1).$$

Relation between Lif & HvLf

- The hvLf metric is conformal to a Lif metric with the same exponent z. Namely, the coordinate transformation

$$r = e^{-\frac{\theta}{d}\bar{r}}, \quad t = \frac{|\theta|}{d}\bar{t}, \quad x^a = \frac{|\theta|}{d}\bar{x}^a$$

the hvLf solution becomes

$$ds^2 = \left(\frac{\theta}{d}\right)^2 e^{\frac{-2\theta\bar{r}}{d}} \left(d\bar{r}^2 - e^{2z\bar{r}}d\bar{t}^2 + e^{2\bar{r}}d\vec{\bar{x}}^2\right), \quad \phi = \mu_h \log r = -\frac{\theta}{d}\mu_h\bar{r} \equiv \mu_L\bar{r}$$

- It follows that the hvLf metric can be written as

$$g_h = e^{-\frac{2\theta}{d\mu_L}\phi} g_L, \quad \mu_L = -\theta\mu_h/d, \quad \ell_L = |\theta|\ell_h/d$$

- This allows us to express any hvLf solution as a Lif solution in a different Weyl frame – cf. “dual frame” for Dp branes with $p \neq 3$. Namely, if $g_h = e^{2\xi\phi}g_L$ is a hvLf metric in the Einstein frame, then g_L is a Lif metric in a Weyl frame with

$$\xi = -\frac{\theta}{d\mu_L} = \frac{1}{\mu_h}$$

Radial Hamiltonian formalism for massive vector-scalar theory

- ADM decomposition

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j$$

- Radial ADM Lagrangian:

$$\begin{aligned} L \quad = \quad & \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-\gamma}Ne^{d\xi\phi}\Big\{R[\gamma] + K^2 - K^{ij}K_{ij} + \frac{2d\xi}{N}K(\dot{\phi} - N^i\partial_i\phi) \\ & -\frac{\alpha_\xi}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - \alpha_\xi\gamma^{ij}\partial_i\phi\partial_j\phi \\ & -Z_\xi(\phi)\left(\frac{2}{N^2}\gamma^{ij}(F_{ri} - N^k F_{ki})(F_{rj} - N^l F_{lj}) + \gamma^{ij}\gamma^{kl}F_{ik}F_{jl}\right) \\ & -W_\xi(\phi)\left(\frac{1}{N^2}\left(A_r - N^i A_i - \dot{\omega} + N^i\partial_i\omega\right)^2 + \gamma^{ij}B_iB_j\right) - V_\xi(\phi)\Big\} \end{aligned}$$

- Hamiltonian:

$$\begin{aligned} H &= \int d^{d+1}x \left(\dot{\gamma}_{ij}\pi^{ij} + \dot{A}_i\pi^i + \dot{\phi}\pi_\phi + \dot{\omega}\pi_\omega\right) - L \\ &= \int d^{d+1}x \left(N\mathcal{H} + N_i\mathcal{H}^i + A_r\mathcal{F}\right) \end{aligned}$$

- where

$$\begin{aligned} \mathcal{H} = -\frac{\kappa^2}{\sqrt{-\gamma}}e^{-d\xi\phi}\left(2\pi^{ij}\pi_{ij} - \frac{2}{d}\pi^2 + \frac{1}{2\alpha}\left(\pi_\phi - 2\xi\pi\right)^2 + \frac{1}{4}Z_\xi^{-1}\pi^i\pi_i + \frac{1}{2}W_\xi^{-1}\pi_\omega^2\right) \\ + \frac{\sqrt{-\gamma}}{2\kappa^2}e^{d\xi\phi}\left(-R[\gamma] + \alpha_\xi\partial^i\phi\partial_i\phi + Z_\xi(\phi)F^{ij}F_{ij} + W_\xi(\phi)B^iB_i + V_\xi(\phi)\right) \end{aligned}$$

$$\mathcal{H}^i = -2D_j\pi^{ji} + F^i{}_j\pi^j + \pi_\phi\partial^i\phi - B^i\pi_\omega$$

$$\mathcal{F} = -D_i\pi^i + \pi_\omega$$

Canonical momenta

- From off-shell Lagrangian:

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(K\gamma^{ij} - K^{ij} + \frac{d\xi}{N}\gamma^{ij}\left(\dot{\phi} - N^k\partial_k\phi\right)\right),$$

$$\pi^{i} = \frac{\delta L}{\delta \dot{A}_i} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}Z_\xi(\phi)\frac{4}{N}\gamma^{ij}\left(F_{rj} - N^k F_{kj}\right),$$

$$\pi_\phi = \frac{\delta L}{\delta \dot{\phi}} = \frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}\left(2d\xi K - \frac{2\alpha_\xi}{N}(\dot{\phi} - N^i\partial_i\phi)\right),$$

$$\pi_\omega = \frac{\delta L}{\delta \dot{\omega}} = -\frac{1}{2\kappa^2}\sqrt{-\gamma}e^{d\xi\phi}W_\xi(\phi)\frac{2}{N}\left(\dot{\omega} - N^i\partial_i\omega - A_r + N^i A_i\right)$$

