

SPLITTING FUNCTIONS AND FEYNMAN INTEGRALS



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10/12/2012 - IFIC

CONTENT

- Introduction
 - Collinear limits
 - Splitting functions
 - Computing splitting functions
 - The $q \rightarrow gq$ splitting amplitude
 - Feynman integrals
 - Definitions and regularization
 - Using IBP identities
 - Conclusions
-

INTRODUCTION

- Today accepted model to describe particle physics: Standard Model (SM)
- Components of SM
 - $SU(2)_L \times U(1)_Y \longrightarrow$ Electroweak interactions
 - $SU(3)_C \longrightarrow$ Color interactions (QCD)
- Main computation technique: perturbation theory
- QCD effects dominate at collider energies \longrightarrow Focus in QCD!
- Hadron collisions involves non-perturbative effects (hadron structure), so we need to add something else \longrightarrow Parton model
 - Hadron cross-sections can be calculated using Parton Distribution Functions (PDFs)
 - Non-perturbative information is encoded inside PDFs

(See: Any QFT textbook; for example, Peskin&Schroeder, *An Introduction to QFT* or Halzen, *Quarks and Leptons*)

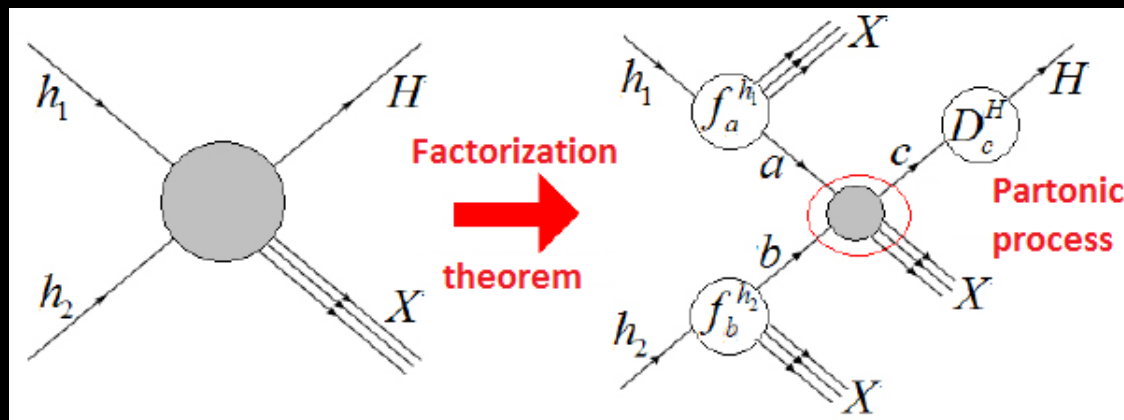
INTRODUCTION

- Cross-sections in the context of parton model

$$d\sigma^{h_1 h_2 \rightarrow X} = \sum_{a,b} \int_0^1 dx \int_0^1 dy f_a^{h_1}(x, Q) f_b^{h_2}(y, Q) d\hat{\sigma}_{ab \rightarrow X}(\mu, Q)$$

PDFs (non-perturbative)

Partonic CS (perturbative)



(See: Field, *Applications of Perturbative QCD*)

INTRODUCTION

- Cross-sections in the context of parton model

$$d\sigma^{h_1 h_2 \rightarrow X} = \sum_{a,b} \int_0^1 dx \int_0^1 dy f_a^{h_1}(x, Q) f_b^{h_2}(y, Q) d\hat{\sigma}_{ab \rightarrow X}(\mu, Q)$$

- OBSERVATIONS:
 1. Partonic cross-sections computable inside pQCD
 2. PDFs can be extracted from experiments (fit data)
 3. Universal behaviour (PDFs are process independent)
 4. No *formal mathematical* proof of “factorization formula” in hadron-hadron collisions (OPE technique allows to prove DIS cross-section formula...)
 5. x and y are called *momentum fractions* (related with the longitudinal momentum carried by partons)

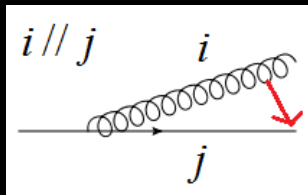
INTRODUCTION

- Higher order corrections to QCD processes involves dealing with divergences
- There are two kinds (of physical divergencies):
 - High-energy or UV singularities
 1. *Origin:* Unknown high-energy physics; very short distance interactions and point-like effects
 2. *Solution:* Renormalization.
 - Low-energy or IR singularities
 1. *Origin:* Emission of collinear (i.e. parallel) or soft (i.e. low energy) particles.
 2. *Solution:* Some theorems (Kinoshita-Lee-Nauenberg) guarantee that they can be cancelled in the final physical result (if we are computing IR safe observables...). For instance, the cancellation can be implemented through the subtraction method.

(See: Muta, *Foundations of Quantum Chromodynamics* for KLN details, and Catani&Seymour, arXiv:hep-ph/9602277 or Frixione,Kunszt&Signer,arXiv:hep-ph/9512328 for subtraction method info)

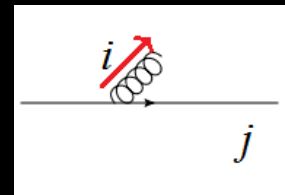
INTRODUCTION

- Focusing on IR singularities: general ideas
 - Related with low-energy and collinear configurations



$$k_i \cdot k_j \rightarrow 0$$

Collinear configuration

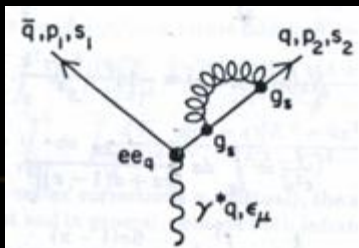


$$k_i \rightarrow 0$$

Soft configuration

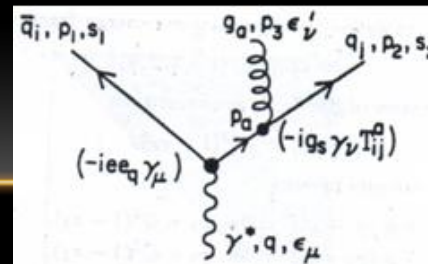
(Note that these quantities appear in denominators of scattering amplitudes...)

- Associated with degenerate states (the experiments are not able to distinguish two particles which are very close neither they can detect low-energy particles)



Virtual corrections

+



Real corrections

=

Finite result

INTRODUCTION

- Focusing on IR singularities: IR-observables
 - Observables implemented through measure functions.

$$\int_{\Omega} d\sigma = \int d\sigma \mathcal{S}_n(p_1^\mu, \dots, p_n^\mu) \longrightarrow \text{Measure function } \mathcal{S}$$

- An observable is IR-safe if it is not sensible to the presence of collinear/soft particles in the final state. These requirements are translated into \mathcal{S} properties:

1. **If** $p_i^0 \rightarrow 0$ **then**

$$\mathcal{S}_{n+1}(p_1^\mu, \dots, p_i^\mu, \dots, p_{n+1}^\mu) = \mathcal{S}_n(p_1^\mu, \dots, p_{n+1}^\mu) \longrightarrow \text{Eliminate soft particles}$$

2. **If** $p_i^\mu = \lambda \xi^\mu$, $p_{i+1}^\mu = (1 - \lambda) \xi^\mu$ **then**

$$\mathcal{S}_{n+1}(p_1^\mu, \dots, \lambda \xi^\mu, \dots, (1 - \lambda) \xi^\mu, \dots, p_{n+1}^\mu) = \mathcal{S}_n(p_1^\mu, \dots, \xi^\mu, \dots, p_{n+1}^\mu) \longrightarrow \text{Merge collinear particles}$$

3. **If** $p_i^\mu = \lambda p_a^\mu$ **then**

$$\mathcal{S}_{n+1}(p_1^\mu, \dots, \lambda p_a^\mu, \dots, p_{n+1}^\mu) = \mathcal{S}_n(p_1^\mu, \dots, p_{n+1}^\mu) \longrightarrow \text{Merge collinear particles}$$

