Reduction of one-loop (and more?) amplitudes at the Integrand level

Who? Ioannis Malamos

From? Radboud Universiteit Nijmegen

When? Valencia,13 September 2011

Introduction

- Phenomenology at new colliders requires NLO (and NNLO) calculations
- One of the parts of these calculations is the virtual part (loop diagrams)
- Reduction methods are used due to the complexity and the large number of Feynman integrals
- One-loop case completely solved,towards a two-loop reduction method

Historical Background

- Already attempts in the 60's (D.B.Melrose (1965),G.Kallen and J.Toll (1965))
- Passarino-Veltman reduction (general applicability, major achievements but not designed at amplitude level)
- Unitarity based methods- Bern,Dixon,Dunbar,Kosower (major advantage: designed to work at amplitude level, limited applications)
- Qadruple and triple cuts- Britto, Cachazo, Feng âĂę (major simplifications)
- Reduction at the integrand level (Ossola,Papadopoulos,Pittau)

Definitions

- Assuming a base in the linear space of Feynman integrals I_i one can reduce any integral as a linear combination of the base integrals

$$\int A = \sum_i c_i I_i \tag{1}$$

- I_i 's are called Master Integrals
- Such a base is known at one-loop (Scalar Integrals up to 4 denominators)

Example of a P-V reduction to scalar integrals

- Assume the Integral

$$\int d^4q \frac{q^\mu}{D_0 D_1} \tag{2}$$

with $D_i = (q + p_i)^2 - m_i^2, p_0 = 0$

- Then,from Lorentz invariance

$$\int d^4q \frac{q^\mu}{D_0 D_1} = c p_1^\mu \tag{3}$$

- Multiplying with $p_{1\mu}$ and using $q.p_i = \frac{1}{2}[D_i - D_0 - Y_{i0}]$ with $Y_{i0} = (p_i^2 - m_i^2 + m_0^2)]$ we have

$$c = \frac{1}{2p_1^2}[\int d^4q \frac{1}{D_0} - \int d^4q \frac{1}{D_1} - Y_{i0} \int d^4q \frac{1}{D_0 D_1}] \tag{4}$$

- In a similar way all tensor integrals can be decomposed to scalar
- Integrals with large number of denominators can be expressed as integrals with up to 4 denominators in 4 dim (due to vanishing Gram determinants when objects linearly dependent)

OPP method-Reduction at the Integrand level

- For the moment we stick to 4 dimensions (this will give us the so-called cut-constructible part)
- The method works at the integrand level
- The integrand of any m-point one-loop amplitude can be written as

$$A(q) = \frac{N(q)}{D_0 ... D_{m-1}} \tag{5}$$

- The OPP reduction formula is

$$N(q) = \sum_{io<i1<i2<i3}^{m-1} d(q, i_o i_1 i_2 i_3) \prod_{i \neq io,i1,i2,i3}^{m-1} D_i + \sum_{io<i1<i2}^{m-1} c(q, i_o i_1 i_2) \prod_{i \neq io,i1,i2}^{m-1} D_i + \sum_{io<i1}^{m-1} b(q, i_o i_1) \prod_{i \neq io,i1}^{m-1} D_i + \sum_{io}^{m-1} a(q, i_o) \prod_{i \neq io}^{m-1} D_i \tag{6}$$

- The coefficients above are q-dependent
- They can be split into constant terms and terms that vanish upon integration (Spurious terms)
- Spurious terms are known and listed(i.e. $d = d_1 + d_2 T(q)$, where T(q) is known)
- Integrating $\int A = \sum(d_1 \int \frac{1}{DDDD} + c_1 \int \frac{1}{DDD} + b_1 \int \frac{1}{DD} + a_1 \int \frac{1}{D})$

Why do we need spurious terms?

