# N2 SUGRA BPS MULTICENTER BHs.
# Mass Formulas, Freudenthal transformations

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**Based on:**

–JHEP 1405( 2014) 081 (JJ,ET)
–N2 BPS Mass formulas, Sub. PRL (JJ,ET)
–Freudenthal and Jordan transformations, ( JJ, A. Marrani,ET)

# INTRO: BPS states. Central Charge.

- BPS STATES ($N > 1$): Short multiplets invariant under a fermionic (1/(2,4,8)) SUSY subalg.
  **-M $=|$ Z $|$.**
  **-QM-stable: the mass formula is exact.**

- EXTREMAL SUGRA BHs: a realization of BPS states by the SUGRA superalgebra.
  $\longrightarrow$ Killing Spinor 1st order eqs+eqs. of motion.

- MASS FORMULAS: necc. conditions for BPS states.

- String Connection: Extremal BHs: bound states of d-branes in compactified space, $\longrightarrow$ microstates, $\longrightarrow$ BH entropy.

# INTRO: BHs Mass formulas. EXAMPLES.

**E-MAXWELL GRAV. (embedded in Pure SUGRA,no scalars)**

- (Static) Majumdar-Papatreou:
  $ds^2 = e^{2U(\mathbf{x})}dt^2 - e^{-2U(\mathbf{x})}dx^2,$
  $$e^{-U} = 1 + \sum_i \frac{M_i}{|x - x_i|},$$
  $\mathbf{BPS: M_i > 0,\ M_i^2 = q_i^2,\ M_{ADM} = \sum M_i = f(q_i)}$

- RN: $\mathbf{M^2 = P_m^2 + Q_e^2}$.
  Borderline between BHs and naked singularities.

- Stationary Kerr-Newman : $\mathbf{M^2 = Q^2 + J^2/M^2}$.

# INTRO: GENERAL SUGRA:

Extremal BH: $\sim$ Balanced Gravity/EM.
SUGRA scalars contribute with an extra attractive long range force ($\sim$ scalar charges)$\longrightarrow$ modified relations

**Maxwell-Einstein axion-dilaton SUGRA** Ferrara98,Kallosh99,Ortin99

- Extremal: $\mathbf{M^2 = \frac{1}{2}\left(Pe^{-\phi_\infty} + Qe^{\phi_\infty}\right)^2}$.
- SUGRA dilaton model: $\mathbf{M^2 = \frac{1}{2}Q^2 \exp 2\phi_\infty}$ Garfinkle:1990qj

**Others:** Gibbons'86,Smarr,PRL73

- $4d$, $N2$−SUGRA, 1c-BH: $M^2 + G^{\infty}_{ab}\Sigma^a\Sigma^b - V_{bh}(p,q,\phi^a_\infty) = 0$
- Smarr M formulas: $dM = TdS + \omega dJ + \Phi dQ$

# INTRO: Multicenter. Existence conditions

Existence and Construction of extremal BHs: a rather trivial problem in a wide number of well known theories.

- 1- center, static n-center solutions: given parameters (and a suitable metric ansatz) $\mathbf{M, Q, z^a_\infty, J, \Sigma^a}$..
  (or a subset of them) satisfiying simple local relations,
  $\longrightarrow$ Build explicitly solutions from them.

- General Stationary Multicenter solutions:

$$\mathbf{M_{ADM}, M_i, Q, q_i, J_{ij}, r_{ij}, z_\infty}$$

$\longrightarrow$ BPS solutions: not a trivial problem for given params.

$\longrightarrow$ **Desiderable**: neccesary and sufficient BPS existence conditions in terms of these"macroscopical" params.
but **this is not known.** $\longrightarrow$ OUR OBJECTIVE HERE.

# THE MODEL: $\mathcal{N}2$ $D4$ SUGRA $+n_v$ vector multiplets

- Bosonic content (essentially) $(\mathbf{g_{\mu\nu}, A_\mu{}^I, z^\alpha})$, $\quad I = n_v + 1, \alpha = n_v$.

- N2 SUGRA Bosonic Action

$$\mathbf{S = \int_{M(4d)} R \star 1 + \mathcal{G}_{\alpha\bar{\beta}} dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} + F^I \wedge G_I}$$

- 
  - $F^I = dA^I, \quad G_I = a_{IJ}(z) F^J + b_{IJ}(z) \star F^J$
  - $dF^I = 0, dG_I = 0 \longrightarrow Sp(2n_v + 2, \mathbb{R})$ Duality symplectic symmetry (Gaillard construction)

- N2 SUSY+Duality:

$\longrightarrow$ $(A^I, z^\alpha)$ same multiplet
$\longrightarrow$ $G_{ab}, a_{IJ}, b_{IJ}$ : determined by Special Kahler-Hodge geometry of the scalar Manifold.

