SOME RECENT DEVELOPMENTS IN QCD RESUMMATION

Eric Laenen

UNIVERSITY OF AMSTERDAM NIKHEF Universiteit Utrecht

Work with D. Bonocore, P. Falgari, E. Gardi, K. Larsen, L. Magnea, S. Melville, G. Stavenga, L. Vernazza, M. Volponi, C. White

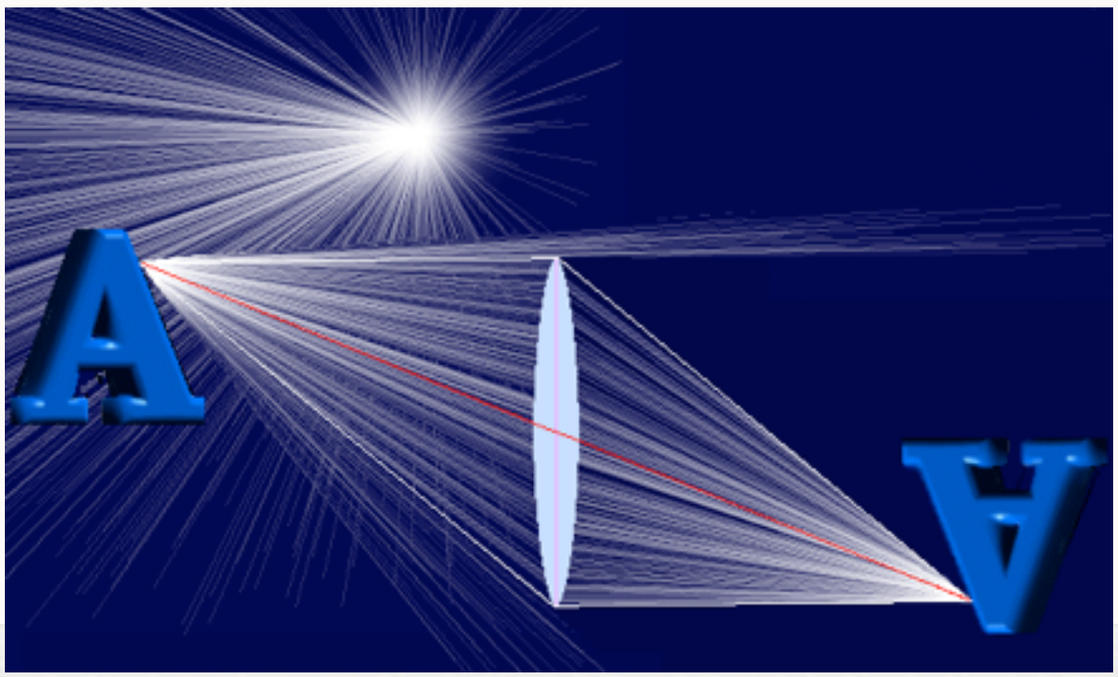
IFIC Valencia, Oct. 22, 2013

HISTORY OF EIKONAL APPROXIMATION

- "Eikon" originally from Greek εικεναι [to resemble]
  - leading to εικον [icon, image]
- Predates quantum mechanics, even Maxwell
  - also known in optics as "ray optics"
    - Rays are straight lines, perpendicular to wave fronts

RAY OPTICS

- Can describe formation of images/eikons
  - wavelength [of emitter] << size of scatterer [recoil agent]
- Cannot describe diffraction, polarization etc
  - these are wave phenomena

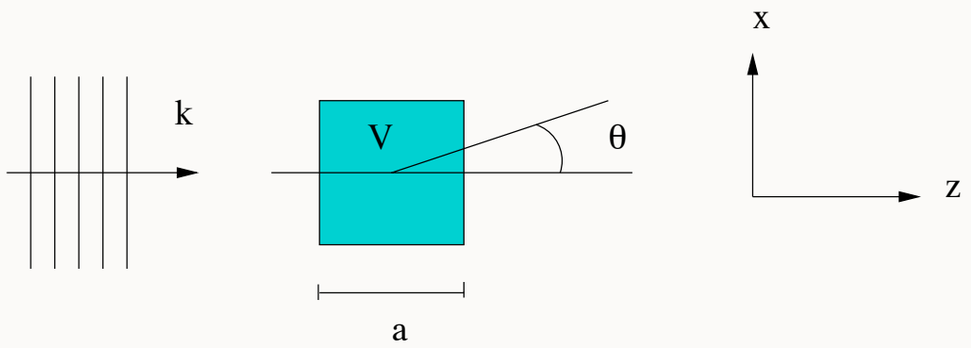


EIKONAL QUANTUM MECHANICS

- Eikonal approximation in QM scattering
  - R.J. Glauber [1959 lecture notes (recommended!)]
  - Highly energetic particle scattering off potential
  - (found use in scattering on nucleus)
  - 2D example
    - scattering on square potential of limited range

QM EIKONAL SCATTERING (IN 2D)

Glauber, 1959 Boulder Lectures



Incoming wave

Scattered wave

$$\psi(x) = e^{ikz} + \frac{\sqrt{k}}{\sqrt{2\pi}} e^{i\frac{\pi}{4}} f(\theta) \frac{e^{ikr}}{\sqrt{r}}$$

Scattering amplitude

Factor high-momentum part

$$\psi(\vec{r}) = e^{ikz}\phi(\vec{r})$$

$$E \gg V \qquad k \gg 1/a$$

Substitute in Schrodinger equation

$$e^{ikz}(-k^2 + 2ik\partial_z + \cancel{\nabla^2} + k^2)\phi(\vec{r}) = e^{ikz}\frac{2m}{\hbar^2}V(\vec{r})\phi(\vec{r})$$

Solve

$$\phi(\vec{r}) = e^{-\frac{i}{\hbar v}\int_{-\infty}^{z} dz' V(x,z')}$$

Exponential form!

QM EIKONAL SCATTERING IN 2D

- Approximation at amplitude level, some "wave" information is preserved. If potential is "black":
  - Optical theorem $\sigma_{tot} = 2\,\mathrm{Im}\, f(\theta = 0) = 2a$
  - Scattering cross section $\sigma_{scattered} = \frac{k}{2\pi} \int d\theta\, |f(\theta)|^2 = a$
  - Resolution: diffraction, shadow of target must be filled in

EIKONAL APPROXIMATION IN QFT

- At amplitude level
  - Reveals new symmetries, new structures in gauge theory
    - Heavy quark spin symmetry in HQET
  - Intuitive interpretation
  - Practical use!
    - Coherence phenomena, resummation, EFT, ...

OUTLINE

- Introduction
- Exponentiation
  - Diagrammatics
  - Path-integrals
  - Next-to-eikonal
  - Drell-Yan check
  - Webs for multiple parton scattering, Imaginary webs
- Summary & outlook

EXPONENTIAL SERIES IN CROSS SECTIONS

When good logs go bad - J. Huston Resummer: log whisperer?

- Typical behavior of cross section $\hat{O}_2 = 1 + \alpha(L^2 + L + 1) + \alpha^2(L^4 + L^3 + L^2 + L + 1) + \ldots$
  - "1" = π^2, 2, etc
  - Effective expansion parameter is αL^2
  - Fix: reorganize/***resum*** such that

$$\begin{aligned}\hat{O} &= 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \\ &= \exp\left(\underbrace{\underbrace{L g_1(\alpha_s L)}_{LL} + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots}_{NLL}\right) \underbrace{C(\alpha_s)}_{\text{constants}} \\ &+ \text{ suppressed terms}\end{aligned}$$

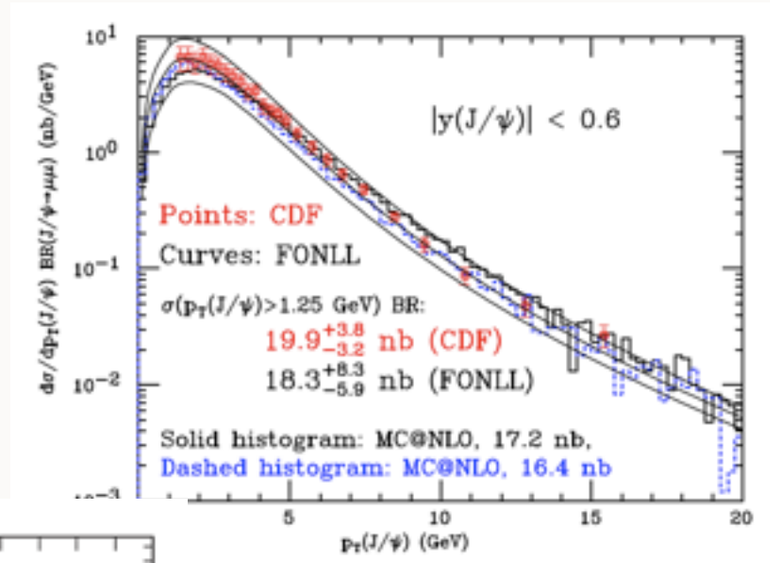
- Just as systematically improvable as PT
  - LO, NLO $\rightarrow$ LL, NLL, etc

RESUMMATION OF WHAT?

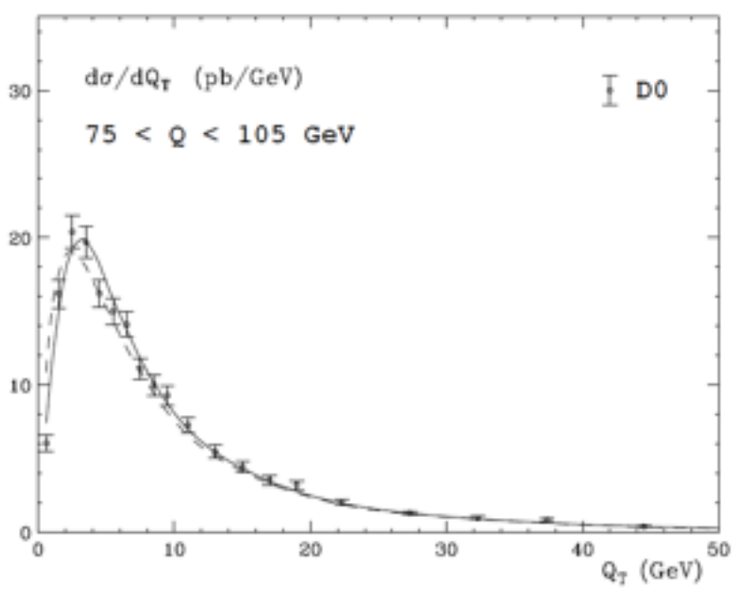
- So many logs...

$$\ln(1-T) \quad \ln(p_T/m_Z) \quad \ln(k_T) \quad \ln(1/x) \quad \ln(1-x) \quad \ln(N) \quad \ln(\mu) \quad \ln(b)$$