COLLINEAR LIMITS

- Let's focus in the collinear limit (1->2 processes)
 - It is useful to introduce some kinematical variables to parametrize collinear momenta.

$$p_i^\mu = x_i p^\mu + k_{\perp i}^\mu - \frac{k_{\perp i}^2}{x_i} \frac{n^\mu}{2p \cdot n}, \quad i = 1, 2$$

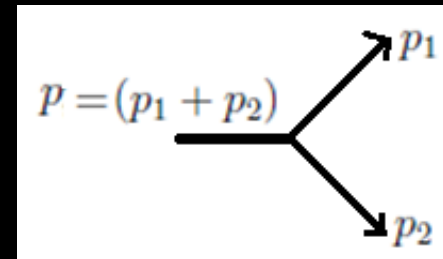
Collinear direction

Transverse component

Approach to the collinear limit

General parametrization

Particles 1 and 2 become collinear. n and p are null-vectors (p is the collinear direction)



$$z_i = \frac{x_i}{x_1 + x_2}, \quad z_1 + z_2 = 1$$

z_i are the momentum fractions

$$\tilde{P}^\mu = (p_1 + p_2)^\mu - \frac{s_{12} n^\mu}{2(p_1 + p_2) \cdot n}$$

Null-vector

COLLINEAR LIMITS

- Let's focus in the collinear limit (1->2 processes)
 - We will work in the LC-gauge (n is the quantization vector). This allows to obtain factorization formulas.
 - Matrix elements have an specific behaviour in the collinear limit:

$$\mathcal{M}(p_1, p_2, \dots, p_n) \underset{s_{12} \rightarrow 0}{\sim} \frac{1}{\sqrt{s_{12}}} \text{mod} (\ln^k s_{12}) \left[1 + \mathcal{O}(\sqrt{s_{12}}) \right]$$

(It can be obtained rescaling the collinear momenta.)

- Matrix elements are divergent in the collinear limit! (Remember previous slides...)
 - Keep only the most divergent contributions when $s_{12} \rightarrow 0$.
- OBSERVATION: These properties can be extended to the multiple collinear limit (see *CdFR* for more information).

COLLINEAR LIMITS

- Color decomposition: We subtract the color structure from the QCD amplitude, so we can introduce the color-ordered amplitudes (which contains kinematical information).

- Tree-level decomposition (only external gluons)

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{\lambda_1}, \dots, n^{\lambda_n}))$$

- One-loop decomposition (only external gluons)

$$\mathcal{A}_n(\{k_i, \lambda_i, a_i\}) = g^n \sum_J n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \text{Gr}_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma)$$

(More details about color decomposition: Dixon, arXiv:hep-ph/9601359v2)

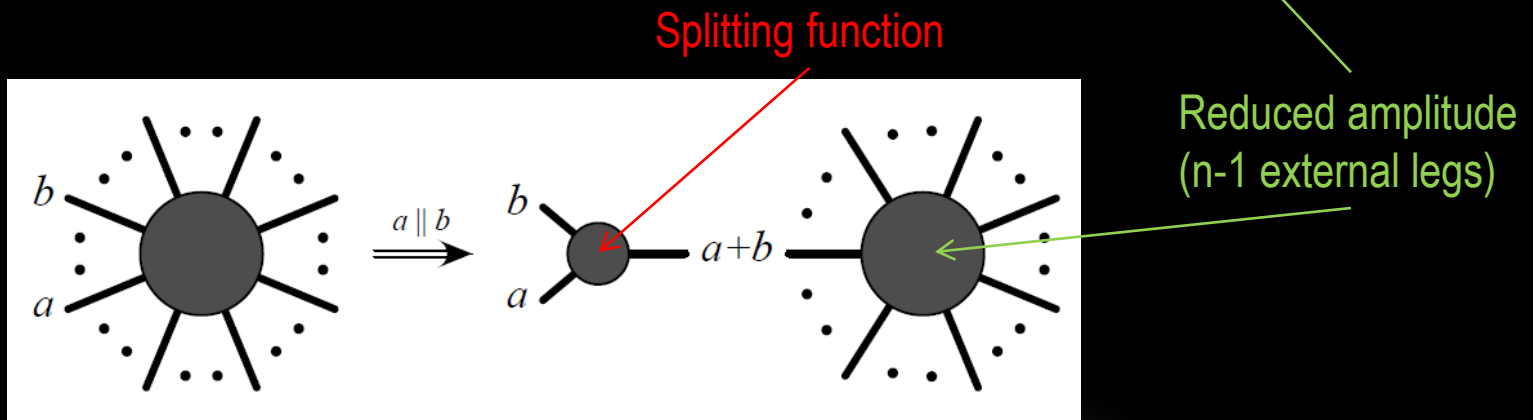
- Properties of color-ordered amplitudes

- *They are singular when two adjacent legs become collinear.*
- *Subleading-color amplitudes are related to leading ones and have a similar collinear behaviour.*

COLLINEAR LIMITS

- Collinear factorization of color-ordered amplitudes: When we are in a collinear configuration, we can introduce universal functions (i.e. process-independent) which describe the collinear behaviour. These are the *splitting functions*.
 - Tree-level factorization

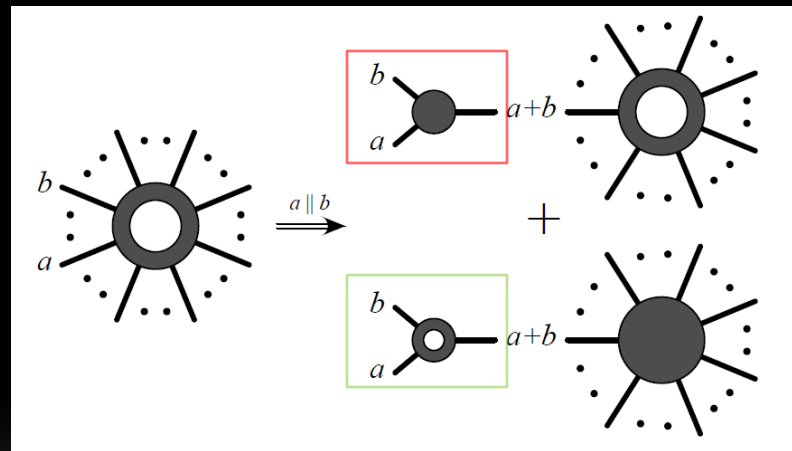
$$A_n^{\text{tree}}(1, \dots, a^{\lambda_a}, b^{\lambda_b}, \dots, n) \xrightarrow{k_a \cdot k_b \rightarrow 0} \sum_{\text{ph. pol. } \sigma} \text{Split}_{-\sigma}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(1, \dots, (a+b)^\sigma, \dots, n)$$



COLLINEAR LIMITS

- Collinear factorization of color-ordered amplitudes
 - One-loop-level factorization

$$A_n^{1\text{-loop}}(1, \dots, a^{\lambda_a}, b^{\lambda_b}, \dots, n) \xrightarrow{a \parallel b} \sum_{\text{ph. pol. } \sigma} \left(\boxed{\text{Split}_{-\sigma}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b})} A_{n-1}^{1\text{-loop}}(1, \dots, (a+b)^\sigma, \dots, n) + \boxed{\text{Split}_{-\sigma}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b})} A_{n-1}^{\text{tree}}(1, \dots, (a+b)^\sigma, \dots, n) \right)$$



COLLINEAR LIMITS

- Collinear factorization of color-ordered amplitudes
 - At one-loop level it is possible to get an explicit formula to compute the splitting function

$$\text{Split}_{-\sigma}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b}) = \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}_{-\sigma}^{\text{tree}}((\ell + a + b)^{-\lambda_2}, (-\ell)^{-\lambda_1}) \frac{i}{(\ell + k_a + k_b)^2} \times A_4^{\text{tree}}((-\ell - a - b)^{\lambda_2}, a^{\lambda_a}, b^{\lambda_b}, \ell^{\lambda_1})$$

- Some remarks:
 - *It is important to sum over the physical polarizations of the intermediate states.*
 - *If we sum over polarization of gluons we get*

$$\sum_{\text{ph. pol. } \sigma} \varepsilon_{\mu}^{\sigma} \varepsilon_{\nu}^{-\sigma} = -g_{\mu\nu} + \frac{q_{\mu} k_{\nu} + k_{\mu} q_{\nu}}{q \cdot k}$$

(Again, see: Dixon, arXiv:hep-ph/9601359v2)

where q is a reference null-vector (not collinear with a or b). This is similar to the light-cone gauge propagator.