# Special Kahler geometry =A Kahler manifold plus...

- A projective embedding, define $X^I/z^\alpha = X^I/X^0$.
- A $(2n_v + 2\ )$ vector complex space $\mathbf{W}$. $\langle \mathbf{X} \mid \mathbf{Y} \rangle$
- An (almost) complex structure on $W$ def by $\mathcal{S}(\mathbf{X})$ ,compatible with the SYMP. prod.:
  $\mathcal{S}^{-1} = -1,\ \mathrm{P}_\pm := \frac{1}{2}(\mathbb{1} \pm i\mathcal{S}) \Rightarrow \quad W = W^+ \oplus W^-$
  **DEF:** $\mathbf{g(X,Y)} \equiv \langle \mathcal{S}\mathbf{X} \mid \mathbf{Y} \rangle = \left\langle \mathbf{Y} \,\middle|\, \mathcal{S}^\dagger \mathbf{X} \right\rangle$
- DEF: $\mathbf{D}_\alpha = \partial_\alpha + \mathbf{pQ}_\alpha,\ \mathbf{Q}_\alpha = \partial_\alpha \mathcal{K}$
- $W$ sympl. sections $\mathbf{V} = (\mathbf{V^I}, \mathbf{V_I}), \boldsymbol{\Omega} \equiv (\mathbf{X^I}, \mathbf{F_I})$.
  $\mathbf{D}_{\bar\alpha}\mathbf{V} = 0.\ \partial_\alpha \boldsymbol{\Omega} = 0.\ \left\langle \mathbf{V} \,\middle|\, \bar{\mathbf{V}} \right\rangle = -\mathbf{i}.\ \boldsymbol{\Omega} = \mathbf{e}^{\mathcal{K}/2}\mathbf{V}.$
- Then

$$\mathcal{G}_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} \mathcal{K} = \left\langle \mathbf{D}_\alpha \mathbf{V} \,\middle|\, \bar{\mathbf{D}}_{\bar\beta} \bar{\mathbf{V}} \right\rangle,$$
$$\mathbf{N_{IJ}} = \mathbf{a_{IJ}} + \mathbf{ib_{IJ}i} = \mathbf{F_{IJ}} + \ldots$$

Where $\mathbf{F_{IJ}} = \partial_\mathbf{I}\partial_\mathbf{J}\mathbf{F},\ \mathbf{F_I} = \partial_\mathbf{I}\mathbf{F}.\ \mathbf{S(F)} = \mathbf{S(F_{IJ})}$

# BPS stationary solutions.

SUSY+BPS CONDITIONS+FIELD EQUATIONS:

- Most general 4d BPS stationary: IWP metric, Tod83
  ($\omega = \omega_i dx^i$)

$$\mathbf{ds^2 = e^{2U(x)}(dt+\omega)^2 - e^{-2U(x)}dx^2}$$

- $dF^I = dG_I = 0$: $\longrightarrow$ $\mathbf{(F^I)_{mn} = \star_3 d_3 I^I, (G_I)_{mn} = \star_3 d_3 I_I}$

- BPS condition: relations $A, U, \omega$ fields. Gauntlett02,Ortin'03
  $\longrightarrow$ $U, \omega, z^\alpha$ solutions in terms of
  $\mathcal{I}(\mathbf{x}) \equiv (\mathbf{I^I}, \mathbf{I_I})$, **a W-valued $\mathbf{R^3}$-harmonic.**

# BPS Solutions: $U, \omega, z^\alpha$, give $\mathcal{I}(\mathbf{x}), \mathcal{S}$:

- $d\omega = 2\langle \mathcal{I} \mid \star_3 d\mathcal{I}\rangle \Rightarrow \langle \mathcal{I} \mid \Delta\mathcal{I}\rangle = 0,\ \omega_\infty \to 0$

- $z^\alpha = \dfrac{(P_-(z)\mathcal{I}(\mathbf{x}))^\alpha}{(P_-(z)\mathcal{I}(\mathbf{x}))^0}$

  $z^\alpha_\infty \sim$: moduli. $\mathbf{z^\alpha_\infty = (P_-\mathcal{I}_\infty)^\alpha/(P_-\mathcal{I}_\infty)^\alpha}$
  $z^\alpha_h$: BPS attractor : $\mathbf{q^a = \mathrm{Re}\left(2i\bar{Z}_f V_f\right)}$

- $e^{-2U} = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I}\rangle$, $\mathbf{e^{-2U_\infty} = \langle \mathcal{S}_\infty\mathcal{I}_\infty \mid \mathcal{I}_\infty\rangle \to 1}$

- CENTRAL Charge $Z(z^\alpha, q) \equiv \langle V \mid q\rangle \quad \Rightarrow$
  $\mathbf{\mid Z(z^\alpha_\infty, Q) \mid^2 = \mid \langle \mathcal{S}_\infty\mathcal{I}_\infty \mid Q\rangle \mid^2 + \mid \langle \mathcal{I}_\infty \mid Q\rangle \mid^2}$ $(= M^2 + N^2)$