- From on-shell action:

$$\pi^{ij} = \frac{\delta\mathcal{S}}{\delta\gamma_{ij}}, \quad \pi^i = \frac{\delta\mathcal{S}}{\delta A_i}, \quad \pi_\phi = \frac{\delta\mathcal{S}}{\delta\phi}, \quad \pi_\omega = \frac{\delta\mathcal{S}}{\delta\omega}$$

- Combining the two expressions for the momenta:

$$\dot{\gamma}_{ij} = -\frac{4\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\left(\gamma_{ik}\gamma_{jl} - \frac{\alpha_\xi + d^2\xi^2}{d\alpha} \gamma_{ij}\gamma_{kl} \right) \frac{\delta}{\delta\gamma_{kl}} - \frac{\xi}{2\alpha}\gamma_{ij}\frac{\delta}{\delta\phi} \right) \mathcal{S},$$

$$\dot{A}_i = -\frac{\kappa^2}{2}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} Z_\xi^{-1}(\phi)\gamma_{ij}\frac{\delta}{\delta A_j}\mathcal{S},$$

$$\dot{\phi} = -\frac{\kappa^2}{\alpha}\frac{1}{\sqrt{-\gamma}} e^{-d\xi\phi} \left(\frac{\delta}{\delta\phi} - 2\xi\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} \right) \mathcal{S},$$

$$\dot{\omega} = -\frac{\kappa^2}{\sqrt{-\gamma}} e^{-d\xi\phi} W_\xi^{-1}(\phi)\frac{\delta}{\delta\omega}\mathcal{S}$$

Zero derivative solution

- The zero order solution of the HJ equation contains no transverse derivatives:

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int d^{d+1}x\sqrt{-\gamma}U(\phi, A_iA^i)$$

- Inserting this ansatz into the Hamiltonian constraint yields a PDE for $U(X,Y)$, where $X := \phi$, $Y := B_iB^i = A_iA^i$ (cf. superpotential equation)

$$\frac{1}{2\alpha}\left(U_X - \xi(d+1)U + 2\xi Y U_Y\right)^2 + Z_\xi^{-1}(X)YU_Y^2$$
$$-\frac{1}{2d}\left((d+1)U + 2(d-1)YU_Y\right)\left(U - 2YU_Y\right) = \frac{1}{2}e^{2d\xi X}\left(W_\xi(X)Y + V_\xi(X)\right)$$

- Lifshitz asymptotics imposes constraints on the asymptotic form of $U(X,Y)$

Constraints from Lifshtiz asymptotics

- Imposing Lif boundary conditions requires that asymptotically the gauge invariant vector field behaves as

$$B_i \sim B_{oi} = \sqrt{\frac{z-1}{2\epsilon}} Z_\xi^{-1/2}(\phi) \mathbb{n}_i$$

where $\mathbb{n}_i$ is the unit normal to the constant t surfaces

- Moreover, the superpotential $U(X, Y)$ must satisfy

$$\begin{aligned} U(X, Y_o(X)) &\sim e^{d\xi X} \left(d(1+\mu\xi) + z - 1\right) \\ U_Y(X, Y_o(X)) &\sim -\epsilon e^{d\xi X} Z_\xi(X) \\ U_X(X, Y_o(X)) &\sim e^{d\xi X} \left(-\mu\alpha_\xi + d\xi(d+z)\right) \end{aligned}$$

- Hence, the asymptotic form of the zero order solution of the HJ equations is

$$\mathcal{S}_{(0)} \sim \frac{1}{\kappa^2} \int_{\Sigma_r} d^{d+1}x \sqrt{-\gamma} e^{d\xi\phi} \left(d(1+\mu\xi) + \frac{1}{2}(z-1) - \epsilon Z_\xi(\phi) B_i B^i \right)$$

Taylor expansion of the superpotential

- Imposing Lif boundary conditions implies that the solution of the HJ equation can be expressed as a Taylor series in $B_i - B_{oi}$
- In particular, the zero derivative solution $\mathcal{S}_{(0)}$ can be Taylor expanded as

$$U = U_0(\phi) + U_1(\phi)(Y - Y_o(\phi)) + U_2(\phi)(Y - Y_o(\phi))^2 + \mathcal{O}(Y - Y_o(\phi))^3$$

where

$$Y - Y_o = 2B_o^i(B_i - B_{oi}) + (B^i - B_o^i)(B_i - B_{oi}), \quad Y_o \equiv B_o^i B_{oi}$$

