Twistor Methods in Quantum Field Theory

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IFIC

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Motivation

LHC

Overview

1. *Section 1:*

   Recall of the basics

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2. *Section 2:*

   Pure Yang-Mills Theory

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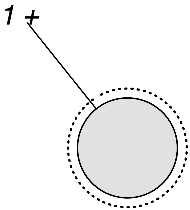
2. *Section 2:*

   Pure Yang-Mills Theory

3. *Section 3:*

   Application to the electroweak gauge boson sector

Explaining MHV



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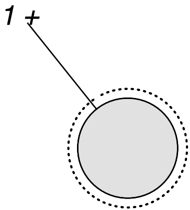

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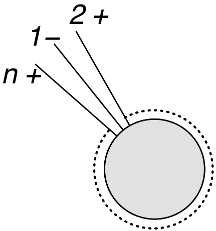

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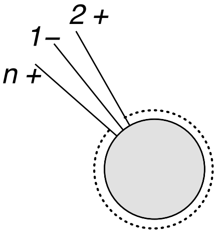

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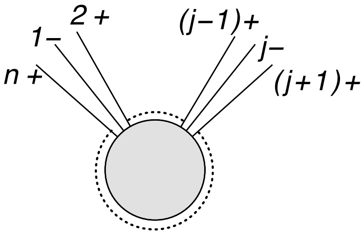

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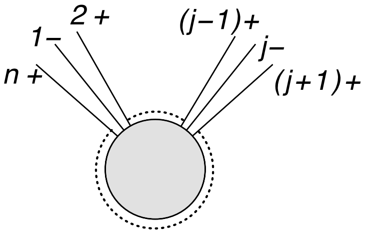

$$= \frac{\langle 1j\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n{-}1,n\rangle\langle n1\rangle}$$

Why bother with MHV?

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- CSW Construction

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- BCF Recursion Relation

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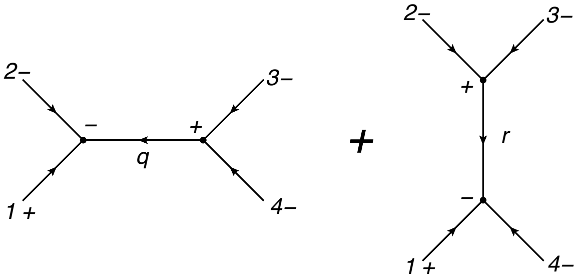
2. Join the vertices together, helicities $+$ to -, using a scalar propagator i/P^2, where P is the momentum flowing between the vertices
3. For each leg of a vertex that joins to a propagator carrying momentum P, define its corresponding holomorphic spinor as $(\lambda_P)_\alpha = P_{\alpha\dot{\alpha}}\tilde{\eta}^{\dot{\alpha}}$, where $\tilde{\eta}$ is some arbitrary spinor

CSW Construction - an example $A(1^+2^-3^-4^-)$



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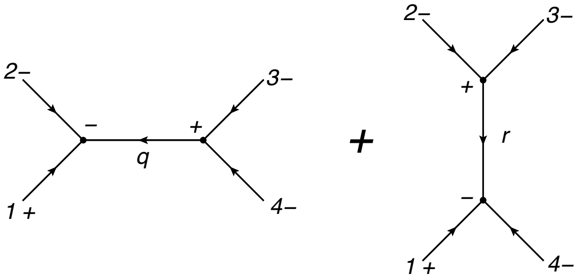
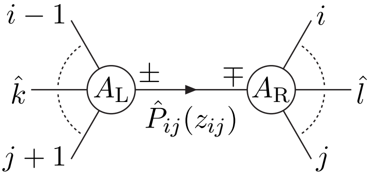


$$= \frac{\langle 2q\rangle^3}{\langle q1\rangle\langle 12\rangle}\frac{1}{q^2}\frac{\langle 34\rangle^3}{\langle 4q\rangle\langle q3\rangle} + \frac{\langle 4q\rangle^3}{\langle q1\rangle\langle 14\rangle}\frac{1}{r^2}\frac{\langle 32\rangle^3}{\langle 2q\rangle\langle q3\rangle} = \cdots = 0$$

BCF Recursion Relation

$$A(1\cdots n) = \sum_{(i,j)\in\mathcal{P}} \sum_{\pm}$$



$$\times \frac{1}{P_{ij}^2}$$

Reducing power

Method	Number of diagrams
Ordinary Feynman Approach	220
Colour Ordering	36
MHV-Vertices	6
BCF-Recursion	2

Table: # of diagrams that contribute to $A_6(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$.