# Multicenter ANSATZ.

- $n_c$ MULTICENTER: NEED $\mathcal{I}(\mathbf{x}) \equiv (I^I, I_I)$
  TAKE: $\mathcal{I} = \mathcal{I}_\infty + \sum_a \frac{\mathbf{q_a}}{|\mathbf{x} - \mathbf{x_a}|}$,
  $\mathcal{I}_\infty, q_a \in (2n_v + 2)$-dim real $\in W$

- • $r_{ab}$ : restricted,
  • TAUB-NUT charge $N = 0 = \langle \mathcal{I}_\infty \mid Q \rangle = \sum_a \langle \mathcal{I}_\infty \mid q_a \rangle = 0$:

$$\mathbf{dd\omega = 0} \quad \longrightarrow \quad \langle \mathcal{I} \mid \mathbf{\Delta} \mathcal{I} \rangle = \mathbf{0} \tag{1}$$

$$\longrightarrow \quad \langle \mathcal{I}_\infty \mid \mathbf{q_b} \rangle + \sum_a \frac{\langle \mathbf{q_a} \mid \mathbf{q_b} \rangle}{\mathbf{r_{ab}}} = \mathbf{0} \tag{2}$$

# BEHAVIOUR AT INFINITY: ADM MASS

- $e^{-2U} = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I} \rangle$. Asymptotically ($\mathbf{x} \to \infty$)

$$\mathbf{e^{-2U} \to 1 + \frac{2\sum_b \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid q_b \rangle}{r} + \frac{\sum_{ab} \langle \mathcal{S}_\infty q_a \mid q_b \rangle}{r^2} + O(\frac{1}{r^3})}$$
$$\mathbf{\equiv 1 + 2\frac{M_{ADM}}{r} + \frac{A_\infty}{r^2} + O(\frac{1}{r^3}).}$$

With: $\mathbf{M_{ADM} = \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid Q \rangle,\ A_\infty = \langle \mathcal{S}_\infty Q \mid Q \rangle.}$

- BPS condition: (as TAUB-NUT $N = \langle \mathcal{I}_\infty \mid Q \rangle = 0$):

$$\mathbf{\mid Z(z^\alpha_\infty, Q) \mid^2 = \mid \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid Q \rangle \mid^2 = M^2_{ADM}}$$

# QUESTIONS:

- The solution is a BH?, How to build a BH?: Not so easy, basically by trial and error...

- Initial configuration parameters: $q_i, \mathcal{I}_\infty, r_{ij}$

– $z_\infty$, are not directly given.
– $r_{ij}$ : cannot be prescribed initially.
–not obvious how can we build a BH with prescribed "macroscopic" params. $(M_{ADM}, Q, z_\infty ...)$??

- FIND MULTICENTER MASS FORMULAS: some necessary conditions for BPS BHs

$\longrightarrow$ **asymptotic flatness,**
$\longrightarrow$ **charge expansions of the vector $\mathcal{I}_\infty$.**

# Simplectic expansions

Well known useful expansion: Bossard,Katmadas'13

- $(2n_v+2)$-W simplectic space, a basis: $\{\mathbf{V}, \mathbf{D}_\alpha \mathbf{V}, \bar{\mathbf{V}}, \bar{\mathbf{D}}_{\bar\alpha}\bar{\mathbf{V}}\}$

By special geometry:

$\langle V \mid \bar V\rangle = -i, \langle \bar V \mid D_\alpha V\rangle = \langle V \mid D_{\bar\beta} V\rangle = 0, \langle D_\alpha V \mid D_{\bar\beta} V\rangle = G_{\alpha\bar\beta}.$

Eigenvectors of $\mathcal{S}(N)$: $\mathcal{S}(N)V = iV, \mathcal{S}(N)D_\alpha V = -iD_\alpha V$.

- Any real $X \in W$

$$X = x_0 V + x^{\bar 0}\bar V + x^{\bar\beta}\bar D_{\bar\beta}\bar V + x^\beta D_\beta V$$

- **ALTERNATIVE: MULTICENTER BHs: MANY SYMPLECTIC VECTORS: $\mathcal{I}_\infty, \mathbf{q_a} \longrightarrow$ TO DEFINE A BASIS IN W ($\mathbf{W}^\pm$)**

# MASS RELATIONS:CHARGE SPACE EXPANSIONS

Fernandez-Melgarejo:2013ksa

- W Basis: $\mathcal{S}_\infty \equiv \mathcal{S}(z^a_\infty)$ eigenvectors, $\mathbf{(P_+ w_k, P_- w_k)}$, $\mathbf{w_k = (q_n, s_a)}$, $2n_a = 2n_V + 2 - 2n_c$, $\mathrm{P}_\pm = (1 \pm i\mathcal{S}_\infty)/2$.