COLLINEAR LIMITS

- Collinear factorization in color space: We introduce the *splitting matrices* (which are the analogous of splitting functions but in color+spin space)

- Tree-level factorization

$$|\mathcal{M}^{(0)}(p_1, p_2, \dots, p_n)\rangle \simeq Sp^{(0)}(p_1, p_2; \tilde{P}) \langle \mathcal{M}^{(0)}(\tilde{P}, \dots, p_n)\rangle$$

Splitting matrix at LO

- One-loop level factorization

$$|\mathcal{M}^{(1)}(p_1, p_2, \dots, p_n)\rangle \simeq Sp^{(1)}(p_1, p_2; \tilde{P}) \langle \mathcal{M}^{(0)}(\tilde{P}, \dots, p_n)\rangle + Sp^{(0)}(p_1, p_2; \tilde{P}) \langle \mathcal{M}^{(1)}(\tilde{P}, \dots, p_n)\rangle$$

Splitting matrix at NLO

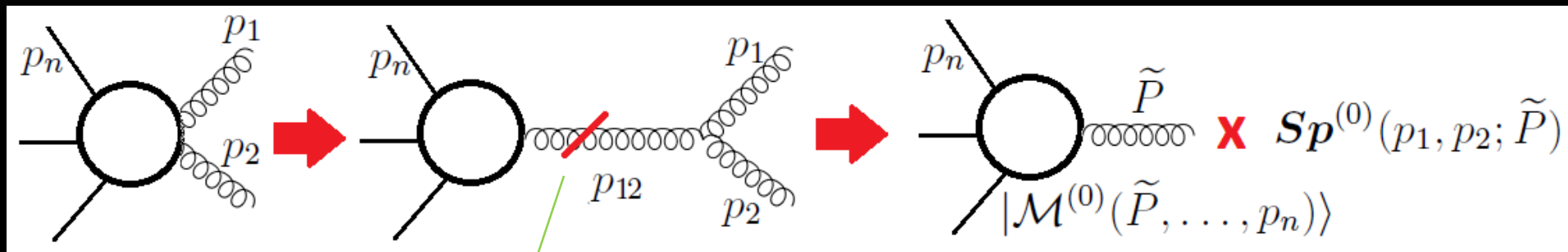
- Some remarks:

- *It is important to note that these expressions are valid up to order $O(\sqrt{s_{12}})$.*
- **Sp** includes color and spin information.
- *At LO we have a relation with splitting functions if we project over color:*

$$Sp^{(0)}(c_1, c_2; c)(p_1, p_2; \tilde{P}) \equiv \langle c_1, c_2 | Sp^{(0)}(p_1, p_2; \tilde{P}) | c \rangle$$

COLLINEAR LIMITS

- Collinear factorization in color space: Graphical motivation (gluon parent)



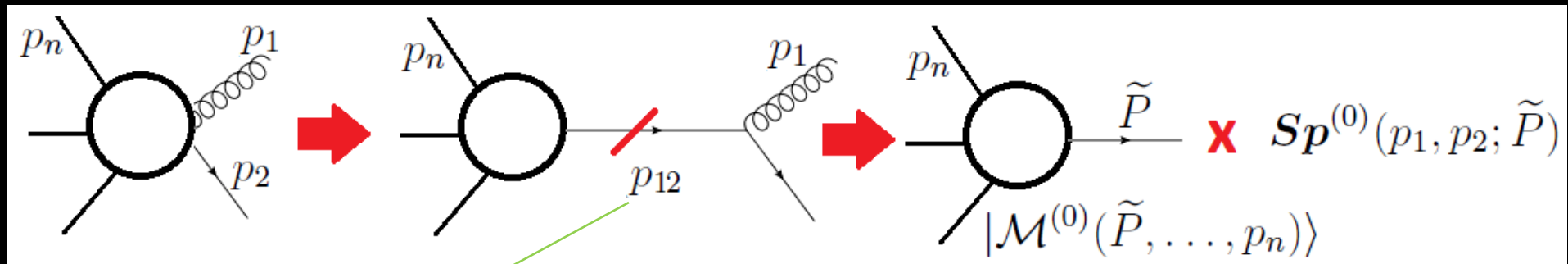
$$\frac{id_{\mu\nu}(p_1 + p_2)}{s_{12}} \approx \frac{id_{\mu\nu}(\tilde{P})}{s_{12}} + \mathcal{O}(s_{12}^0) = \sum_{\text{phys. pol.}} \epsilon_{\mu}^*(\tilde{P}, \sigma) \frac{i\epsilon_{\nu}(\tilde{P}, \sigma)}{s_{12}}$$

Intermediate particle propagator in the collinear limit (dominant contribution)

- Important remarks:
 - *Splitting functions and matrices are computed using a **massless parent** particle.*

COLLINEAR LIMITS

- Collinear factorization in color space: Graphical motivation (quark parent)



$$\frac{i(\not{p}_1 + \not{p}_2)}{s_{12}} \approx \frac{i\not{\tilde{P}}}{s_{12}} + \mathcal{O}(s_{12}^0) = \sum_{\text{phys.pol.}} \frac{v_{u_\sigma}(\tilde{P})}{s_{12}} \bar{u}_\sigma(\tilde{P})$$

Intermediate particle propagator in the collinear limit (dominant contribution)

- Important remarks:
 - Splitting functions and matrices are computed using a **massless parent** particle.

COLLINEAR LIMITS

- Collinear factorization in color space
 - General structure of one-loop splitting matrices

$$Sp^{(1)}(p_1, p_2; \tilde{P}; p_3, \dots, p_n) = Sp_H^{(1)}(p_1, p_2; \tilde{P}) + I_C(p_1, p_2; p_3, \dots, p_n) Sp^{(0)}(p_1, p_2; \tilde{P})$$

One-loop
splitting matrix

Finite
contribution

Singular
contribution

Tree-level
splitting matrix

- More details:
 - Sp_H contains only rational functions of the momenta and only depends on collinear particles.
 - I_C contains transcendental functions and can depend of *non-collinear* particles (through colour correlations). This contribution introduces a violation of *strict* collinear factorization.

COLLINEAR LIMITS

- Relation between splitting functions and Altarelli-Parisi kernels:
 - Altarelli-Parisi kernels are related to the collinear behaviour of squared matrix elements. They also control the evolution of PDF's and FF's (through DGLAP equations).

- LO contribution

$$\langle \hat{P}_{a_1 a_2}^{(0)} \rangle = \left(\frac{s_{12}}{2\mu^2\epsilon} \right) \overline{|Sp_{a_1 a_2}^{(0)}|^2}$$

- NLO contribution

$$\langle \hat{P}_{a_1 a_2}^{(1)} \rangle = \left(\frac{s_{12}}{2\mu^2\epsilon} \right) \left[(Sp^{(0)})^\dagger Sp^{(1)} + (Sp^{(1)})^\dagger Sp^{(0)} \right]$$

- NOTE: Here we are choosing not to include the coupling constant inside the definition of splitting matrices. Also, to get AP kernels, we sum over final-state colors and spins, and average over colors and spins of the parent parton.

$$\langle \hat{P}_{a_1 a_2} \rangle = \langle \hat{P}_{a_1 a_2}^{(0)} \rangle + \frac{\alpha_S}{2\pi} \langle \hat{P}_{a_1 a_2}^{(1)} \rangle + \mathcal{O}(\alpha_S^2)$$

(For more details see: Catani, de Florian and Rodrigo, arXiv:hep-ph/0312067)

