Nuclei as Near-BPS Skyrmions

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Introduction

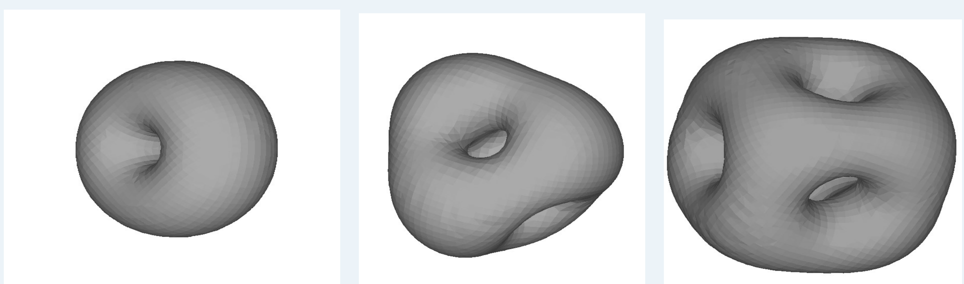
Despite the successes of the Standard Model, we still have no clear explanation on how quarks and gluons form nucleons and nuclei. The Skyrme Model provides an alternative. It is a low-energy QCD effective meson field theory where baryon emerges as topological solitons. However, the Skyrme Model seems unable to reproduce the small binding energy in nuclei which it overestimates by at least an order of magnitude; otherwise, it remains a relatively accurate picture of the nucleons. This suggests that Skyrme-like models that nearly saturate the Bogomol'nyi bound may be more appropriate since their mass is roughly proportional to the baryon number.

Recently, there has been considerable interest in these so-called Near-BPS Skyrme Models [3,4]. Here, we propose an improved Near-BPS Skyrme Model[5]: It consists of terms up to order six in derivatives of the pion fields, including the nonlinear and Skyrme terms which are assumed to be relatively small. Our choice of potential term leads to well-behaved analytical solutions in BPS limit of the model which give rise to approximately constant baryon density configurations, as opposed to the usual shell-like configurations found in most extensions of the Skyrme Model. We used these solutions to evaluate the masses of nuclei. Fitting the 4 model parameters, we find a remarkable agreement for the binding energy per nucleon B/A with respect to experimental data.

Motivation

One of the most attractive ideas to describe low-energy QCD physics is the Skyrme Model[1], an effective meson (pion) field theory motivated by arguments such as $1/N_c$ expansion or more recently holographic QCD[2]. Although the model is usually constructed out of meson fields alone, it can describe baryons and nuclei: they arise as topological solitons (Skyrmions) whose winding number correspond to the baryon number. Indeed, the Skyrme Model:

- Provides an effective meson field theory motivated by low-energy QCD.
- Describes pion and baryon physics within at least 30% accuracy (often to a few %).
- But, fails at multibaryon physics or nuclei.
- Binding energies too large especially for small nuclei (e.g. deuteron $\simeq 40 \times$ observed value).
- Finding lowest energy configurations is numerically challenging so that only some low A solutions are known (e.g. toroidal, tetrahedral, cubic configurations for $A = 2, 3, 4$ Skyrmions, respectively)



$A = 2$ $A = 3$ $A = 4$

- Solutions are not consistent with constant densities observed in nuclei.
- Further analysis of various potential (mass) terms, rotational deformations, higher order terms in derivatives or additional mesons (e.g. ω, ρ, ...) lead to similar configurations and binding energies.

Hint: The mass of of a nucleus almost proportional to that of the nucleon, $M_{\text{nuclei}} \approx A \cdot M_{\text{nucleon}}$, and that BPS-solitons follows this exact pattern, so we propose a Skyrme-like model in a regime where the solutions are **Near-BPS solitons**. In this work, we further fine tune the model in order to reproduce another nuclear property: **constant baryon density.**