- Define "metric": $\mathbf{g_{k\bar{k}} \equiv \langle P_+ w_k \mid P_- w_k \rangle}$ , $\quad \mathbf{g_{i\bar{\jmath}} g^{\bar{\jmath}k} = \delta_i^k}$

- Any real $X$, (i.e. $\mathcal{I}_\infty$) , in "covariant/contravariant" coordinates,

$$\begin{aligned} \mathbf{X} &\equiv \alpha^{\mathbf{k}} \mathbf{P_+ w_k} + \alpha^{\bar{\mathbf{k}}} \mathbf{P_- w_k} \\ &\equiv \alpha_{\mathbf{l}} \mathbf{g^{lk} P_+ w_k} + \alpha_{\bar{\mathbf{l}}} \mathbf{g^{\bar{l}k} P_- w_k} \end{aligned}$$

With: $a_j = \langle X \mid \mathrm{P}_+ w_j \rangle = g_{\bar{\imath}j} \alpha^{\bar{\imath}}$.

- "scalar" product: $\langle \mathrm{P}_+ X \mid \mathrm{P}_- Y \rangle = \alpha_i \beta^{\bar{\imath}} = \alpha_i \beta_{\bar{\jmath}} g^{i\bar{\jmath}}$

# General Mass formulas: INGREDIENTS

- EXPAND $\mathcal{I}_\infty = \alpha_i g^{ik} \mathrm{P}_+ w_k + \alpha_{\bar{\jmath}} g^{\bar{\imath}\bar{k}} \mathrm{P}_- w_k$
- ADVANTAGE: contravariant components are Phys. params.

$$\begin{aligned} a_j(\mathcal{I}_\infty) &= (N_j - iM_j)/2, \quad j = 1, n_c \\ &= \left( \sum \langle q_j \mid q_i \rangle / r_{ij} - iM_j \right) /2 \end{aligned}$$

- ASYMPT. flatness: $e^{-2U} \to \langle \mathcal{S}_\infty \mathcal{I}_\infty \mid \mathcal{I}_\infty \rangle \to 1$,

$$\mathbf{1 = -2i \langle P_+ \mathcal{I}_\infty \mid P_- \mathcal{I}_\infty \rangle = -2i\alpha_i \alpha^{\bar{\imath}} = -2i\alpha_i \alpha_{\bar{\jmath}} g^{i\bar{\jmath}}}$$

with $\alpha_i = \alpha(q_i, M_i, r_{ij})$, $g^{ij} = g^{ij}(q_i, z_\infty)$.

- **GENERAL MASS FORMULA**: $M_{ADM} = \sum_i M_i$, $m_i = M_i / M_{ADM}$

$$\mathbf{1 = aM^2_{ADM} + bM_{ADM} + c}$$

# Mass formulas...

- BPS MASS F: $\mathbf{1 = aM^2_{ADM} + bM_{ADM} + c}$ (dim $W = 2n_c$)

$$a \equiv (1/2)m_i m_j Y^{ij}, \quad b = -m_i N_j X^{ij}, \quad c = (1/2)N_i N_j Y^{ij}$$

  - $a, b, c$: depend on $g^{ij} = X + iY$
    - SYMPL. products of q's ($\sim$ BH $J_{ij}$, $\langle q_i \mid q_j \rangle$ $\langle Sq_i \mid q_j \rangle$),
    - Moduli at infinity, $z^\alpha_\infty$ ( $\mathcal{S}_\infty$ in $\langle \mathcal{S}_\infty q_i \mid q_j \rangle$),
    - $r_{ab}$, intercenter distances, (in $N_i = c_{ij}/r_{ij}$ )
    - and the relative mass parameters $m_i = M_i/M_{ADM}$.

- QUAD. RELATION in $M, 1/r_{ij}, J_{ij}$: $\longrightarrow$ **restrictions on coeffs.**

- MOREOVER: scalar stabilization Eqs. at infinity:

$$\mathbf{2(n_v - 1):} \qquad \mathbf{z^\alpha_\infty = (P_{-}\mathcal{I}_\infty)^\alpha/(P_{-}\mathcal{I}_\infty)^0 = f^\alpha(M_i, r_{ij}, z_\infty).}$$

# GENERAL PROPERTIES: $r_{ij} \to \infty$, Minimal Mass.

$M_{ADM}$ decre. with incre. $r_{ij}$: $\mathbf{1 = aM^2 + b_j N_j M + c_{ij} N_i N_j}$

- If $r_{ij} \to \infty$. Then $N_i = \sum c_{ij}/r_{ij} \to 0$. Mass EQ:

$$\mathbf{1 \;=\; a(m_i) M_\infty^2}$$

$\longrightarrow$ (if $M_\infty^2 > 0$) Minimal ADM mass for a given $q_i, m_i$ configuration.

- MINIMIZE $M_\infty^2 = M^2(m_i, q = cte)$
  $\mathbf{m_{i,min} \simeq \langle \mathcal{S}_\infty Q \mid q_i \rangle \,/\, \langle \mathcal{S}_\infty Q \mid Q \rangle + O(J^2)}$,

- in this case $M_{ADM}(r_{ij} \to \infty)$

$$\mathbf{(M_\infty^2)_{min} = \frac{1}{Im \sum_{ij} \; g^{i\bar{\jmath}}}}$$

$\longrightarrow$ **Minimal ADM mass for a given $q_i$ configuration**.