COMPUTING SPLITTING FUNCTIONS

- General procedure
 - We have to start with the scattering amplitude associated to the process being studied.
 - Replace the incoming polarization vector/spinor by a massless physical one, associated with the light-like vector \hat{P} .
 - Use the usual Feynman rules to write the amplitude and follow as usual.
- Useful identities

$$\begin{aligned} \epsilon^\mu(k_1, +) &= -\frac{1}{\sqrt{2} \langle 1n \rangle} \bar{u}(n, -) \cdot \gamma^\mu \cdot u(k_1, -) \\ \epsilon^\mu(k_1, -) &= -\frac{1}{\sqrt{2} [n1]} \bar{u}(k_1, -) \cdot \gamma^\mu \cdot u(n, -) \end{aligned}$$

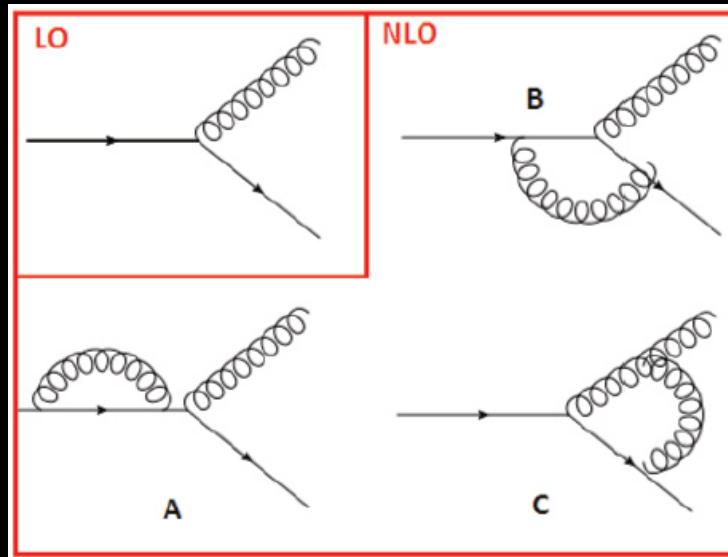
Spinorial representation of massless polarization vectors

$$u(P, +) \rightarrow \frac{1}{[k_P n]} \frac{i \hat{P} \cdot u(n, -)}{P^2} = \frac{1}{[k_P n]} \frac{i (k_1 + k_2) \cdot u(n, -)}{s}$$

Fixing polarizations in splitting amplitudes

COMPUTING SPLITTING FUNCTIONS

- The $q \rightarrow gq$ splitting amplitude
 - We begin drawing all the possible one-loop diagrams



- It is easy to get the LO contribution (it is proportional to quark-gluon vertex)

$$\mathbf{Sp}^{(0)}(p_1, p_2; \tilde{P}) = \frac{g_s}{s_{12}} T^a \bar{u}(p_2) \cdot \not{\epsilon}(p_1) \cdot u(\tilde{P})$$

COMPUTING SPLITTING FUNCTIONS

- The q->gq splitting amplitude
 - To get NLO contribution, we just replace the integrals (available in Kosower&Uwer's paper) and use FeynCalc to handle Dirac algebra.
 - The result can be written as:

C_1 is divergent!

LO structure

$$\mathbf{Sp}^{(1)}(p_1, p_2; \tilde{P}) = \frac{g_s^3}{s_{12}} T^a \left(C_1 \bar{u}(p_2) \cdot \not{\epsilon}(p_1) \cdot u(\tilde{P}) \right. \\ \left. + C_2 \bar{u}(p_2) \cdot \not{\epsilon} \cdot u(\tilde{P}) p_2 \cdot \epsilon(p_1) + C_3 \bar{u}(p_2) \cdot \not{p}_1 \cdot u(\tilde{P}) p_2 \cdot \epsilon(p_1) \right)$$

New structures appear at NLO...

...but C_2 and C_3 are finite!

- It satisfies that the divergent part (in the context of DREG) is proportional to LO!!

FEYNMAN INTEGRALS

- General facts: Definitions
 - Feynman integrals are objects that appear in loop-level computations and have the following structure:

$$I(n_1, \dots, n_N) = \frac{1}{(i\pi^{d/2})^L} \int d^d k_1 \dots d^d k_L \frac{1}{D_1^{n_1} \dots D_N^{n_N}} f(k_1, \dots, k_L, p_1, \dots, p_E)$$

Integration variables (L loops)

Propagators and scalar products (depends on the sign of n)

Function that depends only on scalar variables

- Other way to write integrals:

$$I(p_1, \dots, p_n) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{D_1^{m_1} \dots D_t^{m_t}} S_1^{n_1} \dots S_q^{n_q}$$

FEYNMAN INTEGRALS

- General facts: Usefull notation
 - Shift operators: they act at integrand-level and modifies the power of the denominators

$$\mathbf{n}_i^+ I(n_1, \dots, n_i, \dots, n_N) = I(n_1, \dots, n_i + 1, \dots, n_N)$$

$$\mathbf{n}_i^- I(n_1, \dots, n_i, \dots, n_N) = I(n_1, \dots, n_i - 1, \dots, n_N)$$

$$I(n_1, \dots, n_N) = \frac{1}{(i\pi^{d/2})^L} \int d^d k_1 \dots d^d k_L \frac{1}{D_1^{n_1} \dots D_N^{n_N}}$$

- They appear when we derivate integrals or when multiplying by scalar products

FEYNMAN INTEGRALS

- General facts
 - We can have tensor-type integrals (i.e. with uncontracted Lorentz indices): they can be reduced to scalar integrals (for instance, using Passarino-Veltman method)

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^{\mu_1} q^{\mu_2} \dots q^{\mu_m}}{D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}} = \sum_A F_A(\{k_i\}, \eta_{\mu\nu}) I_A^{\text{scalar}}(\{k_i^\mu k_j^\nu \eta_{\mu\nu}\})$$

Tensorial structure associated with external momenta and metric tensor

- They are divergent! → Use regularization methods (to give sense to these objects...)
- We are going to use dimensional regularization (DREG)

FEYNMAN INTEGRALS

- Classical methods to compute Feynman integrals:
 - Schwinger-Feynman parametrization: this allows to perform the integration in a hypercube. (Formally, we can think that this is the definition of a Feynman integral in d-dimensions)

$$\frac{1}{\prod A_l^{\lambda_l}} = \frac{\Gamma(\sum \lambda_l)}{\prod \Gamma(\lambda_l)} \int_0^1 d\xi_1 \dots \int_0^1 d\xi_L \prod_l \xi_l^{\lambda_l-1} \frac{\delta(\sum \xi_l - 1)}{(\sum A_l \xi_l)^{\sum \lambda_l}}$$
$$A_l = m_l^2 - p_l^2$$

Feynman parameters

$$\frac{1}{(-k^2 + m^2)^\lambda} = \frac{i^\lambda}{\Gamma(\lambda)} \int_0^\infty d\alpha \alpha^{\lambda-1} e^{i(k^2 - m^2)\alpha}$$

Schwinger parameters

- Mellin-Barnes: transform Feynman integrals into complex-value contour integrals (see details in Smirnov's book)

FEYNMAN INTEGRALS: IBP IDENTITIES

- General facts: symmetries
 - Loop momenta reparametrization invariance
 - Lorentz invariance (when working with scalar integrals) \longrightarrow LI identities
 - Scaling invariance \longrightarrow Homogeneity relation
 - IBP relations (using DREG we assume that the integrand vanishes in the boundary of the integration region) \longrightarrow IBP identities
- It is important to note that these relations are not always independent. Redundant information can be used as a consistency check.