Near-BPS Skyrme Models

We consider an extension of the original Skyrme Model with the Lagrangian density

$$\mathcal{L}_{\text{NBPS}} = \underbrace{\mathcal{L}_0 + \mathcal{L}_6}_{\text{BPS-solitons}} \quad + \quad \underbrace{\mathcal{L}_2 + \mathcal{L}_4}_{\text{Skyrme}} \tag{1}$$

such that solutions remain close to saturation of Bogomol'nyi bound without losing link with the Skyrme Model. The pion fields are introduced through the $SU(2)$ matrix $U = \phi_0 + i\tau_i\phi_i$ with $\phi_0^2 + \phi_i^2 = 1$. Then using $L_\mu = U^\dagger \partial_\mu U$, we have

- $\mathcal{L}_0 = -\mu^2 V(U)$: the potential term
- $\mathcal{L}_2 = -\alpha \text{Tr}\left[L_\mu L^\mu\right]$: the NLσ term, quadratic in field derivatives.
- $\mathcal{L}_4 = \beta \text{Tr}\left(\left[L_\mu, L_\nu\right]^2\right)$: the quartic Skyrme term (necessary to stabilize soliton in the Skyrme Model).
- $\mathcal{L}_6 = -\frac{3}{2}\frac{\lambda^2}{16^2} \text{Tr}\left(\left[L_\mu, L_\nu\right]\left[L^\nu, L^\lambda\right]\left[L_\lambda, L_\mu\right]\right)$: the sextic term which remains **quadratic** in time derivatives.

The Near-BPS approch consist in assuming that $\mathcal{L}_0 + \mathcal{L}_6$ dominate and treat $\mathcal{L}_2$ and $\mathcal{L}_4$ are small as perturbations.

Finite energy solutions requires a conserved topological charge which Skyrme identified as the baryon number or mass number A in the context of nuclei

$$A = \int d^3r \mathcal{B}^0 = -\frac{\epsilon^{ijk}}{24\pi^2} \int d^3r \text{Tr}\left(L_i L_j L_k\right). \tag{2}$$

When $\alpha, \beta = 0$, the solutions are BPS-solitons so their masses are exactly proportional to A.

Here, we choose for simplicity an axially symmetric ansatz for U (a BPS solutions in the limit where $\alpha = \beta = 0$)

$$U = \cos F(r) + i\hat{\mathbf{n}} \cdot \tau \sin F(r) \tag{3}$$

where $\hat{\mathbf{n}}$ is the unit vector can be written in terms of the spherical coordinates r, θ, and ϕ.

$$\hat{\mathbf{n}} = \left(\sin\theta\cos A\varphi, \sin\theta\sin A\varphi, \cos\theta\right). \tag{4}$$

Near-BPS Skyrme Models...

Note that for $B > 1$, this is not the absolute static energy minimizer for $\alpha, \beta \neq 0$; they have been conjectured to emerge from "restricted harmonic map"[6]. On the other hand, the Skyrme Model ($\mathcal{L}_2 + \mathcal{L}_4$) leads to "rational map" solutions in which the profile F has no angular dependence. The ansatz (4) preserves this property and should anyhow provide a close upper bound of the static energy for small α, β.

The model has been refined over the last few years:

1. BPS Skyrme Model **[ASW]**[3]: consists of $\mathcal{L}_0 + \mathcal{L}_6$ alone ($\alpha = \beta = 0$) with usual mass term
$$V(U) = -(1/2)Tr\left[I - U\right] = -T_-$$
where $T_\pm = \text{Tr}\left[(2I \pm U \pm U^\dagger)/8\right]$. The solutions are compactons. They saturate the Bogomol'nyi bound leading to zero binding energies and unstable nuclei.
2. Near-BPS Skyrme Model **[BoM]**[4]**:** The choice of potential
$$V_{\text{BoM}}(U) = -T_+ T_-^3$$
gives shell-like baryon density configuration.
3. Modified Near-BPS Skyrme Model **[BHM]**[4]**:** Using the potential
$$V_{\text{BHM}}(U) = -\frac{8}{9}\frac{T_+ T_-^3}{\ln(T_-)},$$
one improves the baryon density which turns out to be gaussian-like.