# GENERAL PROPERTIES: Effective "force" a la Smarr

$M_{ADM}(\sim E) \sim 1/r_{ij}$:

- Smarr-like expressions, From $M_{ADM}(J, z_\infty, q, r)$

$$dM = \Omega dJ + \Sigma dz_\infty + \Phi_i dq_i - F_{ij} dr_{ij}$$

- $dz_\infty$: eliminated from scalar eqs.$\longrightarrow$ (at $q_i$ cte.)

$$dM \,|_{q=cte} = -F_{ij} dr_{ij},$$

- $F$:Effective "force" between centers. TIPICALLY:
  - $F$ :$\sim$ repulsive (for viable $M_{ADM}$).
  - $F \sim 1/r^2$, $r \to \infty$.
  - $F \sim f_0 + f_1/r^2$, $r \to 0$.

# 2 CENTER MASS RELATIONS ( $n_c = 2, n_v = 1$)

ASSUME:

- One scalar $(n_v = 1)$, $dimW = 2n_v + 2 = 4$.

**2 centers: $\mathbf{q_1, q_2} \longrightarrow$ W-Basis $= (\mathbf{P_{\pm} q_{1,2}})$**

Parms(5): $M_1, M_2, r_{12}, z_\infty$. $(q = \text{fixed})$

Eqs(3): $1M{+}2\ z$: $\longrightarrow dof = 5 - 3 = 2$ .

- Metric matrices $g_{ij} = (A + iS)/2, g^{ij} = X + iY$

$$A = \langle q_1 \mid q_2 \rangle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} \langle \mathcal{S}_\infty q_1 \mid q_1 \rangle & \langle \mathcal{S}_\infty q_2 \mid q_1 \rangle \\ \langle \mathcal{S}_\infty q_2 \mid q_1 \rangle & \langle \mathcal{S}_\infty q_2 \mid q_2 \rangle \end{pmatrix}.$$

- $\omega-$ compatibility eqs: $\mathbf{-N_2 = N_1 = \langle q_1 \mid q_2 \rangle / r \equiv J/r}$

- $\mathbf{a_{1,2}(\mathcal{I}_\infty) = (J/r - iM_{1,2})/2, \quad -2ia_i a_j g^{ij} = 1}$

# The 2 CENTER MASS RELATION

- BPS 2-c Mass Equation: $A_\infty = \langle \mathcal{S}_\infty Q \mid Q \rangle$, $M_0^2 = 1/S_{ij}^{-1} m_i m_j$.

$$M_{ADM}^2 = M_0^2 \left(1 + \frac{J^2}{-\det(S)} \left(1 + \frac{2M_{ADM}}{r} + \frac{A_\infty}{r^2}\right)\right).$$

$det(S), A_\infty, M_0^2 : f(q_i, m_i, z_\infty)$

Characteristics:

- 1 real, positive solution: ( $A_\infty > 0$) only if $\det(S) < 0$.
- Unique solution for any $r$ and $J^2$.
- ( $r \to \infty$): $M_\infty^2 = M_0^2 \left(1 + \frac{J^2}{|\det(S)|}\right)$,
- $M \sim M_\infty + J^2/(|\det(S)|r)$ $(r \to \infty)$. $M \sim A/r + Br$ $(r \to 0)$.
- $m_i$: restricted.

# Minimal Mass Formula

- MINIMIZE $M_{ADM}(m_i)$

$$M^2_{ADM} = A_\infty \left(1 + \frac{J^2}{|\det(S)|}\left(1 + \frac{2M_{ADM}}{r} + \frac{A_\infty}{r^2}\right)\right).$$

$$m_{i,\min} = \frac{\langle \mathcal{S}_\infty q_i \mid Q\rangle}{\langle \mathcal{S}_\infty Q \mid Q\rangle}, \qquad A_\infty = \langle \mathcal{S} Q \mid Q\rangle \, , J = \langle q_1 \mid q_2\rangle$$

- At large $r$: $\mathbf{M^2_{ADM} = A_\infty \left(1 + \frac{J^2}{|\det(S)|}\right)}$.

  **MINIMAL MASS FOR A 2 CENTER BH (N2,1scalar)**

- Relative $\mathbf{m_1, m_2}$ **RESTRICTED:**

$$M_0^2 > 0 \longrightarrow m_i \in m_{i,\min} \pm \sqrt{|\det(S)|}/A_\infty$$

# EXAMPLE(1): Pure 4d SUGRA. $F = \frac{-i}{2}(X^0)^2$

- Pure N2 SUGRA $(e_\mu^a, A_\mu, \psi)$: ($\sim$ Einstein-Maxwell) .