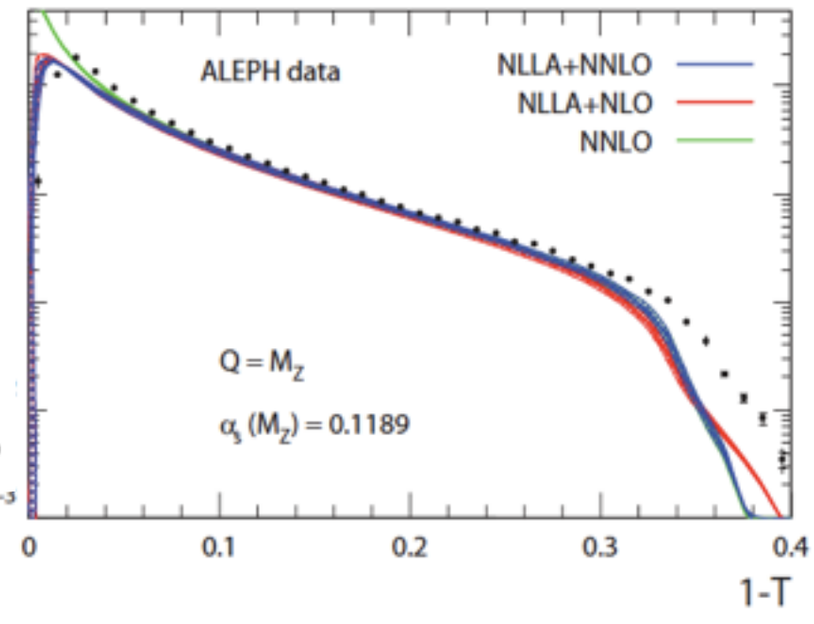
pT of b @ Tevatron



pT of Z @ Tevatron



Thrust T @ LEP



- Many, many results

RESUMMATION IN A FEW EASY STEPS

- Cross section for n extra gluons

Phase space measure **Squared matrix element**

$$\sigma(n) = \frac{1}{2s} \int d\Phi_{n+1}(P, k_1, \ldots, k_n) \times |\mathcal{M}(P, k_1, \ldots, k_n)|^2$$

- When emissions are soft, approximate phase space measure and matrix element [eikonal approximation]

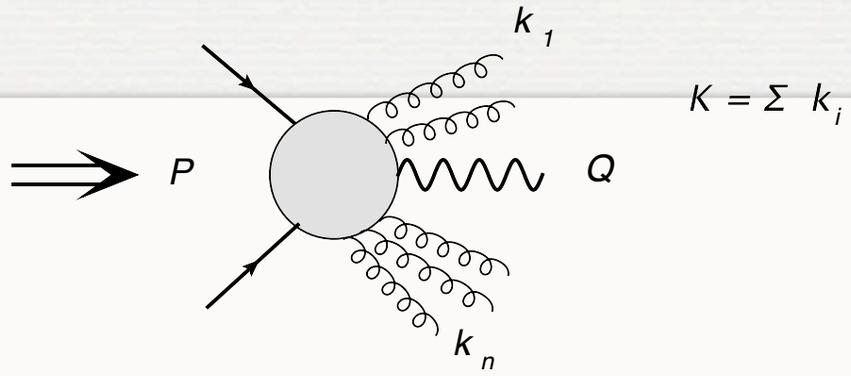
$$d\Phi_{n+1}(P, k_1, \ldots, k_n) \longrightarrow d\Phi(P) \times \Big(d\Phi_1(k)\Big)^n \frac{1}{n!} \qquad |\mathcal{M}(P, k_1, \ldots, k_n)|^2 \longrightarrow |\mathcal{M}(P)|^2 \times \Big(|\mathcal{M}_{1\text{ emission}}(k)|^2\Big)^n$$

- Sum over all orders

$$\sum_n \sigma(n) = \sigma(0) \times \exp\Big[\int d\Phi_1(k) |\mathcal{M}_{1\text{ emission}}(k)|^2\Big]$$

- Incorporate Theta or Delta functions for specific cases
  - but these must factorize similarly, or they cannot go into exponent

PHASE SPACE CONSTRAINTS



- Kinematic condition expresses "z" in terms of gluon energies

$$s = Q^2 - 2P\cdot K - K^2 \qquad\qquad \delta\Big(1 - \frac{Q^2}{s} - \sum_i \frac{2k_i^0}{\sqrt{s}}\Big)$$

- Transform factorizes the phase space

$$\int_0^\infty dw\, e^{-w\,N}\,\delta\Big(w - \sum_i w_i\Big) = \prod_i \exp(-w_i N)$$

- So can go into exponent

$$\sum_n \sigma(n) = \sigma(0) \times \exp\Big[\int d\Phi_1(k)|\mathcal{M}_{1\text{ emission}}(k)|^2(\exp(-w\,N) - 1)\Big]$$

  - Large logs: log(N)

- Exponentiation of amplitude key

RESUMMATION AND FACTORIZATION

- Very generically, if a quantity factorizes, one can resum it
  - Renormalization; factorizes UV modes $G_B(g_B,\Lambda,p) = Z\left(\frac{\Lambda}{\mu}, g_R(\mu)\right) \times G_R\left(g_R(\mu), \frac{p}{\mu}\right)$
  - Evolution equation $\mu\frac{d}{d\mu}\ln G_R\left(g_R(\mu),\frac{p}{\mu}\right) = -\mu\frac{d}{d\mu}\ln Z\left(\frac{\Lambda}{\mu}, g_R(\mu)\right) = \gamma(g_R(\mu))$
  - Solving = resummation $G_R\left(\frac{p}{\mu}, g_R(\mu)\right) = G_R\left(1, g_R(p)\right) \underbrace{\exp\left[\int_p^\mu \frac{d\lambda}{\lambda}\gamma(g_R(\lambda))\right]}_{\text{resummed}}$
- Type of factorization dictates resummation
  - small x $\rightarrow$ k_T factorization
  - large x $\rightarrow$ near-threshold factorization
- Factorization is essentially separating degrees of freedom
  - Systematic approach in Soft Collinear Effective Theory
    - Match&run: run with RG equations

Bauer, Fleming, Pirjol, Stewart

RESUMMED CROSS SECTIONS

Sterman; Catani, Trentadue

Threshold-resummed Drell-Yan cross section

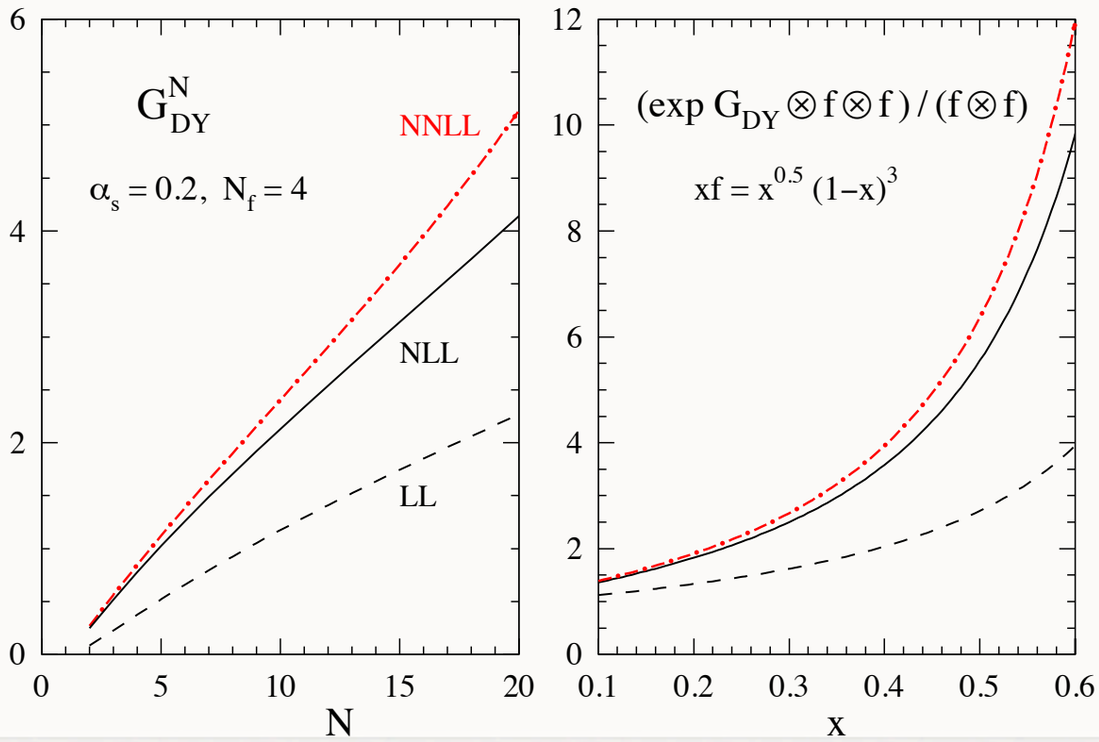
$$\frac{d\sigma^{\rm resum}}{dQ^2}(z) = \int_C \frac{dN}{2\pi\, i}\, z^{-N}\, \hat{\sigma}(N)$$

$$\sigma(N) = \exp\left[-\int_0^1 dx\, \frac{x^{N-1}-1}{1-x} \left\{ \int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu}\, A(\alpha_s(\mu)) + D(\alpha_s((1-x)Q)) \right\} \right] \times (1 + \alpha_s(Q^2)\frac{C_F}{\pi} + \ldots)$$

A. Vogt

$$\hat{\sigma}_{DY}(N,Q^2) = g_0(Q^2)\, \exp\left[G^N_{DY}(Q^2)\right]$$

$$G^N_{DY} = \ln N g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \ldots, \qquad \lambda = \beta_0 \alpha_s \ln N$$