- Parameterizing the coefficients as

$$U_n = e^{(d+1)\xi\phi} Y_o^{-n} u_n(\phi)$$

and inserting this expansion in the superpotential equation for $U(X, Y)$ we get a tower of equations for the functions $u_n(\phi)$

- An additional equation for the functions $u_n(\phi)$ is imposed by the consistency of the Taylor expansion, i.e. requiring that

$$\dot{Y} - \dot{Y}_o = \mathcal{O}(Y - Y_o)$$

- In a bottom up approach these equations can be used to *define* the potentials $V(\phi)$, $Z(\phi)$ and $W(\phi)$ in terms of $u_0(\phi)$ and $u_1(\phi)$, with all $u_n(\phi)$ for $n \geq 2$ being determined in terms of these functions.
- Lifshitz boundary conditions require

$$\boxed{\begin{aligned} u_0(\phi) &\sim (z - 1 + d(1 + \mu\xi))\, e^{-\xi\phi} \\ u_1(\phi) &\sim \frac{1}{2}(z - 1)e^{-\xi\phi} \end{aligned}}$$

- The function $u_2(\phi)$ satisfies a quadratic (Riccati) equation and determines the scaling behavior of the independent mode $Y - Y_o$, while $u_n(\phi)$ with $n \geq 3$ satisfy linear equations.

Recursive solution of the HJ equation

- So far we have discussed the zero derivative solution of the HJ equation

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2}\int_{\Sigma_r} d^{d+1}x\sqrt{-\gamma}U(\phi, B^2)$$

  which is the asymptotically leading one and is related to the boundary conditions.
- In order to determine asymptotically subleading part of the solution of the HJ equation, we will expand $\mathcal{S}$ in a covariant expansion in eigenfunctions of a suitable operator.
- For backgrounds with asymptotic scaling invariance one can use the dilatation operator [I. P. & Skenderis 2004] but in the presence of a linear dilaton this is not sufficient.
- Instead we need an operator such that $\mathcal{S}_{(0)}$ is an eigenfunction for any superpotential $U(\phi, B^2)$.

- In fact there are two mutually commuting such operators:

$$\widehat{\delta} := \int d^{d+1}x \left(2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right), \quad \delta_B := \int d^{d+1}x \left(2Y^{-1} B_i B_j \frac{\delta}{\delta\gamma_{ij}} + B_i \frac{\delta}{\delta B_i} \right)$$

which satisfy

$$\widehat{\delta}\mathcal{S}_{(0)} = (d+1)\mathcal{S}_{(0)}, \quad \delta_B \mathcal{S}_{(0)} = \mathcal{S}_{(0)}, \quad [\widehat{\delta}, \delta_B] = 0$$

- This allows us to seek a solution in the form of a *graded* covariant expansion in simultaneous eigenfunctions of both $\widehat{\delta}$ and δ_B in the form

$$\mathcal{S} = \sum_{k=0}^{\infty} \mathcal{S}_{(2k)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \mathcal{S}_{(2k,2\ell)} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \int d^{d+1}x \mathcal{L}_{(2k,2\ell)}$$

where

$$\widehat{\delta}\mathcal{S}_{(2k,2\ell)} = (d+1-2k)\mathcal{S}_{(2k,2\ell)}, \quad \delta_B \mathcal{S}_{(2k,2\ell)} = (1-2\ell)\mathcal{S}_{(2k,2\ell)}, \quad 0 \leq \ell \leq k+1$$

- The operator $\widehat{\delta}$ counts derivatives
- The operator δ_B annihilates the projection operator $\sigma^i_j := \delta^i_j - Y^{-1} B^i B_j$ and counts derivatives contracted with B_i, which asymptotically become time derivatives since $B_i \sim B_{oi} \propto \mathbb{n}_i$

Linear recursion equations

- Inserting this expansion of $\mathcal{S}$ in the Hamilton-Jacobi equation (Hamiltonian constraint) results in a system of recursive *linear* equations for the higher derivative terms:

$$\frac{1}{\alpha}\left(U_X-(d+1)\xi U+2\xi Y U_Y\right)\frac{\delta}{\delta\phi}\int\mathcal{L}_{(2k,2\ell)}+$$
$$\left((2Y+Z_\xi^{-1})U_Y+\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)\right)B_i\frac{\delta}{\delta B_i}\int\mathcal{L}_{(2k,2\ell)}-$$
$$\left(\frac{1}{d\alpha}\left(\alpha_\xi U-2(\alpha_\xi+d^2\xi^2)YU_Y+d\xi U_X\right)(d+1-2k)+2YU_Y(1-2\ell)\right)\mathcal{L}_{(2k,2\ell)}=$$
$$e^{d\xi\phi}\mathcal{R}_{(2k,2\ell)}$$