Dim $W = 2n_v + 2 = 2$.

$$F = \frac{-i}{2}(X^0)^2, \quad N_{00} = F_{00} = -i, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Static BPS solutions $g_{\mu\nu}, A_\mu$

$$\begin{aligned} ds^2 &= e^{2U(\mathbf{x})}dt^2 - e^{-2U(\mathbf{x})}dx^2 \\ A_t &= e^U \qquad (Killing\ Spinor\ eqs.) \\ \nabla^2 e^{-U} &= 0 \qquad (Maxwell\ eqs.) \end{aligned}$$

- TAKE: ($n_c$ arbitrary): $q_i = (P, Q)_i$.

$$\mathbf{e^{-2U} = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I} \rangle\,, \quad \mathcal{I} = \mathcal{I}_\infty + \sum_{nc} \frac{q_i}{\mid x - x_i \mid}}$$

RN: $\longrightarrow$ TAKE $n_c = 1$

# Pure SUGRA: 1-center: RN Solution

- RN: $\longrightarrow$ TAKE $n_c = 1$. $Q$. Basis W$= (P_{\pm} Q)$.
  $\mathcal{I} = \mathcal{I}_{\infty} + \frac{Q}{r}$. Expand: $\mathcal{I}_{\infty} = \alpha^{0} P_{+} Q + \alpha^{\bar{0}} P_{-} Q$.

- $g_{0\bar{0}} = (i/2) \langle Sq \mid q \rangle$, $g^{0\bar{0}} = 1/g^{\star}_{0\bar{0}}$.

- BPS MASS FORMULA $(\langle S\mathcal{I}_{\infty} \mid \mathcal{I}_{\infty} \rangle = 1)$:

$$1 \quad = \quad \frac{M_{ADM}}{\langle SQ \mid Q \rangle}, \quad M^2_{ADM} = \langle SQ \mid Q \rangle = p^2_m + q^2_e$$

# Pure SUGRA: 2 center: Majumd.-Papapetrou

- $n_c = 2$:$q_1, q_2$,$2n_c >$dim $W$. Many Basis W$= (P_\pm q_i)$,$(P_\pm Q)$...

$\mathcal{I} = \mathcal{I}_\infty + \frac{\mathbf{q_1}}{|\mathbf{x}-\mathbf{x_1}|} + \frac{\mathbf{q_2}}{|\mathbf{x}-\mathbf{x_2}|}$.

- Expand $\mathcal{I}_\infty$: ($\longrightarrow$ 3 simultaneous Mass EQS).

$$
\begin{aligned}
\mathcal{I}_\infty &= \alpha^0 P_+ q_1 + \alpha^{\bar{0}} P_- q_1, \\
&= \alpha^0 P_+ q_2 + \alpha^{\bar{0}} P_- q_2, \\
&= \alpha^0 P_+ Q + \alpha^{\bar{0}} P_- Q,
\end{aligned}
$$

- M Formulas: **( $\mathbf{N_i = \langle \mathcal{I}_\infty \mid q_i \rangle}$, $\mathbf{N = N_1 + N_2 = 0}$, $\mathbf{N_1 = J/r_{12}}$)**

$$
\begin{aligned}
1 &= \frac{M_i^2 + N_i^2}{\langle S q_i \mid q_i \rangle}, \quad \longrightarrow M_i^2 + \frac{J^2}{r^2} = \langle S q_i \mid q_i \rangle \\
1 &= \frac{M_{ADM}^2 + N^2}{\langle S Q \mid Q \rangle}, \quad \longrightarrow M_{ADM}^2 = \langle S Q \mid Q \rangle
\end{aligned}
$$

$\longrightarrow$ CASES: STATIC ($J = 0$), STATIONARY ($J \neq 0$)

# MAJUMDAR-PAPAPETROU (Revisited)

- STATIC (Well-Known): $M_i = \langle Sq_i \mid q_i \rangle$, $M_{ADM}^2 = \langle SQ \mid Q \rangle$,

$$M_{ADM} = M_1 + M_2 \longrightarrow 1 \quad = \quad \frac{\langle Sq_1 \mid q_2 \rangle^2}{\langle Sq_1 \mid q_1 \rangle \langle Sq_2 \mid q_2 \rangle}$$

$\longrightarrow$ $q_1 = \lambda q_2$: (in particular $q_1, q_2$ only elec. or only magn.)
$\longrightarrow$ $r_{12}$ : unrestricted

- STATIONARY (Not so well Known): $M_1, M_2, r_{12}$ fixed by 3 EQs.

$$M_i^2 \quad = \quad \frac{\left(\langle Sq_i \mid q_i \rangle + \langle Sq_i \mid q_j \rangle\right)^2}{\langle SQ \mid Q \rangle}$$

$$r_{12}^2 \quad = \quad M_{ADM}^2 = \langle SQ \mid Q \rangle = p_m^2 + q_e^2$$

# EXAMPLE(2): AXION-DILATON MODEL, $F = -iX_0X_1$.

TOY MODEL: 1-scalar, 2 $A_\mu^{0,1}$: Bellorin:2006xr,F-Melgarejo:2013ksa.