- Imagine trying to decompose a pentagon to boxes without spurious terms at the integrand level
- We can write

$$1 = \alpha_0 D(q) + \alpha_1 D(q+p_1) + ... + \alpha_4 D(q+p_4) \quad (7)$$

- or equivalently

$$1 = \sum_{i=0}^{4} \alpha_i q^2 + 2q_\mu \sum_{i}^{4} \alpha_i p_i^\mu + \sum_{i}^{4} (p_i^2 - m_i^2)\alpha_i \quad (8)$$

- This system of 6 equations and 5 unkowns does NOT have a solution
- We know from Melrose,van Neerven-Vermaseren (and others) this is not the case)
- Notice i.e. that

$$\int d^4q \frac{q_\mu \epsilon^\mu(p_1, p_2, p_3)}{D(q)D(q+p_1)D(q+p_2)D(q+p_3)} = 0 \quad (9)$$

- The term $S = q_\mu \epsilon^\mu(p_1, p_2, p_3)$ is a spurious term for the specific box

- If we now include the spurious terms and rewrite the system

$$1 = \sum_{i=0}^{4} [a_i + \tilde{a}_i S_i] D(q + p_i) \qquad (10)$$

we get a more complicated system,cubic in q,but it's not very hard to show that it has a solution
- With the complete list of spurious terms,every integral can be decomposed a la OPP in scalar integrals up to 4 denominators.
- Such a complete list of one-loop spurious terms is known

Solving the system

- The system can be solved order by order
- The solution can be easily implemented in a computer code (suitable for numerical results) where the only information needed is the numerator (recursive methods)
- For simplicity we show the solution in the case of m=5.The method is general

d-coefficients

- $N(q) = d(q,1234)D_o + ... + d(q,0123)D_4 + \sum(cD_iD_j + ... + aD_iD_jD_kD_l)$
- Find a q_0 such that $D_0 = D_1 = D_2 = D_3 = 0$ putting the propagators on shell.Exactly one solution for q^2
- For this q_0 $N(q_0) = d(q_0,1234)D_4(q_0)$
- $d(q_0,1234) = \frac{N(q_0)}{D_4(q_0)}$ and since the q dependence of d is known we can calculate d(0123)
- In the same way find all d's

c-coefficients

- We try to find q's such that $D_0 = D_1 = D_2 = 0$.Infinite choices
- For such q $N(q) = d(q, 0124)D_3 + d(q, 0123)D_4 + c(q, 012)D_3D_4$
- Since we know d and the q dependence of c we can find c(012)
- In the same way find all c's

b and a coefficients

- Going order by order and putting $D_i = D_j = 0$ we find all b's
- Finally, putting $D_i = 0$ we find all a's.
- Cut-constructible part is calculated
- Solving the system by putting propagators on-shell expalins why this is called Cut-constructible part

- Essentially,one does not need to put propagators to zero to find the solution
- The system becomes easier to be solved that way
- Connection with unitarity methods

Rational terms

- Using dimensional regularisation we depart from 4 dimensions
- The integrand of any m-point one-loop amplitude can now be written as

$$\bar{A}(\bar{q}) = \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}\,, \quad \bar{D}i = (\bar{q} + p_i)^2 - m_i^2\,, \quad (11)$$

  where $\bar{q}$ is the integration momentum.
- Objects denoted with a bar live in $n = 4 + \epsilon$ dimensions while objects denoted with a tilde live in ϵ dimensions

- When an n-dim. index is contracted with a 4-dim. vector,the 4-dim. part is automatically selected

$$\bar{q}^2 = q^2 + \tilde{q}^2 \tag{12}$$

- The numerator function $\bar{N}(\bar{q})$ can be now split into a 4-dim. plus an ϵ-dim. part

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon)\,. \tag{13}$$

- Expanding N(q) in terms of 4-dim. denominators will produce the cut-constructible part
- The price of doing so is adding an extra piece,due to the mismatch of the dimensions,called R_1 and defined as

- $$\mathrm{R}_1 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{f(\tilde{q}^2, q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \,. \qquad (14)$$
- The second term will give rise to the R_2 part defined as
$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \,. \qquad (15)$$
- The cut-constructible part and the R_1 part will be calculated using the OPP method,and we need to find a way to calculate the R_2 part
- $$\int A = C.C. + R_1 + R_2 \qquad (16)$$