Good convergence in exponent

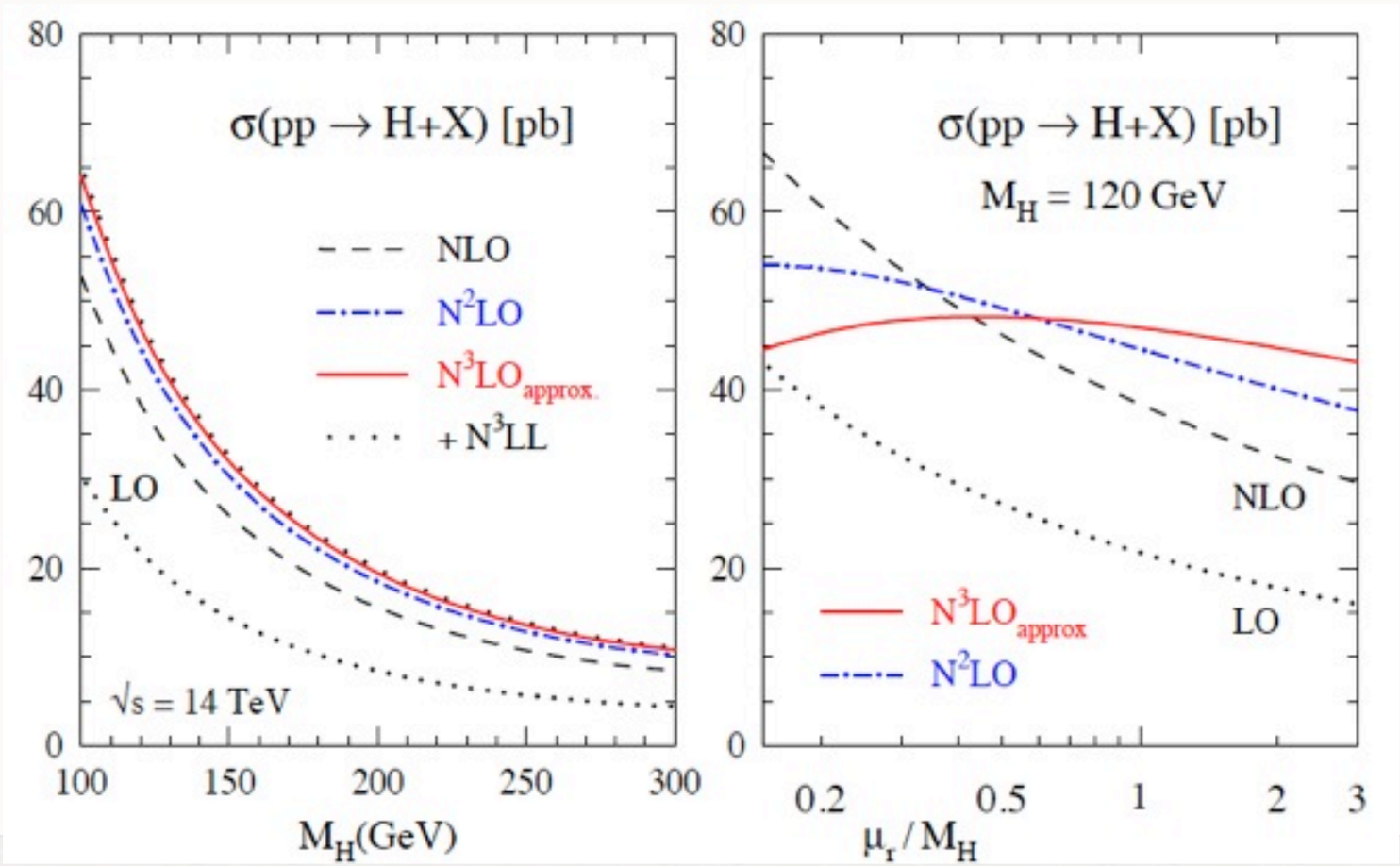
NNNLL THRESHOLD DY AND HIGGS

$$\sigma(N) = \exp\left[\int_0^1 dz\, \frac{z^{N-1}-1}{1-z}\Big\{\int_{\mu_F}^{Q(1-z)} \frac{d\mu}{\mu} A(\alpha_s(\mu)) + D(\alpha_s(Q(1-z))\Big\}\right]$$

EL, Magnea; Moch, Vermaseren Vogt; Ravindran; Idilbi,Ji, Ma, Yuan

$$\underbrace{D(\alpha_s)}_{\text{Single logs}} = 4 \quad \underbrace{B_\delta(\alpha_s)}_{\text{virtual split.fn.}} \quad -2 \quad \underbrace{G(\alpha_s)}_{\text{form factor}} \quad +\beta(\alpha_s)\frac{d}{d\alpha_s}F_{\rm MS}(\alpha_s)$$

Moch, Vermaseren, Vogt



W/Z/H/γ^* RESUMMATION

A large body of work exists...

- N^kLL for logs, k = 1,2,3

Sterman; Catani, Trentadue, Grazzini, de Florian, Nason, Mangano,Forte, Ridolfi, Vogelsang, Kidonakis, EL, Magnea, Moch, Vogt, Ravindran, Becher, Neubert, Ji, Idilbi, Kidonakis...

- Constants

Parisi, Magnea, Sterman, Balulev, Raduyshkin, Stefanis, Kraemer, EL, Spira, Eynck. Ahrens, Becher, Neubert,Yang,..

- Can we do more, possibly with new approaches?

GENERIC LARGE X BEHAVIOR

- For DY, DIS, Higgs, singular behavior when x→ 1

$$\delta(1-x) \qquad \left[\frac{\ln^i(1-x)}{1-x}\right] \qquad \ln^k(1-x)$$

  - singularity structure for plus distributions is organizable to all orders, perhaps also for divergent logarithms?

- After Mellin transform

$$\text{Constants} \qquad \ln^i(N) \qquad \frac{\ln^k(N)}{N}$$

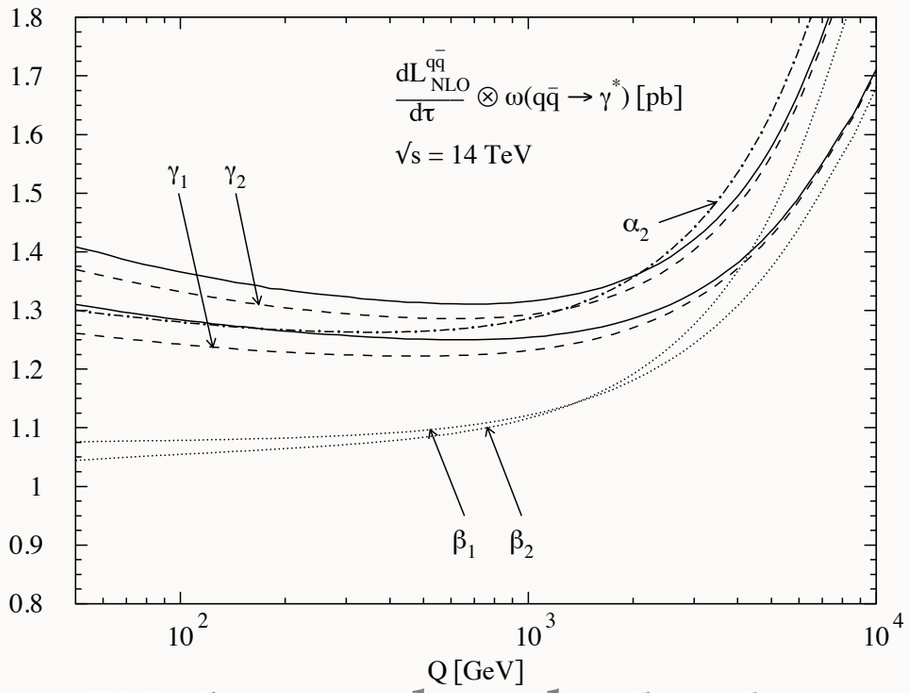
- We know a lot about logs and constants, very little about 1/N

LN(N)/N TERMS

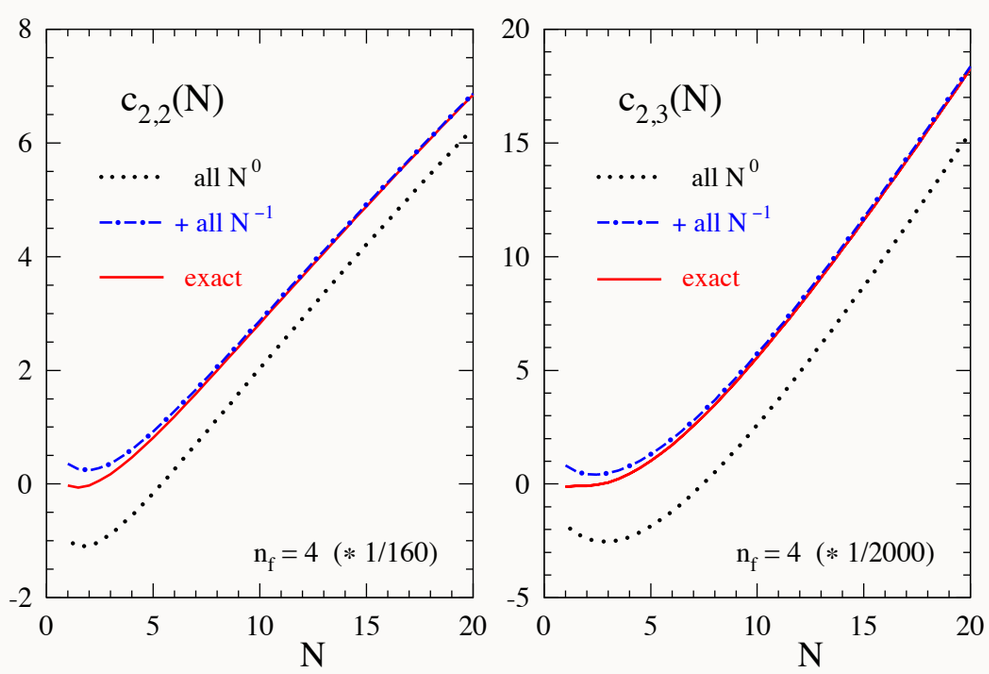
Kraemer, EL, Spira; Catani, De Florian, Grazzini; Kilgore, Harlander

- Can be numerically important

Kraemer, EL, Spira



Moch,Vogt



- We know that the leading series $\ln^i(N)/N$ exponentiates
  - by replacing in resummation formula

$$\exp\left[\int_0^1 dz\,(z^{N-1}-1)\frac{1+z^2}{1-z}\int_{\mu_F}^{Q(1-z)}\dots\right]$$

$$\frac{1+z^2}{1-z} \longrightarrow \frac{2}{1-z} - 2$$

SUCCESFUL LN(N)/N ORGANIZATION

Dokshitzer, Marchesini, Salam
Basso, Korchemsky

$$\gamma_{qq}(N) = A(\alpha_s)\ln N + B(\alpha_s) + C(\alpha_s)\frac{\ln N}{N} + \dots$$

- Moch, Vermaseren, Vogt noted a remarkable relation

$$C_2 = A_1^2 \qquad C_3 = 2A_2A_1$$

- DMS reproduced this by changing DGLAP equation

$$\mu^2\frac{\partial}{\partial\mu^2}\psi(x,\mu^2) = \int_x^1 \frac{dz}{z}\,\psi\left(\frac{x}{z}, z\mu^2\right)\mathcal{P}\left(z,\alpha_s\left(\frac{\mu^2}{z}\right)\right)$$

$$\mathcal{P}(z,\alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s)\,\delta(1-z) + \mathcal{O}((1-z))$$

- Can this be reproduced in threshold resummation?