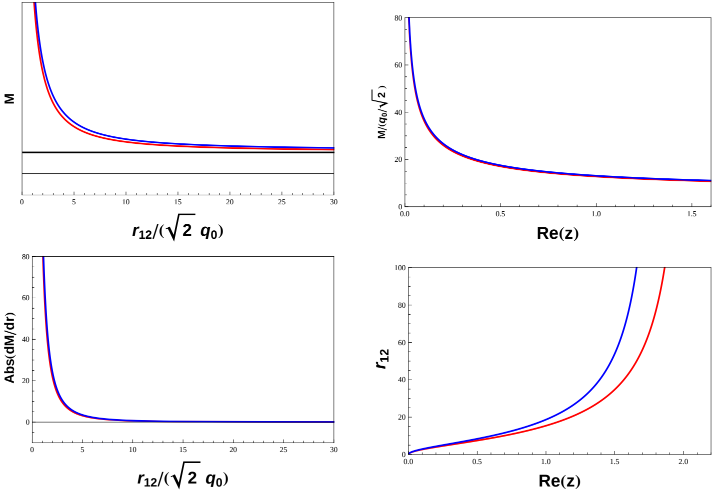
- **$F = -iX_0X_1$, S: scalar independent.**
  $\chi + ie^{-\phi} \equiv -iz$.
  $\mathcal{K} = -\log \mathrm{Re}\,(z)\,, \mathcal{G}_{z\bar{z}} = (2\mathrm{Re}\,(z))^{-2}$. $\mathrm{Re}\,(z) > 0$.

- EXAMPLE: $n_c = 2$. AT FIXED CHARGES:

  Params:5 $(M_1, M_2, r, z_\infty)$. EQS$= 3$: $(M_{ADM}, z_\infty)$.
  $\longrightarrow$ dof$= 2$

  Minimize $M_{ADM}(m_i)$: Eqs.$+1$, dof$= 1$
  $\longrightarrow$ Choose: $r_{12}, \mathrm{Re}\,(z)\,, \mathrm{Im}\,(z)\,\ldots$.

# AXION-DILATON, 2 center.




FIX: $q_1 = (1, 8, 0, -1)q_0$ ,$q_2 = (1, 8, -4, 1)q_0$,
$J^2 > 0$. $A_\infty > 0, -\det(S) > 0$. $m_i^{min} = (9/16, 7/16)$.

# EXAMPLE(3): Stringy models. STU Models

Stringy models: cubic $\mathbf{F = d_{ABC} X^A X^B X^C / X^0}$,
Simplest, STU model: $\mathbf{F = -\frac{X^1 X^2 X^3}{X^0}}$.
$\longrightarrow$ 3 complex scalars: $dimW = 2n_v + 2 = 8$. $S(F)$

- **CASE 2 centers. Basis $\mathbf{W = (P_{\pm} q_{1,2}, P_{\pm} s_{1,2})}$**
  $\sharp$Parms $= 13$: $M_1, M_2, r_{12} : 3, \lambda^i : 4, z^\alpha_\infty : 6$.
  $\sharp$Eqs(7): Mass:1,$z : 6$. dof=6. ($\longrightarrow$ 3 $z^\alpha_\infty$).
  Minimal Mass(1): dof=5.

- **CASE 4 centers. Basis $\mathbf{W = (P_{\pm} q_{1,2,3,4})}$**
  $\sharp$Parms $= 16$: $M_i, r_{ij} : 10, z^\alpha_\infty : 6$.
  $\sharp$Eqs(7): Mass:1,$z : 6$. dof=9 ($\longrightarrow$ ...+ 3 $z^\alpha_\infty$).
  Minimal Mass(3): dof=6.

# Conclusions

- Multicenter BHs in $N2$ $D4$ SUGRA $+n_v$ vector multip.

$\longrightarrow$ new BPS mass formulas

–Properties, Construction and existence of BH multicenter solutions

–Counting of microstates (in stringy inspired models)

- EXAMPLE: minimal $M_{ADM}$, 2 center BPS solutions

$$\mathbf{\mid Z \mid^2 = M^2 \;=\; A_\infty \left(1 + \alpha J^2 \left(1 + \frac{2M}{r} + \frac{A_\infty}{r^2}\right)\right)}$$

- Continuos family, for $q_i$, $r \in (0, \infty)$ ,$M_{ADM} \in (\infty, M_\infty)$ and $z_\infty$
- Minimal mass (Mass Gap)
- Range of values of $m_i$ allowed. etc..

- N2 d4 SUGRA: specially suitable: many restrictions. $\mathcal{M}_z =$ Sp. Kahler.