Computing R_1

- $$\int A = C.C. + R_1 + R_2 \qquad (17)$$
- The R_1 part can be computed in the framework of the OPP method by performing the expansion in terms of n-dim. denominators
- Coefficients receive a $\tilde{q}^2$ dependence
- In practice,once the 4-dim coefficients are determined,one simply redoes the fits for different values of $\tilde{q}^2$ and determine these new coefficients

Calculation of R_2

- $$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}\,. \qquad (18)$$
- Contributions come either from integrals of $\tilde{q}^2$ either from explicit ϵ parts
- Find Feynman rules adding one-loop 1PI diagrams up to 4 external legs
- Perform a tree level calculation using these Feynman rules (like counterterms)

Literature for R_2 Feynman rules

- Ossola,Papadopoulos,Pittau (hep-ph/0802.1876)(Concept and QED example)
- Draggiotis,Garzelli,Papadopoulos,Pittau (hep-ph/0903.0356) (QCD case)
- Garzelli,I.M.,Pittau (hep-ph/0910.3130)(Electroweak Standrad Model)
- Garzelli,I.M.,Pittau (hep-ph/1009.4302)(Calculation in different gauges,exploring the gauge dependence of the rational parts)

Ingredients needed

- External vectors and γ_5 are 4-dim.
- Algebra in $n = 4 + \epsilon$ dimensions (contraction of gamma matrices, metric tensors,etc...)
- Integrals that contribute

List of integrals

- We give as an example the 2-point integrals

$$\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i-p_j)^2}{3}\right]$$

$$P.P.\left(\int d^n\bar{q}\frac{1}{\bar{D}_i\bar{D}_j}\right) = -2\frac{i\pi^2}{\epsilon},$$

$$P.P.\left(\int d^n\bar{q}\frac{q_\mu}{\bar{D}_i\bar{D}_j}\right) = \frac{i\pi^2}{\epsilon}(p_i+p_j)_\mu,$$

$$P.P.\left(\int d^n\bar{q}\frac{q_\mu q_\nu}{\bar{D}_i\bar{D}_j}\right) = \frac{i\pi^2}{3\epsilon}\left\{\frac{(p_i-p_j)^2 - 3(m_i^2+m_j^2}{2}\right.$$

$$-2\,p_{i\mu}p_{i\nu} - 2\,p_{j\mu}p_{j\nu}$$

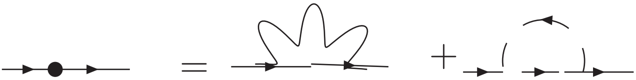
$$\left.-p_{i\mu}p_{j\nu} - p_{j\mu}p_{i\nu}\right\}.$$

Recipe

- Write all possible diagrams for an effective vertex (Generic diagrams)
- Trivial checks to exclude as many as possible that do not contribute
- Calculate the R_2 contribution of the others (FORM)
- Substitute the particles in each case and use the explicit Feynman rules to get the result

Example:Fermion-Fermion effective vertices

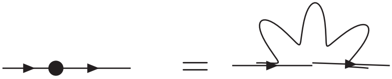
- Write the generic Feynman diagrams



- The second diagram will not contribute

Example:Fermion-Fermion effective vertices

- Contribution comes only from the first generic diagram



- Generic Feynman rule

$$F - F = \frac{ie^2}{\pi^2}\left[\left(C_-\Omega^- + C_+\Omega^+\right)\not{p} + C_0\right] \qquad (20)$$

Example:Fermion-Fermion effective vertices

- Get the real Feynman rules by substituting coupling constants.For leptons

$$
\begin{aligned}
l\bar{l} \; : \; C_- &= \frac{1}{16}\frac{Q_l^2}{c_w^2} \\
C_+ &= \frac{1}{16}\left(\frac{I_{3l}^2}{s_w^2 c_w^2} - \frac{2Q_l I_{3l}}{c_w^2} + \frac{Q_l^2}{c_w^2} + \frac{1}{2s_w^2}\right) \\
C_0 &= \frac{m_l Q_l}{8c_w^2}\left(Q_l - I_{3l}\right) \qquad (21)
\end{aligned}
$$