EXTENDED THRESHOLD RESUMMATION

EL, Magnea, Stavenga

Ansatz: modified resummed expression

$$\ln\Big[\sigma(N)\Big] = \mathcal{F}_{\rm DY}\left(\alpha_s(Q^2)\right) + \int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z}\,D\left[\alpha_s\left(\frac{(1-z)^2Q^2}{z}\right)\right]\right.$$
$$\left. + 2\int_{Q^2}^{(1-z)^2Q^2/z}\frac{dq^2}{q^2}\,P_s\Big[z,\alpha_s(q^2)\Big]\right\}_+$$

where $$P_s^{(n)}(z) = \frac{z}{1-z}A^{(n)} + C_\gamma^{(n)}\ln(1-z) + \overline{D}_\gamma^{(n)}$$

(We constructed a similar expression for DIS). Structure:

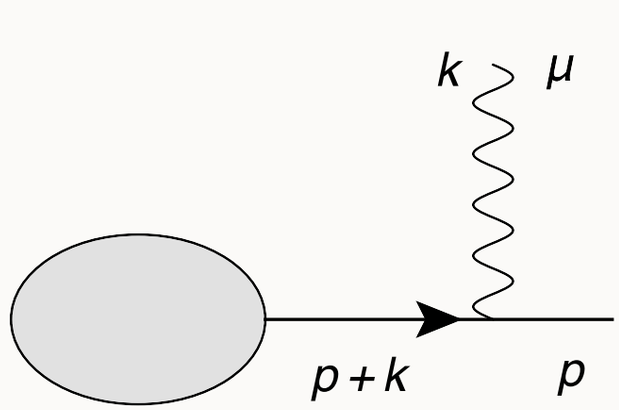
$$\sigma(N) = \sum_{n=0}^{\infty}\left(g^2\right)^n\left[\sum_{m=0}^{2n} a_{nm}\ln^m N + \sum_{m=0}^{2n-1} b_{nm}\frac{\ln^m N}{N}\right] + \mathcal{O}\left(N^{-2}\right)$$

	C_F^2		C_AC_F		n_fC_F	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

Must understand systematics beyond eikonal approximation

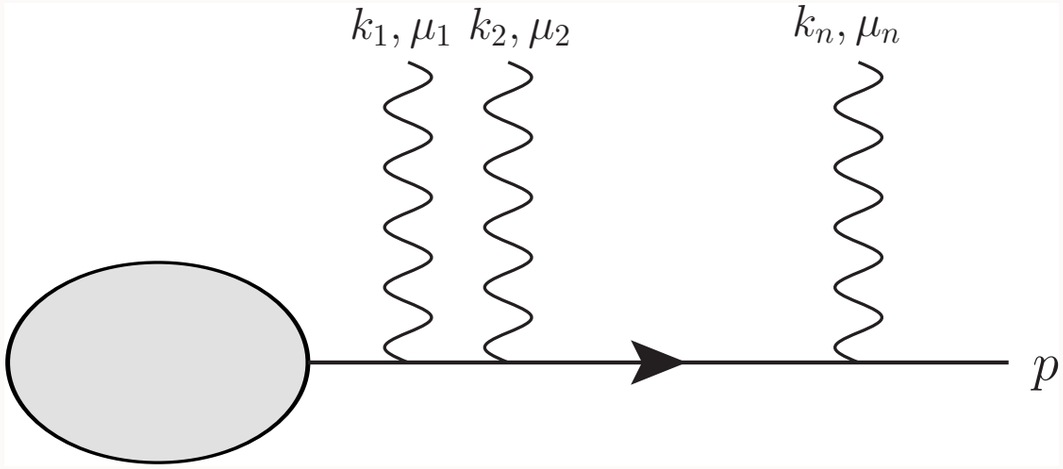
EIKONAL BASICS, QED

- Soft emission by charged particle
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k}$$

BASICS QED, CONT'D



Exact: $$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1}\ldots\frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n},\quad K_i=\sum_{m=i}^{n}k_m.$$

Approx: $$\frac{1}{2pK_1}2p^{\mu_1}\ldots\frac{1}{2pK_n}2p^{\mu_n}$$

Eikonal identity: $$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's: $$\prod_i\frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

NON-ABELIAN EIKONAL APPROXIMATION

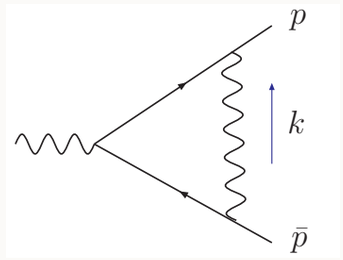
- Same methods as for QED, but organization harder: SU(3) generator at every vertex
  - no obvious decorrelation

Order the T_a according to λ

- Key "object": Wilson line $$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A^a(\lambda n)\, T_a\right]$$
  - Order by order in "g", it generates QCD eikonal Feynman rules

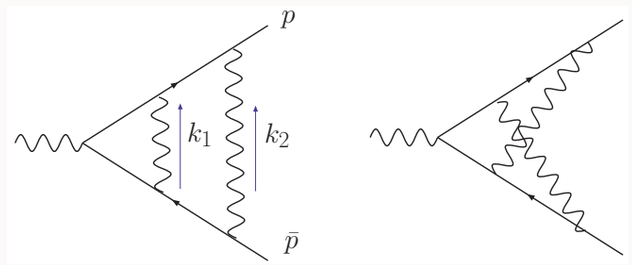
ABELIAN ALGEBRAIC EXPONENTIATON

One loop vertex correction, in eikonal approximation



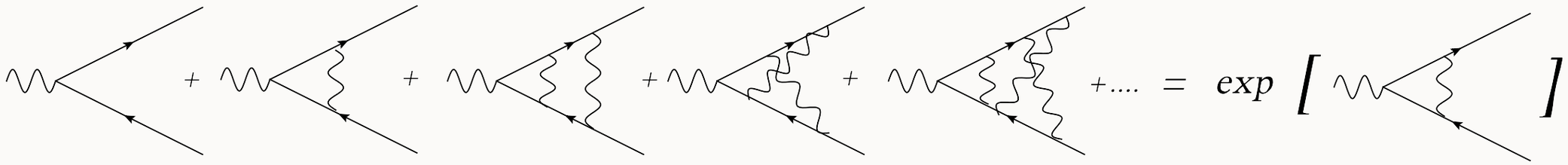
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

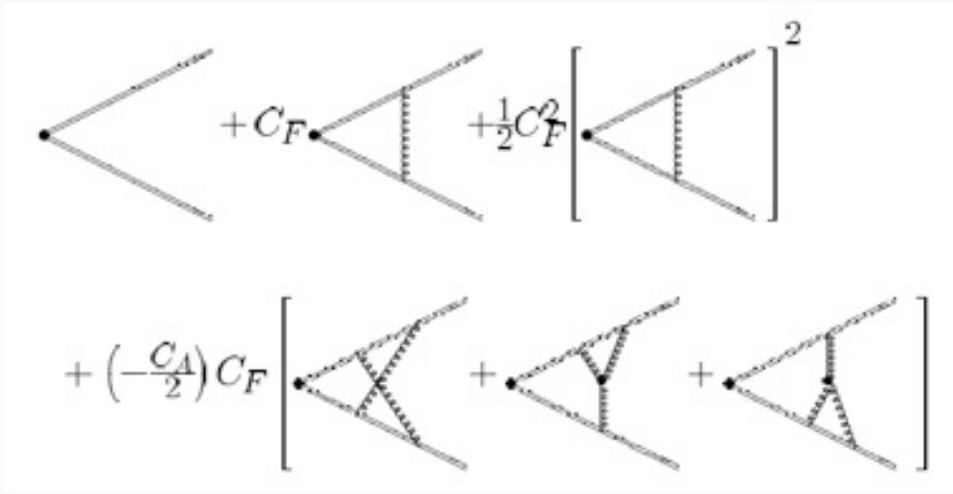
Exponential series



NON-ABELIAN EXPONENTIATION: WEBS

Gatheral; Frenkel, Taylor

- Take quark - antiquark line, connect with soft gluons in all possible ways, use eikonal approximation
- Exponentiation still occurs (without path ordering)!
  - A selection of diagrams in exponent, but with modified color weights: "webs"

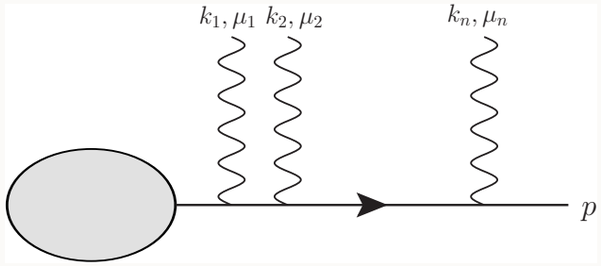


  - Webs are two-eikonal line irreducible
  - Proof by induction; recursive definition of color weights

DIAGRAMMATIC APPROACH

EL, Magnea, Stavenga, White

Recall: Abelian case, multiple emission, and sum over permutations



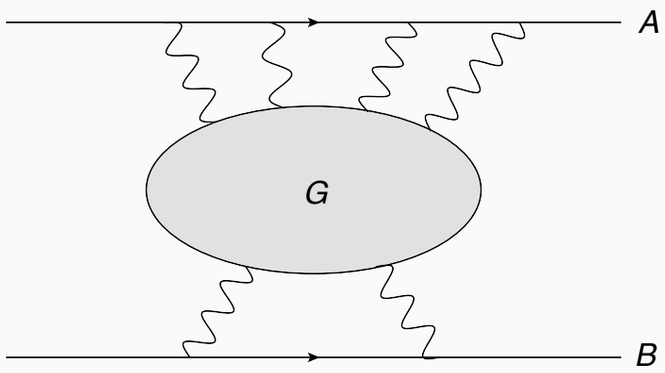
Eikonal identity:

$$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

For many emissions

$$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Consider graphs of type:

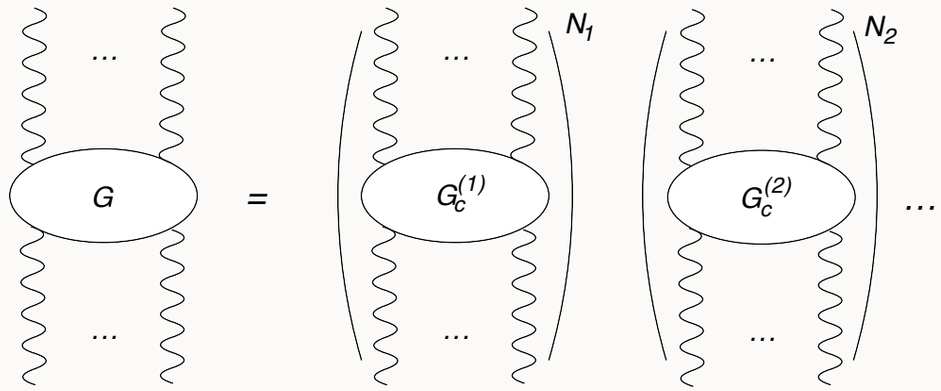


$$\mathcal{F}_{AB}=\sum_G\left[\prod_i\frac{p_A^{\mu_i}}{p_A\cdot k_i}\right]\left[\prod_j\frac{p_B^{\nu_j}}{p_B\cdot l_j}\right]G_{\mu_1\ldots\mu_n;\nu_1\ldots\nu_m}(k_i,l_j)\,,$$