FEYNMAN INTEGRALS: IBP IDENTITIES

- Lorentz identities (LI)
 - Since the integral is a Lorentz scalar, it does not change when we perform a transformation in external momenta:

$$p^\mu \rightarrow p^\mu + \delta p^\mu = p^\mu + \delta \epsilon_\nu^\mu p^\nu$$

Transformation parameters

- First order Taylor expansion:

$$I(p_1 + \delta p_1, \dots, p_n + \delta p_n) = I(p_1, \dots, p_n)$$



$$I(p_1 + \delta p_1, \dots, p_n + \delta p_n) = I(p_1, \dots, p_n) + \delta p_1^\mu \frac{\partial}{\partial p_1^\mu} I(p_1, \dots, p_n) + \dots + \delta p_n^\mu \frac{\partial}{\partial p_n^\mu} I(p_1, \dots, p_n)$$

FEYNMAN INTEGRALS: IBP IDENTITIES

- Lorentz identities (LI)
 - We factor the generator and we get

$$\delta\epsilon_{\nu}^{\mu} \left(p_1^{\nu} \frac{\partial}{\partial p_1^{\mu}} + \dots + p_n^{\nu} \frac{\partial}{\partial p_n^{\mu}} \right) I(p_1, \dots, p_n) = 0$$

- Since $\epsilon_{\mu\nu}$ is a 4-dimensional antisymmetric tensor, then we have a maximum of 6 independent LI identities that can be obtained if we contract with antisymmetric tensors:

$$A(p_{i\mu} p_{j\nu}) \left(p_1^{\nu} \frac{\partial}{\partial p_{1\mu}} - p_1^{\mu} \frac{\partial}{\partial p_{1\nu}} + \dots + p_n^{\nu} \frac{\partial}{\partial p_{n\mu}} - p_n^{\mu} \frac{\partial}{\partial p_{n\nu}} \right) I(p_1, \dots, p_n) = 0$$

Antisymmetrized product of external momenta

Remember

$$\delta\epsilon_{\nu}^{\mu} = -\delta\epsilon_{\mu}^{\nu}$$

FEYNMAN INTEGRALS: IBP IDENTITIES

- IBP identities

Vector build using external or loop momenta

Scalar function

$$\int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{\partial}{\partial k_r^\mu} v^\mu f(k_s, p_i) = 0$$

Loop-momentum derivative

- We assume that the expression is integrable (valid because of DREG)
- Generally, we have $L(L+E)$ IBP identities for each integral \rightarrow Many equations!! (although not all of them are independent)

FEYNMAN INTEGRALS: IBP IDENTITIES

- Rescaling identity
 - Assuming that we only have quadratic propagators, then integrals are homogeneous in the masses and external momenta:

$$\left(\sum_i p_i \cdot \frac{\partial}{\partial p_i} + \sum_i m_i \frac{\partial}{\partial m_i} \right) I = \left(Ld - 2 \sum_i n_i \right) I$$

- The property is also valid at integrand level:

$$\left(\sum_i q_i \cdot \frac{\partial}{\partial q_i} + \sum_i m_i \frac{\partial}{\partial m_i} \right) f = \left(-2 \sum_i n_i \right) f$$

Includes loop momenta

$$I_{t,r,s}(s_{12}, s_{13}, s_{23}, d) = \lambda^{-\alpha(d,r,s)} I_{t,r,s}(\lambda^2 s_{12}, \lambda^2 s_{13}, \lambda^2 s_{23}, d)$$

FEYNMAN INTEGRALS: IBP IDENTITIES

- Some applications:
 - *Master integrals reduction*: Hard integrals can be decomposed in terms of easier ones (using Laporta's algorithm). Computer code FIRE implements this technique.

$$\begin{array}{c} \overrightarrow{q} \\ \overrightarrow{p_1} \\ \overleftarrow{p_1} \end{array} \begin{array}{c} \overrightarrow{p_2} \\ \overrightarrow{p_3} \\ \overleftarrow{p_3} \end{array} = \frac{d-3}{p_2 \cdot (p_1 + p_3)} \left[\frac{1}{(p_1 + p_2 + p_3)^2} \begin{array}{c} p_{123} \\ \circ \end{array} - \frac{1}{(p_1 + p_3)^2} \begin{array}{c} p_{13} \\ \circ \end{array} \right]$$

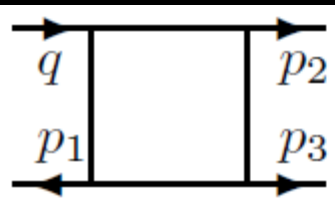
(For more details see: Grozin, arXiv:1104.3993 [hep-ph])

- *Finding new hypergeometric identities*: Since IBP method gives a lot of equivalent differential equations for a Feynman integral and they usually involves hypergeometric or Appell functions, it is possible to explore new relations among them.

(For more details see: Kniehl&Tarasov, arXiv:1108.6019 [math-ph])

FEYNMAN INTEGRALS: IBP IDENTITIES

- Box-integral with massless propagators and 1 off-shell leg


$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k - p_2)^2 (k - p_2 - p_3)^2 (k - p_1 - p_2 - p_3)^2}$$

- Note that we have the same number of propagators and scalar products
- *Master integral*: using LI and IBP equations we are not able to reduce this integral in term of triangles or bubbles.
- However, we will use IBP method to get a solvable differential equation for this integral.

FEYNMAN INTEGRALS: IBP IDENTITIES

- Box-integral with massless propagators and 1 off-shell leg
 - *Step 1:* Apply IBP and relate momenta derivatives with derivatives in terms of s_{ij}
 - *Step 2:* Derivate in terms of external momenta and contract with them

$$\begin{aligned}
 p_1^\mu \frac{\partial}{\partial p_1^\mu} \text{Box}(q, p_1, p_2, p_3) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} \\
 &\quad \left(\frac{2p_1^\mu (k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} \right) \\
 p_2^\mu \frac{\partial}{\partial p_2^\mu} \text{Box}(q, p_1, p_2, p_3) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} \\
 &\quad \left(\frac{2p_2^\mu (k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} + \frac{2p_2^\mu (k-p_2-p_3)_\mu}{(k-p_2-p_3)^2} + \frac{2p_2^\mu (k-p_2)_\mu}{(k-p_2)^2} \right) \\
 p_3^\mu \frac{\partial}{\partial p_3^\mu} \text{Box}(q, p_1, p_2, p_3) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p_2)^2(k-p_2-p_3)^2(k-p_1-p_2-p_3)^2} \\
 &\quad \left(\frac{2p_3^\mu (k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} + \frac{2p_3^\mu (k-p_2-p_3)_\mu}{(k-p_2-p_3)^2} \right)
 \end{aligned}$$


FEYNMAN INTEGRALS: IBP IDENTITIES

- Box-integral with massless propagators and 1 off-shell leg
 - Step 3: Use IBP to simplify (2)
 - Step 4: Introduce the variable $s_{123} = s_{12} + s_{13} + s_{23}$ and rewrite the corresponding differential equation:

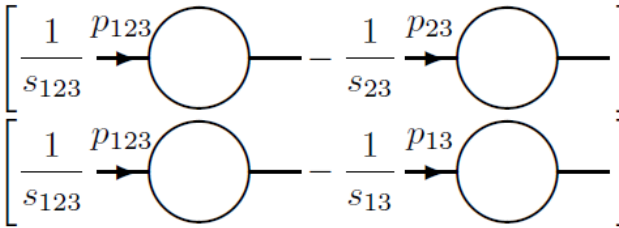
$$\begin{aligned}
 & \frac{\partial}{\partial s_{123}} \left[\text{Box}(q, p_1, p_2, p_3) \right] + \frac{d-4}{2(s_{123} - s_{13} - s_{23})} \text{Box}(q, p_1, p_2, p_3) = \\
 & + \frac{2(d-3)}{(s_{123} - s_{23})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Bubble}(p_{123}) - \frac{1}{s_{23}} \text{Bubble}(p_{23}) \right] \\
 & + \frac{2(d-3)}{(s_{123} - s_{13})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Bubble}(p_{123}) - \frac{1}{s_{13}} \text{Bubble}(p_{13}) \right]
 \end{aligned}$$

FEYNMAN INTEGRALS: IBP IDENTITIES

- Box-integral with massless propagators and 1 off-shell leg
 - *Step 5: Performing the identification*

$$s_{123} = x \quad y(x) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - p_1^2)(q^2 - p_2^2)(q^2 - p_3^2)} \quad f(x) = \frac{d-4}{2(s_{123} - s_{13} - s_{23})}$$


we get

$$g(x) = \frac{2(d-3)}{(s_{123} - s_{23})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Diagram 1} - \frac{1}{s_{23}} \text{Diagram 2} \right] + \frac{2(d-3)}{(s_{123} - s_{13})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Diagram 3} - \frac{1}{s_{13}} \text{Diagram 4} \right]$$


And we finally arrive to a first order differential equation:

$$\frac{\partial y(x)}{\partial x} + f(x)y(x) = g(x) \quad \longrightarrow \quad \text{Known general solution!}$$

CONCLUSIONS

- Understanding IR behaviour of QCD amplitudes is important to calculate higher-order corrections.
 - Splitting amplitudes describe collinear factorization properties and are process-independent quantities (except, maybe, in some kinematical configurations).
 - Being able to compute Feynman integrals is crucial to get higher-order corrections.
 - Traditional methods have problems when increasing the number of legs and/or loops; IBP identities might help to simplify the situation.
-

GRACIAS POR LA ATENCION!