**BUT SIMPLE ESSENTIAL INGREDIENTS: BPS condition, asymptotical flatness... $\longrightarrow$ generalizable other contexts.**

## $\mathcal{S}$ and Freudenthal transformations

- $\mathcal{S}$-transformations

$$X \quad \longrightarrow \quad X' = f(\mathcal{S})X, \qquad\qquad f(\mathcal{S}) = a + b\mathcal{S} \equiv \lambda e^{\theta\mathcal{S}}$$

  - $\langle X' \mid Y' \rangle = \lambda^2 \langle X \mid Y \rangle$

- Freudenthal dual

Borsten,Dahanayake,Duff,Ruben'09
Ferrara,Marrani,Yeranyan'11

$$X \quad \longrightarrow \quad \tilde{X} = T(X)|\Delta_4(X)|^{-1/2}, \qquad\qquad \tilde{\tilde{X}} = -X$$

such that $\Delta_4(\tilde{X}) = \Delta_4(X)$, where $S_{4,bh} = \pi\sqrt{|\Delta_4(x)|}$

- Matching Freudenthal/$\mathcal{S}$ transformations

$$\tilde{X} \equiv \exp\left(\tfrac{\pi}{2}\mathcal{S}\right) X$$

# $\mathcal{S}$-transformations

- Scalings

$$X \longrightarrow \lambda e^{\theta \mathcal{S}} X \Rightarrow \begin{cases} S & \to & \lambda^2 S \\ M_{ADM} & \to & \lambda M_{ADM} \\ r_{ab} & \to & \lambda r_{ab} \end{cases}$$

- Quartic invariant

$$\Delta_4(X) = \Delta_4(\tilde{X}) = \frac{1}{4} \langle X \mid X \rangle^2$$

- Generalized Freudenthal transformation

$$\Delta_4(aX + b\tilde{X}) = (a^2 + b^2)^2 \Delta_4(X)$$

# Freudenthal triple system

- Definition. A Freudenthal Triple System (FTS) is axiomatically defined as a finite dimensional vector space $\mathfrak{F}$ over a field $\mathbb{F}$, such that:

  1. $\mathfrak{F}$ possesses an antisymmetric bilinear form $\{x, y\}$
  2. $\mathfrak{F}$ possesses a symmetric four-linear form $\Delta(x, y, z, w)$
  3. $\mathfrak{F}$ possesses a ternary product $T(x, y, z)$ defined by $\{T(x, y, z), w\} = \Delta(x, y, z, w)$. Then $3\{T(x, x, y), T(y, y, y)\} = \{x, y\}\Delta\{x, y, y, y\}$

- Freudenthal dual: $\tilde{X} = T(X)|\Delta(X)|^{-1/2}$, $\tilde{\tilde{X}} = -X$
- $\Delta_4(X)$: example of $\Delta$, where $\Delta_4(X) \equiv \Delta(X, X, X, X)$

# Jordan algebras

- Definition. A Jordan algebra $\mathfrak{J}$ is a vector space over a ground field $\mathbb{F}$ equipped with a bilinear product such that $\forall X, Y \in \mathfrak{J}$,

$$X \bullet Y = Y \bullet X$$
$$X \bullet (X \bullet Y) = X \bullet (X^2 \bullet Y)$$

- Integral cubic Jordan algebra, cubic form $N : \mathfrak{J} \longrightarrow \mathbb{F}$

$$N(X, Y, Z) := \frac{1}{6}[N(X+Y+Z) - N(X+Y) - N(X+Z) - N(Y+Z) + N(X) + N(Y) + N(Z)]$$

- Define a quadratic map $^\sharp : \mathfrak{J} \longrightarrow \mathfrak{J}, \quad Tr(X^\sharp, Y) = 3N(X, X, Y)$

- Jordan dual: $X^* = X^\sharp N(X)^{-1/3}, \qquad X^{**} = X$

- $S_{5,\text{black string}} = 2\pi\sqrt{N(X)} \qquad S_{5,\text{black hole}} = 2\pi\sqrt{N(Y)}$

# 4D/5D uplift

- Given $(\mathfrak{J}, N)$, one can construct an integral FTS,

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J}$$

- Duality scheme

Borsten,Dahanayake,Duff,Ruben'09
Gaiotto,Strominger,Yin'05

$$\begin{array}{ccc} \text{4D black hole } x & \xrightarrow{\text{4D/5D lift}} & \text{5D black string } A \sim \tilde{B}^\star \\ \text{Freudenthal dual} \big\downarrow & & \big\downarrow \text{Jordan dual} \\ \text{dual 4D black hole } \tilde{x} & \xrightarrow[\text{4D/5D lift}]{} & \text{dual 5D black hole } \tilde{B} \sim A^\star \end{array}$$

# 4D/5D uplift

- Given $(\mathfrak{J}, N)$, one can construct an integral FTS,

$$\mathfrak{F}(\mathfrak{J}) = \mathbb{F} \oplus \mathbb{F} \oplus \mathfrak{J} \oplus \mathfrak{J}$$

- Duality scheme

Borsten,Dahanayake,Duff,Ruben'09

Gaiotto,Strominger,Yin'05

4D black hole $x$ $\xrightarrow{\text{4D/5D lift}}$ 5D black string $A \sim \tilde{B}^\star$

Generalised ↓ Freudenthal dual — Generalised ↓ Jordan dual

dual 4D black hole $ax + b\tilde{x}$ $\xrightarrow[\text{4D/5D lift}]{}$ dual 5D black hole $\alpha\tilde{B} + \beta A^\star$