DIAGRAMMATIC APPROACH

EL, Magnea, Stavenga, White

G will have disconnected pieces, each with a multiplicity N_i



Symmetry factor: $$S_G = \prod_i S_i^{N_i} (N_i)!$$

Amplitude is now, to all orders $$\mathcal{F}_{AB} = \sum_{\{N_i\}} \prod_i \frac{1}{N_i!} \left[\mathcal{F}_c^{(i)}\right]^{N_i} = \exp\left[\sum_i \mathcal{F}_c^{(i)}\right]$$

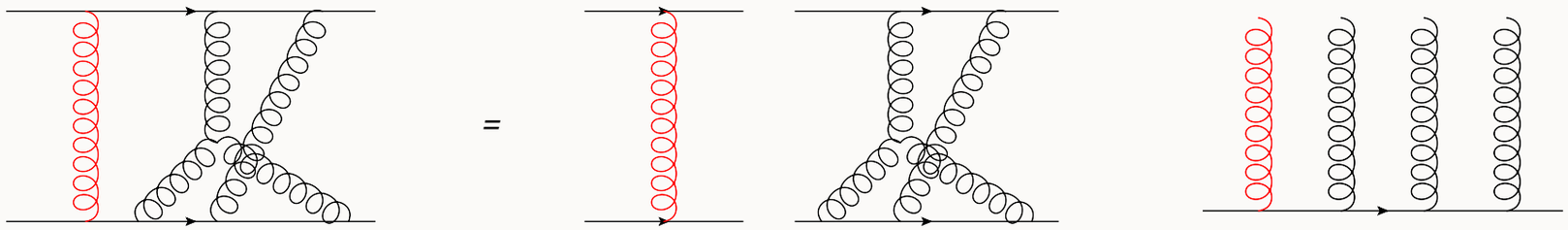
$$\mathcal{F}_c^{(i)} = \frac{1}{S_i} \left(\prod_q \frac{p_A^{\mu_q}}{p_A \cdot k_q}\right) \left(\prod_r \frac{p_B^{\nu_r}}{p_B \cdot l_r}\right) G^{(i)}_{c,\,\mu_1 \ldots \mu_{n_q};\nu_1 \ldots \nu_{m_r}}(k_q, l_r)$$

DIAGRAMMATIC APPROACH

EL, Magnea, Stavenga, White

Non-abelian case; define

- web: two-eikonal irreducible graph
- "group" : projection of web on external line



Web decomposition Two groups

Generalization of eikonal identity

- tilde π: permutation leaving order in each group fixed
- sum over these still yields factorization

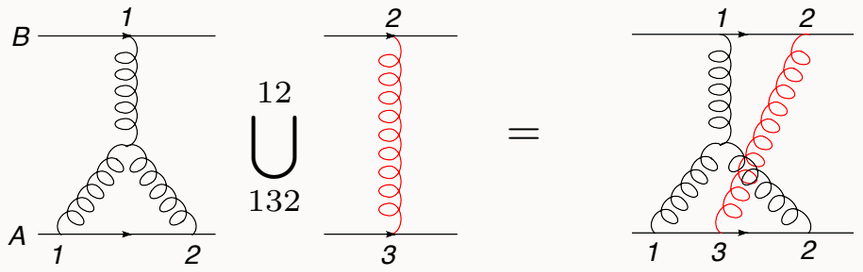
(shuffle product)

$$\sum_{\tilde{\pi}} \frac{1}{2p\cdot k_{\tilde{\pi}_1}} \frac{1}{2p\cdot(k_{\tilde{\pi}_1}+k_{\tilde{\pi}_2})} \cdots \frac{1}{p\cdot(k_{\tilde{\pi}_1}+\ldots+k_{\tilde{\pi}_n})} = \prod_g \frac{1}{2p\cdot k_{g_1}} \frac{1}{2p\cdot(k_{g_1}+k_{g_2})} \cdots \frac{1}{2p\cdot(k_{g_1}+\ldots+k_{g_m})}$$

EXPONENTIATION BY INDUCTION

Can also use this in reverse, as "merging" (shuffling)

$$E(H_1)\,E(H_2) = \sum_{\pi_A}\sum_{\pi_B} E(H_1 \cup^{\pi_B}_{\pi_A} H_2)$$



E(G): eikonal amplitude of G, without color

Collect identical diagrams $\quad E(H_1)\,E(H_2) = \sum_G E(G)\,N_{G|H_1H_2}$

$$\exp\left\{\sum_i \bar{c}_H E(H)\right\} = \prod_H \left(\sum_n \frac{1}{n!}[\bar{c}_H E(H)]^n\right) = \sum_G c_G E(G)$$

Prove that, for normal color factors on rhs, those on left side are the reduced color factors of webs

Proof uses

- induction
- combinatorics
- simplicity of color structure

PATH INTEGRAL METHOD

EL, Stavenga, White

Represent propagator as particle path integral, between coord. and momentum states

$$\tilde{\Delta}_F(p_f^2) = \frac{1}{2}\int_0^\infty dT \frac{\langle p_f|U(T)|x_i\rangle}{\langle p_f|x_i\rangle} = -\frac{i}{p_f^2+m^2-i\varepsilon}$$

where

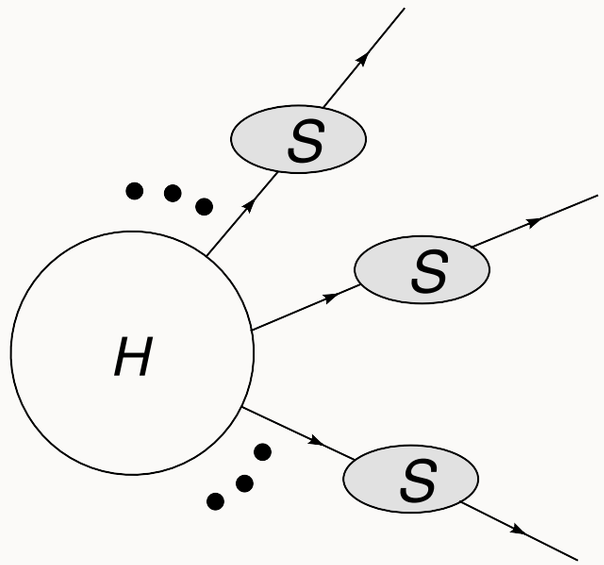
$$\langle p_f|U(T)|x_i\rangle = e^{-ip_f x_i - i\frac{1}{2}(p_f^2+m^2)T}\int_{x(0)=0}^{p(T)=0}\mathcal{D}p\mathcal{D}x\, e^{i\int_0^T dt(p\dot{x}-\frac{1}{2}p^2)}$$

Scalar particle with (abelian) gauge field

$$\langle p_f|U(T)|x_i\rangle = \int_{x(0)=x_i}^{p(T)=p_f}\mathcal{D}p\mathcal{D}x\exp\Big[-ip(T)x(T)+i\int_0^T dt(p\dot{x}-\frac{1}{2}(p^2+m^2)+p\cdot A+\frac{i}{2}\partial\cdot A-\frac{1}{2}A^2)\Big]$$

n-point Green's function

$$G(p_1,\ldots,p_n) = \int \mathcal{D}A_s^\mu\, H(x_1,\ldots,x_n,A_s)$$
$$\times\langle p_1|(S-i\varepsilon)^{-1}|x_1\rangle\ldots\langle p_n|(S-i\varepsilon)^{-1}|x_n\rangle$$



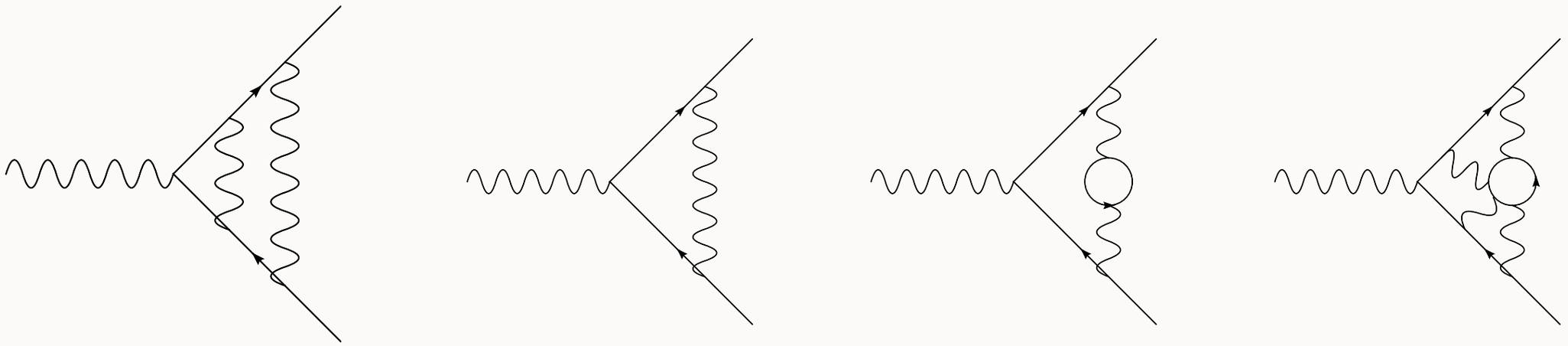
PATH INTEGRAL METHOD

Truncate external lines for S-matrix element $i(p_f^2+m^2)\langle p_f| - i((p-A)^2-i\varepsilon)^{-1}|x_i\rangle = e^{-ip_f x_i} f(\infty)$

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A_s^\mu H(x_1,\ldots,x_n)\, e^{-ip_1x_1} f_1(\infty)\ldots e^{-ip_nx_n} f_n(\infty)\, e^{iS[A_s]}$$

$$f(\infty) = \int_{x(0)=0} \mathcal{D}x\, e^{i\int_0^\infty dt\left(\frac{1}{2}\dot{x}^2+(p_f+\dot{x})\cdot A(x_i+p_ft+x(t))+\frac{i}{2}\partial\cdot A(x_i+p_ft+x)\right)}$$

Eikonal vertices act as sources for gauge bosons along path



Disconnected

Connected

QED: exponentiation now textbook result:
all diagrams = exp (connected diagrams)

PATH INTEGRAL METHOD, NON-ABELIAN

- Not immediately obvious how this could work (the path integral must be a real exponential), since
  - Source terms have non-abelian charges, so don't commute
  - External line factors are path-ordered exponentials
  - External lines are colored, path integral as color tensor..?
- We will use nice trick (from statististical physics): using "replica's"

REPLICA TRICK

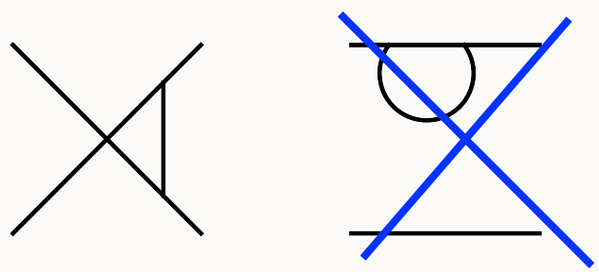
- Relates exponentiation of soft gauge fields to that of connected diagrams in QFT.
- Consider a N copies of a scalar theory

$$Z[J]^N = \int \mathcal{D}\phi_1 \dots \mathcal{D}\phi_N e^{iS[\phi_1]+..+iS[\phi_N]+J\phi_1+..J\phi_N}$$

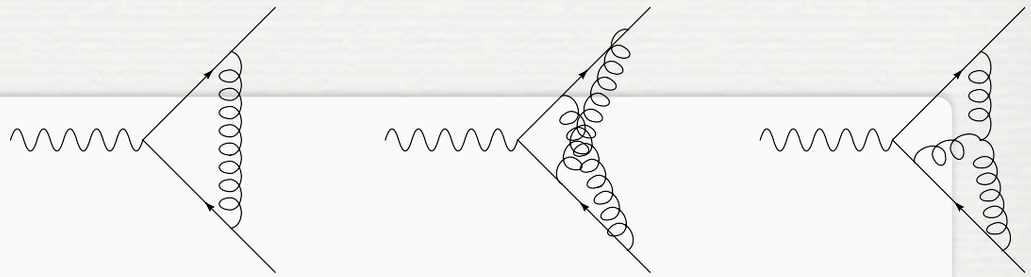
- If Z is exponential, find out what contributes to log Z

$$Z^N = 1 + N \log Z + \mathcal{O}(N^2)$$

- Amounts to diagrams that allow only one replica → connected!