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- Catani, S., de Florian, D. and Rodrigo, G. – *Space-like (vs. time-like) collinear limits in QCD: is factorization violated?* (arXiv:1112.4405)
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- Smirnov, V. A. – *Feynman Integral Calculus* (Springer, 2010)

BACKUP SLIDES

CONTENT

- Regularization methods: DREG
 - IBP Example: Vacuum Diagram
 - IBP Example: Box-integral
 - More IBP applications
 - Feynman integrals: General case
 - Computing splitting amplitudes: Results
-

REGULARIZATION METHODS: DREG

- UV regularization
 - Cut-off: perform the integration until we reach certain maximum momenta K
 - Pauli-Villars: modify propagators to include very massive particles
 - *Dimensional regularization*
- IR regularization
 - Mass regularization: add a tiny mass to internal massless particles
 - *Dimensional regularization*
- Some remarks:
 - When using LC-gauge, some additional, spurious, divergencies can arise. In that case, other regularization method have to be use.
 - Note that dimensional regularization can be used to treat UV and IR divergencies, but, generally, in two steps.

REGULARIZATION METHODS: DREG

- Dimensional regularization
 - Proposed by Giambiagi&Bollini and t'Hooft&Veltman (70's)
 - The main idea is to change the dimension of the space where loop-momentum lives.
 - It has to be thought as an abstract operator (since the «dimension» can be a real number).

$$\mathcal{O}_d[F] = \int d^d \mathbf{x} F(\mathbf{x})$$

$$d = 4 - 2\varepsilon$$



1. linearity $\int d^d \mathbf{x} (aF(\mathbf{x}) + bG(\mathbf{x})) = a \int d^d \mathbf{x} F(\mathbf{x}) + b \int d^d \mathbf{x} G(\mathbf{x})$
2. scaling $\int d^d \mathbf{x} F(s\mathbf{x}) = s^{-d} \int d^d \mathbf{x} F(\mathbf{x})$
3. translational invariance $\int d^d \mathbf{x} F(\mathbf{x} + \mathbf{y}) = \int d^d \mathbf{x} F(\mathbf{x})$

- *Divergencies appear as poles in epsilon*

REGULARIZATION METHODS: DREG

- Dimensional regularization
 - More properties

- $\int d^{d-J} \mathbf{p}_T F(\mathbf{p}) = K_{d-J} \int_0^\infty dp_T p_T^{d-J-1} F(\mathbf{p})$ with $K_{d-J} = \frac{2\pi^{(d-J)/2}}{\Gamma((d-J)/2)}$

- given α, β and M complex numbers

$$\int d^d \mathbf{p} \frac{(\mathbf{p}^2)^\alpha}{(\mathbf{p}^2 + M^2)^\beta} = \pi^{d/2} M^{d+2\alpha-2\beta} \frac{\Gamma(\alpha + d/2) \Gamma(\beta - \alpha - d/2)}{\Gamma(d/2) \Gamma(\beta)}$$

- if F depends on the variable q

$$\frac{\partial}{\partial \mathbf{q}} \int d^d \mathbf{p} F(\mathbf{p}, \mathbf{q}) = \int d^d \mathbf{p} \frac{\partial}{\partial \mathbf{q}} F(\mathbf{p}, \mathbf{q})$$

- if the integral $\int d^n x F(x)$ exists with n natural number, then

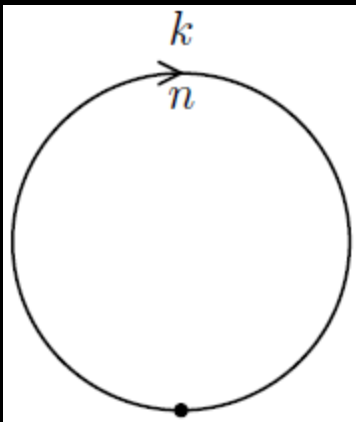
$$\int d^d \mathbf{x} F(\mathbf{x}) \rightarrow \int d^n x F(x)$$

- integrals over different variables in the d -dimensional space commute, i.e

$$\int d^d \mathbf{x} \int d^d \mathbf{y} F(\mathbf{x}, \mathbf{y}) = \int d^d \mathbf{y} \int d^d \mathbf{x} F(\mathbf{x}, \mathbf{y})$$

IBP EXAMPLE: VACUUM DIAGRAM

- Massive 1-loop vacuum diagram



Diagram

$$I(n) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D^n} = V(n) m^{d-2n}$$

Dimensional dependence

Using IBP

$$(d - 2n + 2n1^+) V(n) = 0$$

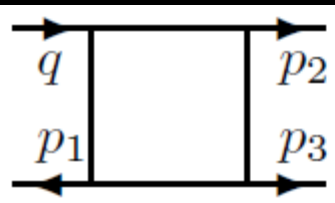
Solving the recurrence (for n greater or equal to 1)

$$I(n) = \frac{1}{\Gamma(n)} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(1 - \frac{d}{2})} V(1) m^{d-2n}$$

$$D = m^2 - k^2 - i0 \quad \text{Propagator}$$

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg



The diagram shows a square loop with four internal propagators. The top-left vertex is labeled q with an arrow pointing right. The top-right vertex is labeled p_2 with an arrow pointing right. The bottom-right vertex is labeled p_3 with an arrow pointing right. The bottom-left vertex is labeled p_1 with an arrow pointing left. The internal loop momentum is k .

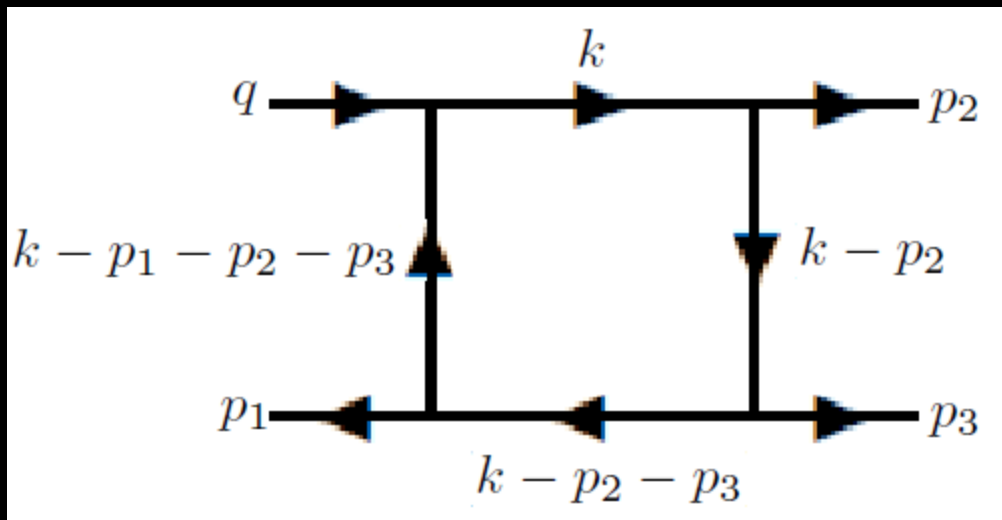
$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k - p_2)^2 (k - p_2 - p_3)^2 (k - p_1 - p_2 - p_3)^2}$$

- Note that we have the same number of propagators and scalar products ($t=4$)
- *Master integral*: using LI and IBP equations we are not able to reduce this integral in term of triangles or bubbles.
- However, we will use IBP method to get a solvable differential equation for this integral.