- All the Feynman rules needed for the calculation of the R_2 part are obtained and tested by means of independent calculations and comparisons
- Gauge invariance of the full rational part $R_1 + R_2$ has been checked
- Explicit test of the gauge invariance of the Four Dimensional Helicity scheme in the complete one-loop standard model was a byproduct of the calculation
- With this method an extra tree-level calculation is needed but it adds no time comparing to the whole 1-loop calculation
- The same method could be extended for other theories as well
- With the OPP method (and the Feynman rules of the R_2 part) a full one-loop calculation can be performed.After one does the reduction at the integrand level, one can use a package available for the scalar one loop integrals and take the result.

Towards a reduction at two-loops

- For many cases NNLO calculations are also needed
- After the succes of one loop people are trying to extend the methods in two loops(Similarities and differences from one to two loops)
- A reduction at the integrand level of two loop amplitudes was recently proposed (Mastrolia,Ossola-hep-ph/1107.6041)

Similarities and differences between one and two loops

- A generic two loop diagram has three kind of propagators

$$
\begin{aligned}
&D(l_1+p_1), D(l_1+p_2), ..., D(l_1+p_n),\\
&D(l_2+q_1), D(l_2+q_2), ..., D(l_2+q_k),\\
&D(l_1+l_2+r_1), D(l_1+l_2+r_2), ..., D(l_1+l_2+r_s),
\end{aligned} \tag{22}
$$

- Due to these mixed propagators the problem is not just a double copy of a one-loop case

Similarities and differences between one and two loops

- At one-loop the base of Master Integrals is known a priori.At two loop not
- We know that the base cannot be just scalar integrals (i.e. $l_1.q_1$ is not reducible,notice the difference with one loop)
- Integration of two-loops integrals is also difficult (We mostly deal with the reduction problem here-at one loop there are packages that one can use after the reductions to take the final result)

What did Mastrolia and Ossola propose?

- Reduction starts at 8 propagators (Reasonable cause in 4 dimensions 2 loop momenta have in total 8 components).Ocatple cut freezes completely the loop momenta
- They introduce the notion of Irreducible Scalar Products (ISP)-The reduction at the integrand level in two loops icludes these ISP which are linear,quadratic,... in the loop momenta.The same happens at one loop but there only the spurious terms are irreducible and thus vanish upon integration leaving only scalar integrals
- We can also have spurious terms here but only for lower cases (when the diagram has at most three independent external momenta)
- They apply their reduction to known cases from literature (doublebox/pentabox in $N = 4$ SYM) and they find agreements

- The reduction formula of Mastrolia and Ossola as logical as it seems has to be proven
- Add things needed to make the method d-dimensional
- There are other methods that people must/are investigating (Duality-tree theorem or FTT,Integration of one of the loop momenta first and then second integration and more...)
- Also programs giving numerical results of two loop integrals must be constructed
- Not that far from starting automatising two loop calculations.

One-loop Conclusions

- We presented the necessary tools to perform a complete one-loop calculation
- We described the OPP method and how one can solve the system to obtain the coefficients needed
- We made clear why one need spurious terms
- When we depart from 4 dimensions Rational terms are needed(that adds an extra tree level calculation)
- Method and programs(i.e HELAC 1-loop) have already produced a lot of significant for collider-physics results.One loop is completely automatised)

Two-loop Conclusions

- We mainly present the Mastrolia-Ossola ideas about two-loop reduction at the integrand level(Kind of OPP in two loops)
- Other methods are also built at the same time
- There is still a lot of work to be done there until all the tools are constructed
- Of course people are mainly interested not only in the virtual part of calculations but at the full NLO /NNLO results