APPLICATION TO QCD



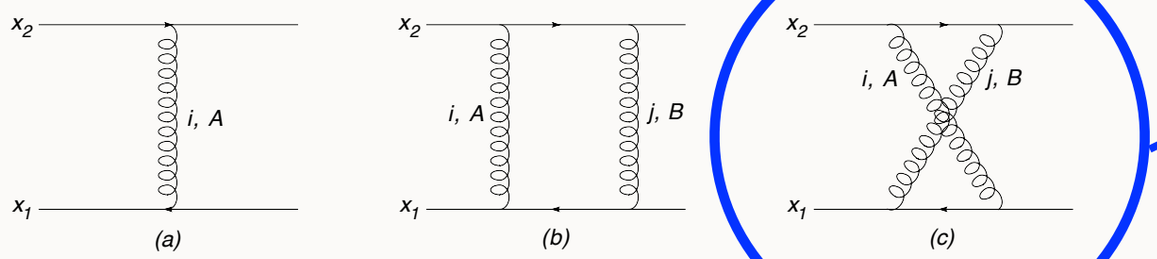
Amplitude for two colored lines

$$S(p_1,p_2) = H(p_1,p_2) \int \mathcal{D}A_s f(\infty) e^{iS[A_s]}$$

Replicate, and introduce replica ordering operator R

$$f(\infty) = \mathcal{P}\exp\left[\int dx \cdot A(x)\right] \qquad \prod_{i=1}^{N} \mathcal{P}\exp\left[\int dx \cdot A_i(x)\right] = \mathcal{R}\mathcal{P}\exp\left[\sum_{i=1}^{N}\int dx \cdot A_i(x)\right]$$

Look for diagrams of replica multiplicity N. These will go into exponent



Web
Modified color factor

(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2C_F^2$

(c) for equal replica number (i=j): C_F^2-C_F C_A /2.
For i≠j C_F^2. Term linear in N: $N\left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_F C_A}{2}\right)$

WEBS TO ALL ORDERS

- Can give an all-order, *non-recursive* formula for modified color factors

  - Consider general diagram G, with a number of connected pieces

  - Distribute replica numbers in all possible ways, and count for each such partition P the multiplicity to linear order in N

$$\overline{C}(G) = \sum_P (-)^{n(P)-1}(n(P)-1)! \prod_g C(g)$$

- Examples

$$\begin{aligned}\overline{C}(X) &= C(X) - C(I)C(I) \\ &= (C_F^2 - \frac{1}{2}C_A C_F) - C_F^2\end{aligned}$$

$$\overline{C}(IX) = C(IX) - C(I)C(X) = 0$$

- More examples later

NEXT-TO-EIKONAL CORRECTIONS

- Keep 1 term more in k expansion beyond eikonal approximation

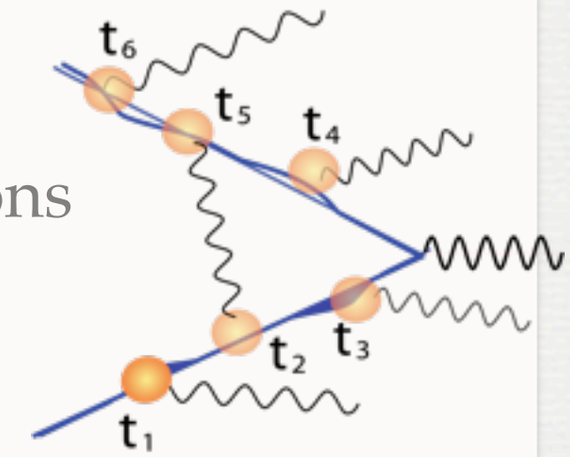
$$\text{scalar}: \quad \frac{2p^\mu + k^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k} + \frac{k^\mu}{2p\cdot k} - k^2 \frac{2p^\mu}{(2p\cdot k)^2}$$

$$\text{fermion}: \quad \frac{\not{p} + \not{k}}{2p\cdot k + k^2}\gamma^\mu u(p) \longrightarrow \left[\frac{2p^\mu}{2p\cdot k} + \frac{\not{k}\gamma^\mu}{2p\cdot k} - k^2 \frac{2p^\mu}{(2p\cdot k)^2}\right] u(p)$$

- Becomes emitter-spin dependent
- Decorrelation not obvious
- Can one make systematic statements (exponentiation, factorization) about them?

NEXT-TO-EIKONAL EXPONENTIATION

- Wilson lines are classical solutions of path integral
- Fluctuations around classical path lead to NE corrections
  - This class of NE corrections exponentiates
  - Keep track via scaling variable λ $\quad p^\mu = \lambda n^\mu$



$$f(x_i, p_f, A_s) = \int_{x(0)=0} \mathcal{D}x \, \exp\Big[i \int_0^\infty dt \Big(\frac{\lambda}{2}\dot{x}^2 + q_i \, (n + \dot{x}) \cdot A(x_i + nt + x) + \frac{i}{2\lambda} q_i \, \partial \cdot A(x_i + nt + x) \Big) \Big]$$

Use 1-D field theory propagators

$$\langle x(t) x(t') \rangle = G(t, t') = \frac{i}{\lambda} \min(t, t')$$

$$\langle \dot{x}(t) \dot{x}(t') \rangle = \frac{i}{\lambda} \delta(t - t')$$

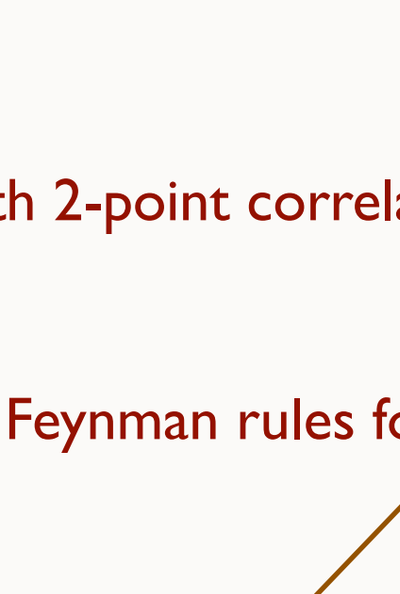
to write f(..) as the exponent of 1-D connected graphs:

FEYNMAN RULES

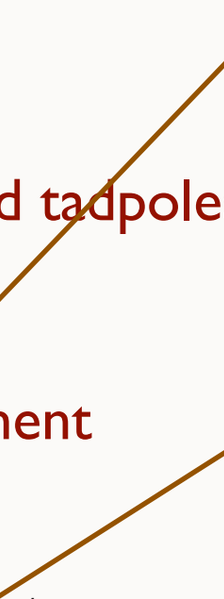
$$f(\infty) = \exp\left[- \int \frac{d^d k}{(2\pi)^d} \frac{n^\mu}{n\cdot k} \tilde{A}_\mu(k) + \frac{1}{2\lambda} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{n\cdot k} \tilde{A}_\mu(k) + \sum \bullet\!\!-\!\!-\!\!\bullet + \sum \bullet\!\!\bigcirc \right]$$

Both 2-point correlators and tadpoles contribute

NE Feynman rules for exponent



$$\frac{k^\mu}{2p\cdot k} - k^2 \frac{p^\mu}{2(p\cdot k)^2}$$



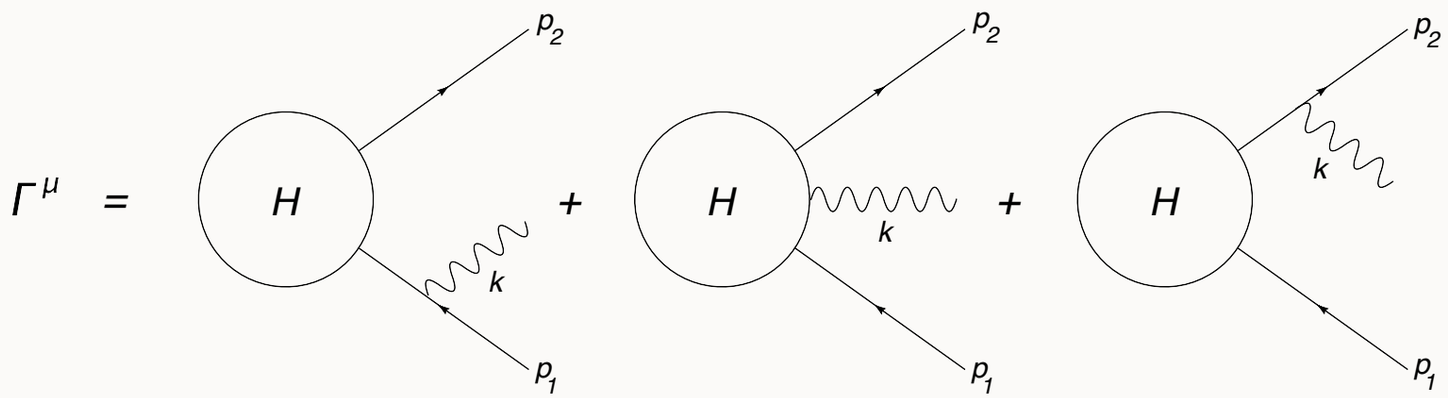
$$+\frac{\eta^{\mu\nu}}{p\cdot(k+l)}$$



$$-\frac{l^\mu p^\nu p\cdot k + k^\nu p^\mu p\cdot l}{p\cdot(k+l)p\cdot k p\cdot l}$$

LOW'S THEOREM

- So far we only looked at emissions from external lines. At next-to-eikonal order, also 1 "internal" emission contributes