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg

$$\begin{array}{c} \overrightarrow{q} \\ \overleftarrow{p_1} \end{array} \left[\begin{array}{c} \overrightarrow{p_2} \\ \overrightarrow{p_3} \end{array} \right] = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k - p_2)^2 (k - p_2 - p_3)^2 (k - p_1 - p_2 - p_3)^2}$$



● Squared propagator

× Cancelled propagator

Momentum conventions

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - Step 1: Apply IBP (four equations)

$$\int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \frac{v^\mu}{k^2 (k - p_2)^2 (k - p_2 - p_3)^2 (k - p_1 - p_2 - p_3)^2} = 0$$

- Step 2: Relate momenta derivatives with derivatives in terms of s_{ij}

$$\begin{aligned} s_{12} \frac{\partial}{\partial s_{12}} &= \frac{1}{2} \left(+p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu} \right) \\ s_{13} \frac{\partial}{\partial s_{13}} &= \frac{1}{2} \left(+p_1^\mu \frac{\partial}{\partial p_1^\mu} - p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_3^\mu} \right) \\ s_{23} \frac{\partial}{\partial s_{23}} &= \frac{1}{2} \left(-p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu} + p_3^\mu \frac{\partial}{\partial p_3^\mu} \right) \end{aligned}$$

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - Step 3: Derivate in terms of external momenta and contract with them

$$\begin{aligned}
 p_1^\mu \frac{\partial}{\partial p_1^\mu} \text{Box}(q, p_1, p_2, p_3) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2} \\
 &\quad \left(\frac{2p_1^\mu (k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} \right) \\
 p_2^\mu \frac{\partial}{\partial p_2^\mu} \text{Box}(q, p_1, p_2, p_3) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2} \\
 &\quad \left(\frac{2p_2^\mu (k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} + \frac{2p_2^\mu (k-p_2-p_3)_\mu}{(k-p_2-p_3)^2} + \frac{2p_2^\mu (k-p_2)_\mu}{(k-p_2)^2} \right) \\
 p_3^\mu \frac{\partial}{\partial p_3^\mu} \text{Box}(q, p_1, p_2, p_3) &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2} \\
 &\quad \left(\frac{2p_3^\mu (k-p_1-p_2-p_3)_\mu}{(k-p_1-p_2-p_3)^2} + \frac{2p_3^\mu (k-p_2-p_3)_\mu}{(k-p_2-p_3)^2} \right)
 \end{aligned}$$

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - Step 4: Use IBP to simplify (3)

$$\begin{aligned}
 p_1^\mu \frac{\partial}{\partial p_1^\mu} \text{Box}(q, p_1, p_2, p_3) &= - \text{Box}(q, p_1, p_2, p_3) + \text{Box}(q, p_1, p_2, p_3)_{\text{cut}} \\
 p_2^\mu \frac{\partial}{\partial p_2^\mu} \text{Box}(q, p_1, p_2, p_3) &= - \text{Box}(q, p_1, p_2, p_3) + \text{Box}(q, p_1, p_2, p_3)_{\text{cut}} \\
 p_3^\mu \frac{\partial}{\partial p_3^\mu} \text{Box}(q, p_1, p_2, p_3) &= (d-6) \text{Box}(q, p_1, p_2, p_3) - \text{Box}(q, p_1, p_2, p_3)_{\text{cut}} - \text{Box}(q, p_1, p_2, p_3)_{\text{cut}}
 \end{aligned}$$

Comes from using an IBP identity with $v=k$
(loop momentum)

IBP EXAMPLE: BOX-INTEGRAL

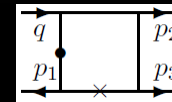
- Box-integral with massless propagators and 1 off-shell leg
 - Previous results (known)

$$\begin{aligned}
 & \left[\text{Box diagram with external momenta } q, p_1, p_2, p_3 \right] = \frac{d-3}{p_2 \cdot (p_1 + p_3)} \left[\frac{1}{(p_1 + p_2 + p_3)^2} \text{ (circle with } p_{123} \text{)} - \frac{1}{(p_1 + p_3)^2} \text{ (circle with } p_{13} \text{)} \right] \\
 & \left[\text{Circle diagram with external momentum } p \right] = \left[\frac{(4\pi)^{\frac{4-d}{2}} \Gamma(3-d/2) \Gamma^2(d/2-1)}{16\pi^2 \Gamma(d-3)} \right] \frac{-2i}{(d-4)(d-3)} (-p^2)^{\frac{d-4}{2}} \equiv A_{2,LO} (-p^2)^{\frac{d-4}{2}}
 \end{aligned}$$

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg

- Step 5: Join (2) and (4), together with the known expansion of



$$\begin{aligned}
 s_{13} \frac{\partial}{\partial s_{13}} \text{Box}(q, p_1, p_2, p_3) &= \frac{d-6}{2} \text{Box}(q, p_1, p_2, p_3) - \frac{2(d-3)}{s_{12} + s_{13}} \left[\frac{1}{s_{123}} \text{Bubble}(p_{123}) - \frac{1}{s_{23}} \text{Bubble}(p_{23}) \right] \\
 s_{23} \frac{\partial}{\partial s_{23}} \text{Box}(q, p_1, p_2, p_3) &= \frac{d-6}{2} \text{Box}(q, p_1, p_2, p_3) - \frac{2(d-3)}{s_{12} + s_{23}} \left[\frac{1}{s_{123}} \text{Bubble}(p_{123}) - \frac{1}{s_{13}} \text{Bubble}(p_{13}) \right] \\
 s_{12} \frac{\partial}{\partial s_{12}} \text{Box}(q, p_1, p_2, p_3) &= -\frac{d-4}{2} \text{Box}(q, p_1, p_2, p_3) + \frac{2(d-3)}{s_{12} + s_{13}} \left[\frac{1}{s_{123}} \text{Bubble}(p_{123}) - \frac{1}{s_{23}} \text{Bubble}(p_{23}) \right] \\
 &\quad + \frac{2(d-3)}{s_{12} + s_{23}} \left[\frac{1}{s_{123}} \text{Bubble}(p_{123}) - \frac{1}{s_{13}} \text{Bubble}(p_{13}) \right]
 \end{aligned}$$

IMPORTANT: Boundary conditions can be read from these equations (make zero the invariant s_{ij})

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - Step 6: Introduce the variable $s_{123} = s_{12} + s_{13} + s_{23}$ and rewrite the corresponding differential equation:

$$\begin{aligned}
 & \frac{\partial}{\partial s_{123}} \left[\text{Box Diagram with external momenta } q, p_1, p_2, p_3 \right] + \frac{d-4}{2(s_{123} - s_{13} - s_{23})} \left[\text{Box Diagram with external momenta } q, p_1, p_2, p_3 \right] = \\
 & + \frac{2(d-3)}{(s_{123} - s_{23})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \left[\text{Bubble Diagram with external momenta } p_{123} \right] - \frac{1}{s_{23}} \left[\text{Bubble Diagram with external momenta } p_{23} \right] \right] \\
 & + \frac{2(d-3)}{(s_{123} - s_{13})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \left[\text{Bubble Diagram with external momenta } p_{123} \right] - \frac{1}{s_{13}} \left[\text{Bubble Diagram with external momenta } p_{13} \right] \right]
 \end{aligned}$$