An Improved Near-BPS Model

We are interested in a model that can reproduce the constant baryon density in nuclei. We propose an **improved the Near-BPS Model [BeM]**[5]**:**

Our approach proceeds through of the following steps:

Step 1: Introduce an appropriate $V(U)$ in order to generate constant baryon density. We found by inspection that

$$V_{\text{BeM}}(U) = \frac{224}{45} T_+ T_-^3 \frac{(1 - (14/5)\ln(T_-))}{1 - \sqrt{1 - (14/5)\ln(T_-)}} \tag{5}$$

obeys this criterion.

Step 2: Find the **analytical solution (**$\alpha = \beta = 0$**)** where $x = ar$ with $a = \left(\mu/(18A\lambda)\right)^{1/3}$

$$F_{\text{BeM}}(x) = \arcsin[\exp(-x^2 - 7x^4/5)] \tag{6}$$

The proposed model with $V_{\text{BeM}}(U)$ leads to a constant baryon density.

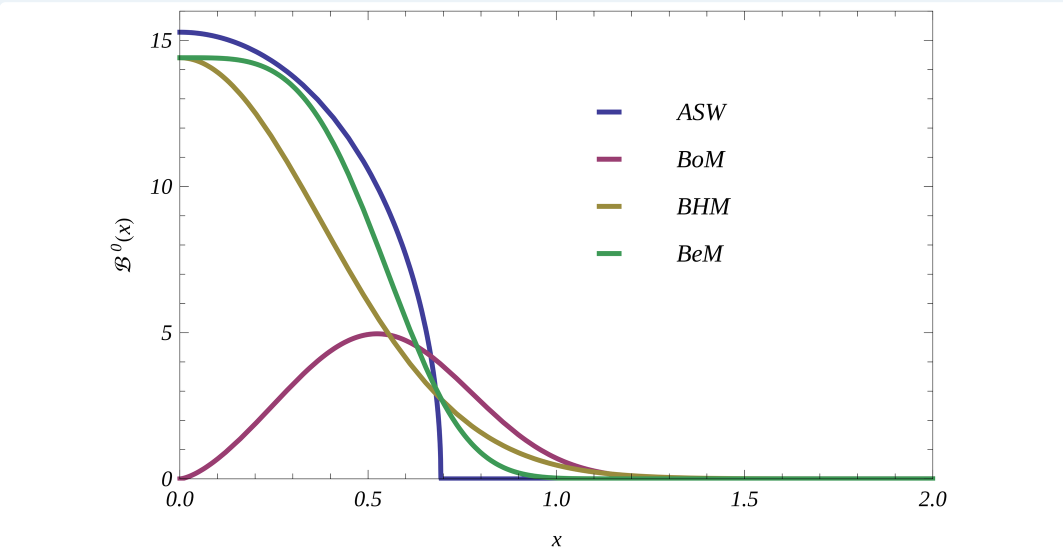


Figure : Baryon densities $\mathcal{B}^0(x)$ for various models ASW, BoM, BHM, and the constant baryon density Near-BPS model BeM.

Step 3: Relax the constraint α, $\beta = 0$ and include $\mathcal{L}_2 + \mathcal{L}_4$ as small perturbations. The static energy then reads

$$E_{\text{s}} = 15.93\,\mu\lambda\,A + \alpha a^{-1}\left(a_2 + b_2 A^2\right) + \beta a\left(a_4 + b_4 A^2\right)$$

where $a_2 = 79.746$, $b_2 = 12.189$, $a_4 = 1033.1$, $b_4 = 1411.5$. Note that E_{s} grows like $\alpha A^{7/3}$ and $\beta A^{5/3}$ but for α,β small enough, the nuclei remain bound states for a $A < 250$.(see next figure).

Step 4: Add the rotational energy

$$E_{\text{r}} = \frac{1}{2}\left[\frac{j(j+1)}{V_{11}} + \frac{i(i+1)}{U_{11}} + \left(\frac{1}{U_{33}} - \frac{1}{U_{11}} - \frac{A^2}{V_{11}}\right)\kappa^2\right]$$

where U_{33}, U_{11}, V_{11} are (iso)rotational moments of inertia.

Step 5: Add Coulomb energy E_{C} from charge density $\rho(\mathbf{x}) = J^0_{EM} \equiv \frac{1}{2}\mathcal{B}^0(\mathbf{x}) + i_3\frac{\mathcal{U}_{33}(\mathbf{x})}{U_{33}}$.