- Low's theorem (scalars, generalization to spinors by Burnett-Kroll)
  - Work, expand to order k, and use gauge invariance

$$\Gamma^\mu = \left[\frac{(2p_1-k)^\mu}{-2p_1\cdot k}+\frac{(2p_2+k)^\mu}{2p_2\cdot k}\right]\Gamma+\left[\frac{p_1^\mu(k\cdot p_2-k\cdot p_1)}{p_1\cdot k}+\frac{p_2^\mu(k\cdot p_1-k\cdot p_2)}{p_2\cdot k}\right]\frac{\partial\Gamma}{\partial p_1\cdot p_2}$$

- Elastic amplitude still determines the emission to NE accuracy,
  - needs derivative
  - but not knowledge of internal part

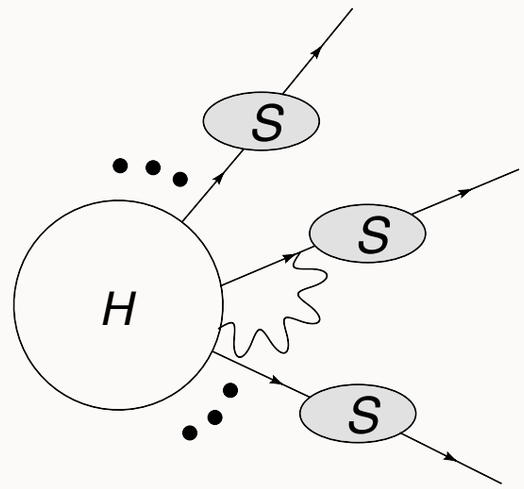
LOW-BURNETT-KROLL

For massless external lines, internal emission take form

$$\Gamma_{\mu}^{int} = -\sum_i \frac{\partial \Gamma}{\partial p_i^{\mu}}$$

Analyzed in context of jet-soft factorization by Del Duca

One emission from H still missing in our approach

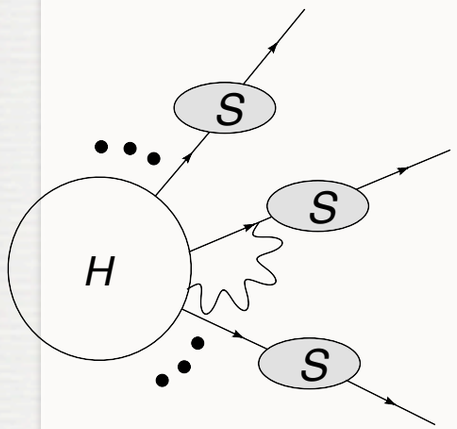


Path integral formulation is also very convenient for this class of next-to-eikonal corrections

LOW-BURNETT-KROLL

Path integral method provides elegant way to derive Low's theorem

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A_s H(x_1,\ldots,x_n;A_s) e^{-ip_1x_1} f(x_1,p_1;A_s)\ldots e^{-ip_nx_n} f(x_1,p_1;A_s) e^{iS[A_s]}$$



Gauge transformation must cancel between f's and H

$$f(x_i,p_f;A) \to f(x_i,p_f;A+\partial\Lambda) = e^{-iq\Lambda(x_i)} f(x_i,p_f;A)$$

Opposite transformation in H, expand to first order in A and Λ

Low's contribution is then:

$$S(p_1,\ldots,p_n) = \int \mathcal{D}A \left[\int \frac{d^dk}{(2\pi)^d} \sum_j^n q_j \left(\frac{n_j^\mu}{n_j\cdot k} k_\nu \frac{\partial}{\partial p_{j\,\nu}} - \frac{\partial}{\partial p_{j\,\mu}} \right) H(p_1,\ldots,p_n) A_\mu(k) \right]$$
$$\times f(0,p_1;A)\ldots f(0,p_n;A)$$

First term is due to displacement of f(x,p,A), second due to expansion of H

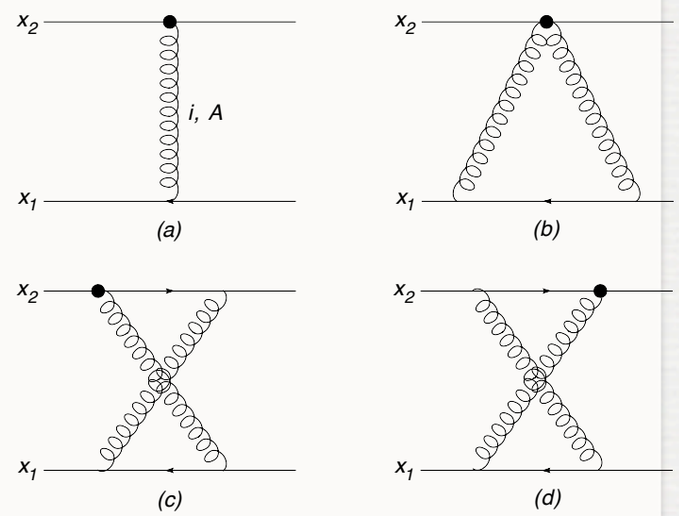
Analagous result in non-abelian case, for n=2

For scalar case, agrees precisely with Low's original answer

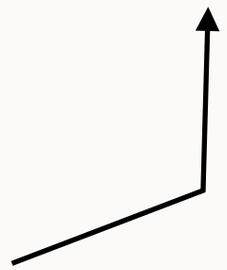
UPSHOT

- Exponentiation of soft emissions for matrix elements as "connected diagrams"
  - For both eikonal and next-to-eikonal contributions from external lines
  - Replica trick both for exponentiation, and for explicit expression for webs. New NE Webs.
  - 1 emission from hard part also included
  - QCD: 2 lines ok. 3 lines also easy. 4+ (later)
- Can we arrive at the same results using diagrammatic analysis, and inductive reasoning?
  - Combinatorics challenging....

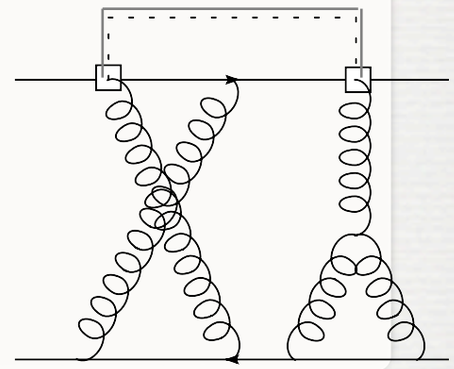
NEXT-TO-EIKONAL DIAGRAMMAR



- As for eikonal case earlier
  - identify next-to-eikonal vertices
  - show that they "decorrelate", after summing over all (shuffled) perm's. Use induction again
    - as eikonal webs (2 eik. line irreducible), but now with new vertices
    - they become spin-sensitive

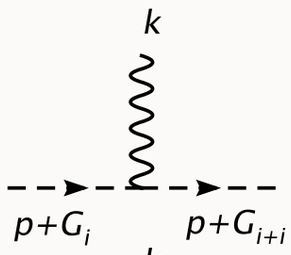


- New 2-gluon correlations between eikonal webs → NE webs



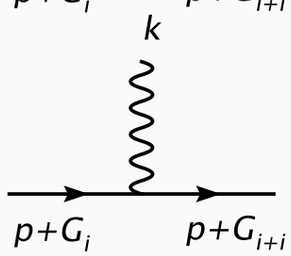
NEXT-TO-EIKONAL DIAGRAMMAR

scalar NE
1 emission



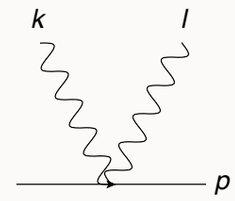
$$= \; t^A \left(\frac{2G_i^\mu - k^\mu}{2p\cdot G_i} - \frac{G_i^2\, p^\mu}{2(p\cdot G_i)^2} \right) = \frac{t^A}{2p\cdot G_i} \left(G_i^\mu + G_{i+1}^\mu - \frac{G_i^2\, p^\mu}{p\cdot G_i} \right) .$$

spinor NE
1 emission



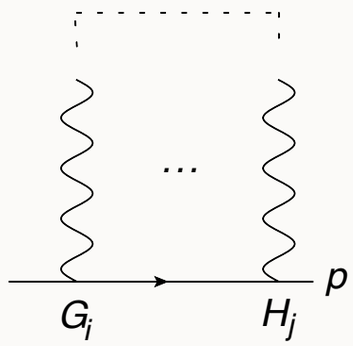
$$= \; t^A \left(\frac{2G_i^\mu + \gamma^\mu \not{k}}{2p\cdot G_i} - \frac{G_i^2\, p^\mu}{2(p\cdot G_i)^2} \right) = \frac{t^A}{2p\cdot G_i} \left(G_i^\mu + G_{i+1}^\mu + \not{k}\, \gamma^\mu - k^\mu - \frac{G_i^2\, p^\mu}{p\cdot G_i} \right) ,$$

spinor NE
2 emissions



$$= \; \frac{g^{\mu\nu}\, \{t^A, t^B\}}{p\cdot(k+l)} + \frac{[\gamma^\mu, \gamma^\nu]\, [t^A, t^B]}{2p\cdot(k+l)} ,$$

2-group
correlation



$$= \; t^A \otimes t^B \left(\frac{-\, G_i^\mu\, p^\nu\, (p\cdot H_j) - H_j^\nu\, p^\mu\, (p\cdot G_i) + p^\mu p^\nu\, G_i\cdot H_j}{2p\cdot(G_i + H_j)\, p\cdot G_i\, p\cdot H_j} \right) .$$

All these are precisely reproduced in the path integral formalism

DRELL-YAN CHECK

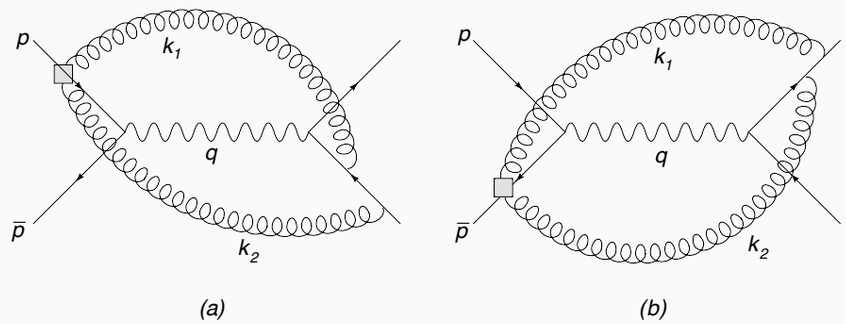
- Check use of NE Feynman rules for Drell-Yan double real emission
  - for amplitude, expand

$$\mathcal{A} = \mathcal{A}^{(0)} \exp\left[\bar{\mathcal{A}}^{(1)\mathrm{E}} + \bar{\mathcal{A}}^{(1)\mathrm{NE}} + \bar{\mathcal{A}}^{(2)\mathrm{NE}}\right]$$

  - to 2nd order, and integrate each term with exact 3-particle phase space. Cross term leads to

$$\left(\frac{\alpha_S C_F}{4\pi}\right)^2 \left[\frac{-1024\log^3(1-z)}{3} - \frac{512\log^2(1-z)}{\epsilon} - \frac{512\log(1-z)}{\epsilon^2} - \frac{256}{\epsilon^3}\right]$$