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - Step 7: Performing the identification

$$s_{123} = x \quad y(x) = \text{Diagram of a box integral with external momenta } q, p_1, p_2, p_3 \quad f(x) = \frac{d-4}{2(s_{123} - s_{13} - s_{23})}$$

we get

$$g(x) = \frac{2(d-3)}{(s_{123} - s_{23})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Diagram} - \frac{1}{s_{23}} \text{Diagram} \right] + \frac{2(d-3)}{(s_{123} - s_{13})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Diagram} - \frac{1}{s_{13}} \text{Diagram} \right]$$

$$\frac{\partial}{\partial s_{123}} \text{Diagram} + \frac{d-4}{2(s_{123} - s_{13} - s_{23})} \text{Diagram} = \frac{2(d-3)}{(s_{123} - s_{23})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Diagram} - \frac{1}{s_{23}} \text{Diagram} \right] + \frac{2(d-3)}{(s_{123} - s_{13})(s_{123} - s_{13} - s_{23})} \left[\frac{1}{s_{123}} \text{Diagram} - \frac{1}{s_{13}} \text{Diagram} \right]$$

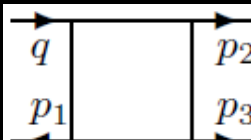


$$\frac{\partial y(x)}{\partial x} + f(x)y(x) = g(x)$$

First order differential equation (known general solution)

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - *Solution*

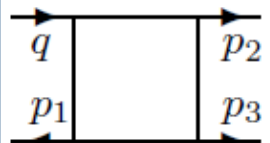


$$(s_{123}, s_{13}, s_{23}) = 2(d-3)A_{2,LO} (s_{13} + s_{23} - s_{123})^{2-\frac{d}{2}} \int^{s_{123}} ds'_{123} (s_{13} + s_{23} - s'_{123})^{\frac{d}{2}-3}$$

$$\left[\frac{(-s_{13})^{\frac{d}{2}-3}}{s_{13} - s'_{123}} + \frac{(-s_{23})^{\frac{d}{2}-3}}{s_{23} - s'_{123}} - \frac{2s'_{123} - s_{13} - s_{23}}{(s_{13} - s'_{123})(s_{23} - s'_{123})} (-s'_{123})^{\frac{d}{2}-3} \right]$$

IBP EXAMPLE: BOX-INTEGRAL

- Box-integral with massless propagators and 1 off-shell leg
 - *Solution (in terms of hypergeometric functions)*



$$\begin{aligned}
 &= -\frac{4(d-3)}{d-4} A_{2,LO} \frac{1}{s_{13}s_{23}} \\
 &\left[\left(\frac{s_{13}s_{23}}{s_{13} - s_{123}} \right)^{\frac{d}{2}-2} {}_2F_1 \left(d/2 - 2, d/2 - 2; d/2 - 1; \frac{s_{123} - s_{13} - s_{23}}{s_{123} - s_{13}} \right) \right. \\
 &+ \left(\frac{s_{13}s_{23}}{s_{23} - s_{123}} \right)^{\frac{d}{2}-2} {}_2F_1 \left(d/2 - 2, d/2 - 2; d/2 - 1; \frac{s_{123} - s_{13} - s_{23}}{s_{123} - s_{23}} \right) \\
 &\left. - \left(\frac{-s_{123}s_{13}s_{23}}{(s_{13} - s_{123})(s_{23} - s_{123})} \right)^{\frac{d}{2}-2} {}_2F_1 \left(d/2 - 2, d/2 - 2; d/2 - 1; \frac{s_{123}(s_{123} - s_{13} - s_{23})}{(s_{123} - s_{13})(s_{123} - s_{23})} \right) \right]
 \end{aligned}$$

MORE IBP APPLICATIONS

- Applications: *master integrals reduction*
 - We get overdetermined linear systems
 - Using IBPs we can modify the exponents that appear in Feynman integrals in one unit.

$$(N_{\text{IBP}} + N_{\text{LI}})N(I_{t,r,s})$$

Available equations

$$N(I_{t,r+1,s+1}) + N(I_{t,r+1,s})$$

Variables (difficult integrals)



If we put together the equations and variables until we reach adequate values of r and s , we can produce an overdetermined linear system



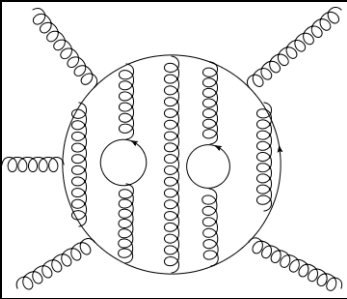
Reduction to easier integrals

$$N(I_{t,r,s}) = \binom{r-1}{r-t} \binom{8-t+s}{s}$$

Number of integrals in each class

FEYNMAN INTEGRALS: GENERAL CASE

- Counting invariant products
 - Let's consider an amplitude with L loops and $E+1$ external legs



$$q_i = k_1, \dots, k_L, p_1, \dots, p_E \quad (i \in [1, M], M = L + E)$$

Momenta

Loop

External

$$s_{ij} = q_i \cdot q_j \quad (j \geq i)$$

Scalar products

- We can have more scalar products (N) than denominators in the integrals.

$$N_E = E(E + 1)/2 \quad s_{ij} \text{ with } i > L$$

$$N = L(L + 1)/2 + LE \quad s_{ij} \text{ with } i \leq L$$

Number of scalar products

Irreducible
denominators

$$D_a = \sum_{i=1}^L \sum_{j=i}^M A_a^{ij} s_{ij} + m_a^2 \quad s_{ij} = \sum_{a=1}^N A_{ij}^a (D_a - m_a^2)$$

Scalar
products

FEYNMAN INTEGRALS: GENERAL CASE

- General facts: Definitions
 - Feynman integrals are objects that appear in loop-level computations and have the following structure:

$$I(n_1, \dots, n_N) = \frac{1}{(i\pi^{d/2})^L} \int d^d k_1 \dots d^d k_L \frac{1}{D_1^{n_1} \dots D_N^{n_N}} f(k_1, \dots, k_L, p_1, \dots, p_E)$$

Integration variables (L loops)

Propagators and scalar products (depends on the sign of n)

Function that depends only on scalar variables

- Other way to write integrals:

$$I(p_1, \dots, p_n) = \int \frac{d^d k}{(2\pi)^d} \frac{d^d l}{(2\pi)^d} \frac{1}{D_1^{m_1} \dots D_t^{m_t}} S_1^{n_1} \dots S_q^{n_q}$$

COMPUTING SPLITTING AMPLITUDES: RESULTS

- The $q \rightarrow gq$ splitting amplitude
 - Explicitly, the coefficients are

$$\tilde{C}_1 = \frac{f_2(\epsilon(6 - \epsilon(\delta\epsilon + N_C^2(\delta\epsilon - 1) + 3)) - 2) - (\epsilon - 1)(2\epsilon - 1)(N_C^2 z_1 f_1(z_1) + (z_1 - 1)f_1(1 - z_1))}{2N_C(\epsilon - 1)(2\epsilon - 1)}$$

$$\tilde{C}_2 = -\frac{\epsilon(N_C^2 z_1^2 f_1(z_1) + (z_1 - 1)^2 f_1(1 - z_1) - 2f_2((N_C^2 - 1)z_1 + 1))}{2N_C n P(z_1 - 1)z_1(2\epsilon - 1)}$$

$$\tilde{C}_3 = \frac{\epsilon}{N_C s(z_1 - 1)z_1(\epsilon - 1)(2\epsilon - 1)} \left((\epsilon - 1)(N_C^2 z_1^2 f_1(z_1) + (z_1 - 1)^2 f_1(1 - z_1)) + f_2(\delta(N_C^2 + 1)(z_1 - 1)) \right) \\ \times z_1 \epsilon^2 - \epsilon(z_1(N_C^2(z_1 + 1) + z_1 - 3) + 2) + 2(N_C^2 - 1)z_1 + 2)$$

with $C_i = -(-s)^{-\epsilon} \tilde{C}_i$ and

$$f_1(z) = \frac{2c_\Gamma}{\epsilon^2} \left(-\Gamma[1 - \epsilon]\Gamma[1 + \epsilon]z^{-1-\epsilon}(1 - z)^\epsilon - \frac{1}{z} + \frac{(1 - z)^\epsilon}{z} {}_2F_1[\epsilon, \epsilon, 1 + \epsilon, z] \right)$$

$$f_2(z) = -\frac{c_\Gamma}{\epsilon^2},$$