Step 6: Add isospin breaking term E_{I} to account for proton-neutron mass difference using

$$E_{\text{I}} = a_I i_3$$

Step 7: The values of the parameters μ, α, β and λ remain to be fixed. This is done by fitting these 4 parameters μ, α, β, and λ using our result for the nuclear mass

$$E_{\text{t}} = E_{\text{s}} + E_{\text{r}} + E_{\text{C}} + E_{\text{I}}. \tag{7}$$

Results

We consider three approches to set the parameters of of the model μ, α, β, and λ :

Fit I: We set $\alpha = \beta = 0$ and fit μ and λ using the mass of the nuclei H and ^{40}Ca. This corresponds to the limit where the minimization of the static energy leads to the exact analytical BPS solution in (6) and provides a good estimate for the values of μ, α, β, and λ required in Fits II and III.

Fit II: The fit optimizes the 4 parameters to better reproduce the masses of 140 most stable isotopes.

Fit III: Same as Fit II but here we choose to optimize the binding energy per nucleon B/A.

	Fit I	Fit II	Fit III
μ (10^4 MeV^2)	1.23223	1.02259	1.33515
α (10^{-3} MeV^2)	0	1.48244	0.508933
β (10^{-8} MeV^0)	0	1.20427	1.31582
λ (10^{-3} MeV^{-1})	4.74078	5.70373	4.36994

Results...

One can then compute the masses of the nuclei using Eq. (7).The results are presented in the figure below which displays the general behavior of B/A as a function of the baryon number for our model [BeM] for the 3 sets of predictions.

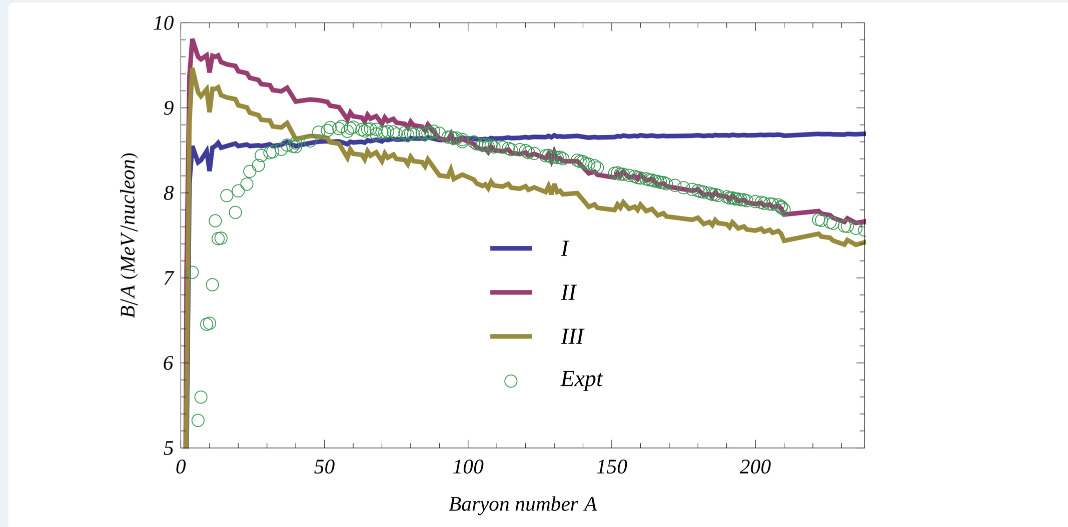


Figure : Binding energy per nucleon B/A r for model BeM shown for the parameters of Fits I, II, III, and experimental values.

We find that:

- Mass predictions that are accurate to at least 0.4%, even for heavier nuclei
- Predictions for binding energy per nucleon B/A, that are within 10% of expt. values. Fit II is almost exact for $B \gtrsim 50$ nuclei.
- The model shows an significant improvement over predictions of the Skyrme model which overestimates the B/A by at least an order of magnitude.

It is interesting to compare how the contributions from static, (iso)rotational, Coulomb, and isospin breaking energies affect binding energy per nucleon B/A.(next figure). Note that these results emerge from a low-energy QCD effective meson field theory but are arguably better than that of the more empirical Bethe-Weizsäcker mass formula.