- Special vertex (2-group correlation)



$$R^{\mu\nu}(p;k_1,k_2) = -\frac{(p\cdot k_2)p^\mu k_1^\nu + (p\cdot k_1)k_2^\mu p^\nu - (p\cdot k_1)(p\cdot k_2)g^{\mu\nu} - (k_1\cdot k_2)p^\mu p^\nu}{p\cdot(k_1+k_2)}$$

  - gives zero after azimuthal integration

DRELL-YAN CHECK

- Combine with exact phase space

$$K^{(2)NE} = \left(\frac{\alpha_S C_F}{4\pi}\right)^2 \left[\frac{1024\mathcal{D}_3}{3} - \frac{1024\log^3(1-z)}{3} + 640\log^2(1-z) + \frac{512\mathcal{D}_2 - 512\log^2(1-z) + 640\log(1-z)}{\epsilon} + \frac{512\mathcal{D}_1 - 512\log(1-z)}{\epsilon^2} + \frac{256\mathcal{D}_0 - 256}{\epsilon^3}\right]$$

$$\mathcal{D}_i = \left[\frac{\log^i(1-z)}{1-z}\right]_+$$

- Agrees with equivalent exact result

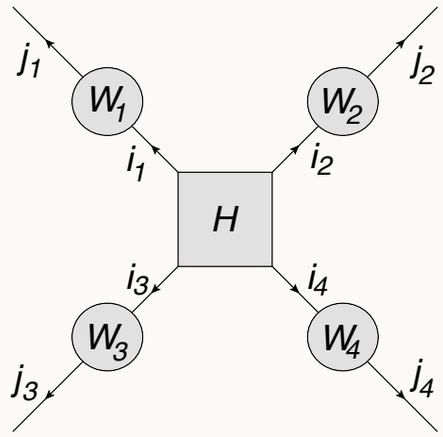
DRELL-YAN CHECK

- Next step: reproduce subleading log(1-z) terms

  Bonocore, Falgari, EL, Magnea, Melville, Vernazza, Volponi, White, ..

  - $\log^2(1-z)$ ok, log(1-z) almost
  - Low theorem contribution subtle. Emission from collinear and hard regions only

MULTIPLE COLORED LINES



Gardi, EL, Stavenga, White

Mitov, Sterman, Sung

$$Z_{IJ}^{N} = \int \mathcal{D}\mathbf{A^1} \dots \mathcal{D}\mathbf{A^N} e^{i \sum S[A^i]} \times \left[(\mathbf{W}_1^{(1)} \dots \mathbf{W}_N^{(1)}) \dots \right]$$

- Can we write correlator of multiple Wilson lines as a pure exponential of full (eikonal) diagrams? Valuable:
  - Insight into singularity structure for amplitudes
  - Webs have more information than anomalous dimensions (finite parts)
  - Coherent multi-parton wide-angle radiation organized
- Use replica trick for multiple colored lines; find again order N terms
  - Even when "present" might turn out to be zero

Aybat, Dixon, Sterman

MULTIPLE COLORED LINES

- We find

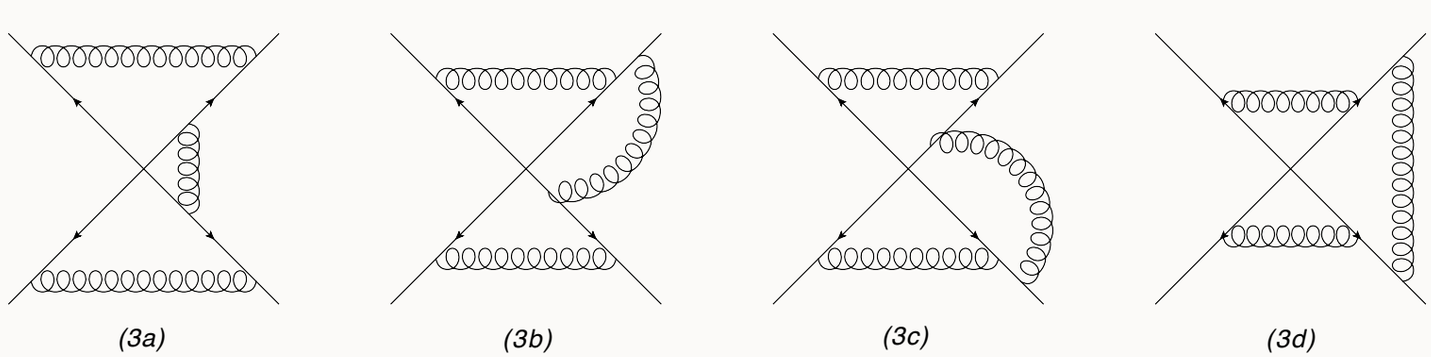
Projector matrix

$$\sum_{d'} R_{dd'} = 0$$

Eigenvalues 0 or 1

$$\sum \mathcal{F}(D)C(D) = \exp[\sum_{d,d'} \mathcal{F}(d) R_{dd'} C(d')]$$

  - multi-parton webs are "closed sets" of diagrams, with modified color factors



= Web !!

$$\frac{1}{6}\Big[C(3a) - C(3b) - C(3c) + C(3d)\Big] \times \Big[M(3a) - 2M(3b) - 2M(3c) + M(3d)\Big]$$

- Closed form solution for modified color factor

$$\overline{C}(G) = \sum_P \frac{(-)^{n(P)-1}}{n(P)} \sum_\pi C(g_{\pi_1}) \ldots C(g_{\pi_{n(P)}})$$

PROJECTOR MATRIX

$$\sum \mathcal{F}(D)C(D) = \exp[\sum_{d,d'} \mathcal{F}(d) R_{dd'} C(d')]$$

- Projects out contributions that come from exponentiation of lower order diagrams
  - Interesting combinatorial aspects (Stirling numbers)
  - Proof of idempotency and zero sum row property Gardi, White
- Find it through replica trick, semi-automatically
- Removes leading subdivergences in sum of diagrams that make up web

4-LOOP PROJECTOR

$$\frac{1}{24}\begin{pmatrix}
6 & -6 & 2 & 2 & -2 & 4 & -4 & 2 & -2 & -2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-6 & 6 & -2 & -2 & 2 & -4 & 4 & -2 & 2 & 2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 6 & -2 & 2 & 4 & -4 & -2 & 2 & -6 & 4 & 4 & -4 & -4 & 0 & 0 \\
2 & -2 & -2 & 6 & 2 & 4 & -4 & -2 & -6 & 2 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & 2 & 6 & 4 & -4 & -6 & -2 & -2 & 4 & -4 & 4 & -4 & 0 & 0 \\
2 & -2 & 2 & 2 & 2 & 4 & -4 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & -2 & -2 & -2 & -4 & 4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & -2 & -2 & -6 & -4 & 4 & 6 & 2 & 2 & -4 & 4 & -4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -6 & -2 & -4 & 4 & 2 & 6 & -2 & 4 & 4 & -4 & -4 & 0 & 0 \\
-2 & 2 & -6 & 2 & -2 & -4 & 4 & 2 & -2 & 6 & -4 & -4 & 4 & 4 & 0 & 0 \\
-2 & 2 & 2 & -2 & 2 & 0 & 0 & -2 & 2 & -2 & 4 & 0 & 0 & -4 & 0 & 0 \\
2 & -2 & 2 & -2 & -2 & 0 & 0 & 2 & 2 & -2 & 0 & 4 & -4 & 0 & 0 & 0 \\
-2 & 2 & -2 & 2 & 2 & 0 & 0 & -2 & -2 & 2 & 0 & -4 & 4 & 0 & 0 & 0 \\
2 & -2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 2 & -4 & 0 & 0 & 4 & 0 & 0 \\
-18 & -6 & -6 & -6 & -18 & 12 & 12 & -6 & -18 & -18 & 12 & 12 & 12 & 12 & 24 & 0 \\
-6 & -18 & -18 & -18 & -6 & 12 & 12 & -18 & -6 & -6 & 12 & 12 & 12 & 12 & 0 & 24
\end{pmatrix}
\begin{pmatrix}
C[[1,2],[3,1],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[4,1]] \\
C[[1,2],[3,2],[3,4],[4,1]] \\
C[[1,2],[2,3],[3,4],[1,4]] \\
C[[1,2],[3,2],[4,3],[1,4]] \\
C[[1,2],[1,3],[4,3],[4,2]] \\
C[[1,2],[3,2],[3,4],[1,4]] \\
C[[1,2],[1,3],[3,4],[4,2]] \\
C[[1,2],[3,1],[4,3],[4,2]] \\
C[[1,2],[1,3],[4,3],[2,4]] \\
C[[1,2],[1,3],[3,4],[2,4]] \\
C[[1,2],[2,3],[4,3],[1,4]] \\
C[[1,2],[3,1],[4,3],[4,2]] \\
C[[1,2],[3,2],[4,3],[4,1]] \\
C[[1,2],[3,1],[4,3],[2,4]] \\
C[[1,2],[2,3],[3,4],[4,1]]
\end{pmatrix}$$



Matrix is indeed a projector (exercise: check this)

"Escher staircase" diagrams have no subdivergences

IMAGINARY WEBS (IN PROGRESS)

EL, Larsen, Rietkerk

- Can one find a rule to compute imaginary parts of webs?
  - To compute non-abelian phases (issues in factorization)

    Catani, de Florian, Rodrigo
    Forshaw, Seymour, Siodmok
  - Generalize "largest time equation" to eikonal diagrams
  - "Unitarity methods" in exponent?
- Key ingredients:
  - Use coordinate space
  - Formula (f has no poles)

$$\int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \frac{f(x_1,x_2)}{D(x_1,x_2) \pm i\delta} = \int_{a_1}^{b_1} dx_1 \mathrm{PV} \int_{a_2}^{b_2} dx_2 \frac{f(x_1,x_2)}{D(x_1,x_2)} \mp \frac{1}{2} \int_{a_1}^{b_1} dx_1 \oint_C dx_2 \frac{f(x_1,x_2)}{D(x_1,x_2)} \,.$$

  - Have result for L-loop ladder diagrams. More subtle: 3-gluon vertex diagrams

SUMMARY

- Exponentiation of soft radiation contribution interesting, can we extend it?
  - Eikonal approximation important ingredient
- Next-to-eikonal contributions form new webs, and exponentiate*
  - using path integrals, or diagrams
    - Feynman rules for exponent of scattering amplitude
  - *classified “Low’s theorem” contributions
- Multi-parton webs defined, with rule for exponentiated color factors

OUTLOOK

- Recent
  - Application to soft graviton emission — White
  - Combinatorics in multi-parton webs (Stirling numbers) — Gardi, White
- Lots to do: — Beneke, Chapovsky, Diehl, Feldmann
  - Divergence structure in loops with NE vertices
  - Resummation for large variety of NE effects, next order predictions
  - Relation to SCET now being clarified; clear at 1 loop