- The inhomogeneous term $\mathcal{R}_{(2k,2\ell)}$ involves derivatives of lower order terms as well as the 2-derivative sources from the Hamiltonian constraint

Lifshitz boundary conditions

- The covariant expansion in eigenfunctions of $\widehat{\delta}$ and δ_B is independent of the specific choice of boundary conditions
- To impose Lifshitz boundary conditions we must additionally expand $\mathcal{S}_{(2k,2\ell)}$ in $B_i - B_{oi}$ at each order of the covariant expansion as

$$\begin{aligned}\mathcal{L}_{(2k,2\ell)} &= \mathcal{L}^0_{(2k,2\ell)}[\gamma(x),\phi(x)] \\ &+ \int d^{d+1}x'(B_i(x') - B_{oi}(x'))\mathcal{L}^{1i}_{(2k,2\ell)}[\gamma(x),\phi(x);x'] + \mathcal{O}\left(B - B_o\right)^2\end{aligned}$$

- Inserting this Taylor expansion in the above recursion relations eliminates the derivative with respect to B_i, resulting in linear functional differential equations in ϕ only

Solution of the recursion relations up to $\mathcal{O}(B - B_o)$

- The inhomogeneous solution of these linear functional differential equations takes the form

$$\begin{aligned}
\mathcal{L}^0_{(2k,2\ell)} &= e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} \mathcal{R}^0_{(2k,2\ell)}, \\
\varpi^i_j \mathcal{L}^{1j}_{(2k,2\ell)} &= Z_\xi^{\frac{1}{2}} e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} Z_\xi^{-\frac{1}{2}} \varpi^i_j \mathcal{R}^{1j}_{(2k,2\ell)}, \\
B_{oj}(x)\mathcal{L}^{1j}_{(2k,2\ell)} &= \Omega^{-1} e^{-\mathcal{C}_{k,\ell}\mathcal{A}(\phi)} \int^{\phi} d\bar{\phi}\mathcal{K}(\bar{\phi}) e^{\mathcal{C}_{k,\ell}\mathcal{A}(\bar{\phi})} \Omega B_{oj} \widehat{\mathcal{R}}^{1j}_{(2k,2\ell)}
\end{aligned}$$

where $\mathcal{C}_{k,\ell} := d + 1 - 2k + (z-1)(1-2\ell)$,

$$\mathcal{K}(\phi) := \frac{\alpha}{e^{\xi\phi}\left(u_0' + \frac{Z'}{Z}u_1\right)} \sim -\frac{1}{\mu}, \quad e^{\mathcal{A}(\phi)} = Z_\xi^{-\frac{1}{2(\epsilon - z)}} \sim e^{\phi/\mu}$$

and the $\Omega(\phi)$ can be expressed in terms of u_0, u_1 and u_2.

- These relations provide a very general recursive algorithm for solving the Hamilton-Jacobi equation for gravity coupled to specific matter fields
- This is equivalent to constructing the most general asymptotic solution of Einstein's equations with given boundary conditions
- It also amounts to computing the Hawking wavefunction of quantum gravity in the WKB approximation
- The complete holographic dictionary, including the operator/field map, the Fefferman-Graham asymptotic expansions, the Ward identities, the boundary counterterms required for holographic renormalization and the conformal anomalies can be derived from this asymptotic solution of the HJ equation
- The algorithm is ideal for implementing in a symbolic computation package such as xAct

Conformal anomalies

- As an illustration of the power of the method, we can provide a general prediction for the occurence of conformal anomalies in Lif and hvLf theories
- Conformal anomalies correspond to poles of the solution of the HJ equation. From the solution we obtained we know such poles occur whenever any of the two quantities vanishes:

$$\boxed{d+1-2k+(z-1)(1-2\ell)-\theta, \quad d+1-2k+(z-1)(1-2\ell)-\theta-\Delta_+}$$

  where Δ_+ is the scaling dimension of the operator dual to the mode $Y - Y_o$.
- For example, for $k = 1$, $d = z = 2$ and $\theta = 0$, there are two potential conformal anomalies, a 2-derivative one ($k = 1,\ \ell = 1$) and a 4-derivative one ($k = 2,\ \ell = 0$)

Concluding remarks

- We have developed a general recursive algorithm for solving the radial Hamilton-Jacobi equation for an Einstein-Proca-scalar theory with arbitrary scalar couplings
- The full holographic dictionary for asymptotically Lifshitz or hyperscaling violating Lifshitz backgrounds can be derived from the resulting solution of the Hamilton-Jacobi equation
- This provides the necessary tools to rigorously explore holography for such backgrounds by properly identifying the dual operator spectrum and computing the corresponding correlation functions