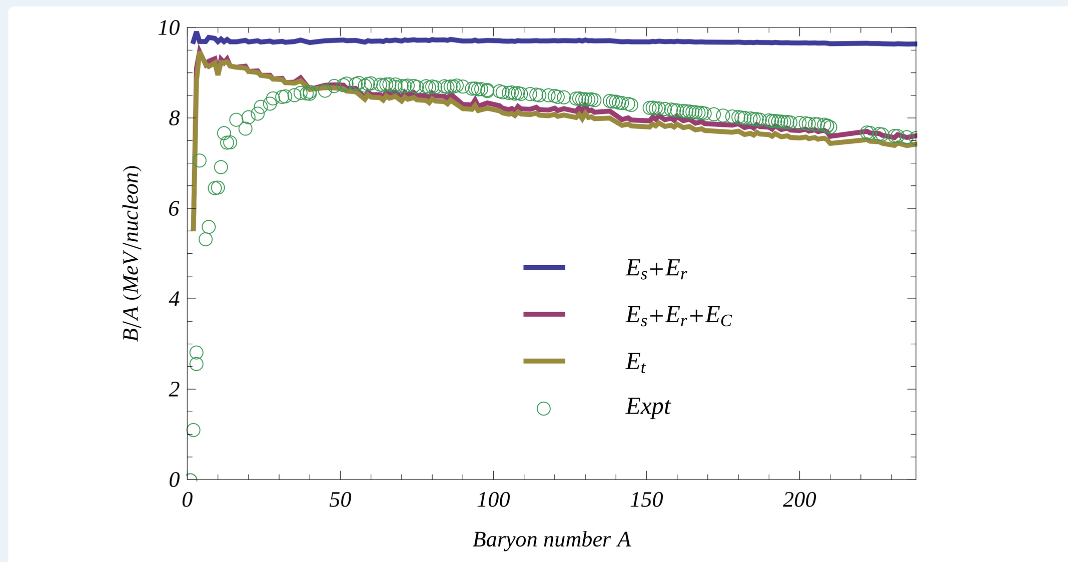


Figure : Static E_s, (iso)rotational E_r, Coulomb E_C and total E_ti contributions to B/A shown for Fit III anlong with experimental values.

Further analysis is required:

- The size of nuclei or rms radius for the charge density is, as observed, proportional to $A^{1/3}$ but is slightly too large:
$$\left\langle r^2_{\text{em}}\right\rangle^{1/2} = (1.90 \text{ fm})\,A^{1/3} \qquad (\text{Expt} = (1.23 \text{ fm})\,A^{1/3}).$$
- Somewhat related is the rather small (iso)rotational energy, especially for large nucleus. It predicts for example that the mass difference between the Δ and the nucleon is $M_\Delta - M_N \sim 50$ MeV.
- Given our choice of potential and parameters, we find a very small pion decay constant and no pion mass
$$F_\pi = 0.09 \text{ MeV} \qquad (F_\pi^{\text{expt}} = 186 \text{ MeV})$$
$$m_\pi = 0 \qquad (m_\pi^{\text{expt}} = 138 \text{ MeV})$$
so the link to soft-pion physics is yet unclear.
- The solution that minimizes the static energy E_{s} or.the mass of the nuclei E_{t} for a given A remains unknown.
- Several aspects of the model remain to be studied (e.g. magnetic moments, vibrational and rotational excitations....).

Conclusion

Near-BPS Skyme Models add both the NLσ terms and the Skyrme term to the BPS model in order to retain some of the successes of the Skyrme Model. We approximate the lowest energy solution by an analytical axial solution, that of the BPS Model, and compute directly the relevant physical quantities. This is a big advantage over the original Skyrme Model where it remains a challenge to even find the lowest energy solution.

In this work, we show that it is possible to construct constant baryon and charge densities for all A. Although it remains a prototype model, it leads to a remarkably accurate description of B/A and other properties of the nuclei. More generally, it clearly supports the idea that nuclei could be Near-BPS Skyrmions.

References/Acknowledgements

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