Limitations

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- No corresponding formula for loops

Final aim

Electroweak Gauge Boson Sector	$\xleftarrow{2}$	Transformation on MHV-Lagrangian	$\xrightarrow{1}$	Pure Yang-Mills Sector

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6. Extract a recursion relation and solve it
7. Assemble the pieces and obtain the MHV-Lagrangian

Step 1: Action

$$S = \frac{1}{2g^2} \int d^4x \,\mathrm{tr}\, \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}$$

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$$\text{and} \quad \mathcal{A}_\mu = -igA^a_\mu T^a$$

Step 2: Light cone coordinates

$$x_+ = \frac{1}{\sqrt{2}}\left(t - x^3\right) \qquad x_- = \frac{1}{\sqrt{2}}\left(t + x^3\right)$$

$$x = \frac{1}{\sqrt{2}}\left(x^1 + ix^2\right) \qquad \bar{x} = \frac{1}{\sqrt{2}}\left(x^1 - ix^2\right)$$

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$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$$

Step 3: Integrating Out

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$$\begin{aligned}
\mathcal{L}^{+-} =& \operatorname{tr} \int d^3x \bar{\mathcal{A}} \left(\partial\bar{\partial} - \partial_+ \partial_- \right) \mathcal{A} \\
\mathcal{L}^{-++} =& - \operatorname{tr} \int d^3x \left(\bar{\partial} \partial_-^{-1} \mathcal{A} \right) [\mathcal{A}, \partial_- \bar{\mathcal{A}}] \\
\mathcal{L}^{--+} =& - \operatorname{tr} \int d^3x [\bar{\mathcal{A}}, \partial_- \mathcal{A}] \left(\partial \partial_-^{-1} \bar{\mathcal{A}} \right) \\
\mathcal{L}^{--++} =& - \operatorname{tr} \int [\bar{\mathcal{A}}, \partial_- \mathcal{A}] \partial_-^{-2} [\mathcal{A}, \partial_- \bar{\mathcal{A}}]
\end{aligned}$$

Step 4: Eliminate non MHV vertex

$$\mathcal{L}^{-+}[\mathcal{A}, \bar{\mathcal{A}}] + \mathcal{L}^{-++}[\mathcal{A}, \bar{\mathcal{A}}] \stackrel{!}{=} \mathcal{L}^{-+}[\mathcal{B}, \bar{\mathcal{B}}]$$

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using a canonical transformation

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- Technically easier to solve the master equation
- Canonical transformations leave the dynamics of the physical system untouched

Step 5: Power series expansion

$$A(\mathbf{x}) = \sum_{n=2}^{\infty} \int d^3x_1 \dots d^3x_n \Upsilon(\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_n) B(\mathbf{x}_1) \dots B(\mathbf{x}_n)$$

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- Calculation in momentum space

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$$\Upsilon(\mathbf{p}_1, \ldots, \mathbf{p}_n) = \frac{\left(\sqrt{2}g\right)^{n-1}}{\langle p_1 p_2 \rangle \ldots \langle p_{n-1} p_n \rangle} \frac{p_1^- + \cdots + p_n^-}{\sqrt{p_1^- p_n}}$$

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- Proof by induction

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- James Ettle: MHV Lagrangians for Yang Mills and QCD

Electroweak gauge boson sector

$$S = \frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, \mathcal{W}^{\mu\nu}\mathcal{W}_{\mu\nu} + \frac{1}{2g'^2}\int d^4x \, \mathrm{tr}\, \mathcal{B}^{\mu\nu}\mathcal{B}_{\mu\nu}$$
$$+ \int d^4x \, (D_\mu\phi)^\dagger D^\mu\phi - \int d^4x \, m^2\phi^\dagger\phi + \int d^4x \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$$

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with

$$\phi(x) = \phi_0 + \phi'(x)$$

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Theorem

Let $\mathcal{F}$ be the space of all smooth functions that vanish at infinity. Let $\mathcal{G}$ be all constant functions and $\mathcal{F}$. Furthermore $\phi'(x) \in \mathcal{F}$ and $\phi(x) \in \mathcal{G}$.

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Possible

$\int dx \phi'_1 \partial \phi'_2 = - \int dx \partial \phi'_1 \phi'_2$

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Possible

$\int dx \phi'_1 \partial \phi'_2 = - \int dx \partial \phi'_1 \phi'_2$

Impossible

$\int dx \phi_1 \partial \phi_2 \neq - \int dx \partial \phi_1 \phi_2$

Spin-0-problem

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Count ϕ as "+" and $\phi^\dagger$ as "-".

Gauge Boson Scalar Coupling

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  These have the rough structure $\mathcal{W}\phi\phi = \mathcal{W}(\phi_0 + \phi')(\phi_0 + \phi')$.

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Shift of the minimum of the Higgs potential

Consider the 4ϕ terms in $\mathcal{L}^{--++}$. $(\phi\overleftrightarrow{\partial}\psi := \phi\partial\psi - \partial\phi\psi)$

$$\frac{1}{4}\left(g^2 - g'^2\right)\partial_-^{-1}\left(\phi^\dagger\overleftrightarrow{\partial}_-\phi\right)\partial_-^{-1}\left(\phi^\dagger\overleftrightarrow{\partial}_-\phi\right) \text{ and}$$

$$-\frac{g^2}{2}\mathrm{Tr}\partial_-^{-1}\left(\phi\overleftrightarrow{\partial}_-\phi^\dagger\right)\partial_-^{-1}\left(\phi\overleftrightarrow{\partial}_-\phi^\dagger\right)$$

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Shift of the minimum of the Higgs potential

$$\partial_-^{-1}\left(\phi_0^\dagger\partial_-\phi' + \phi'^\dagger\partial_-\phi' - \partial_-\phi'^\dagger\phi_0 - \partial_-\phi'^\dagger\phi'\right)$$
$$= \left[\phi_0^\dagger\phi' - \phi'^\dagger\phi_0\right] + \partial_-^{-1}\left[\phi'^\dagger\partial_-\phi' - \partial_-\phi'^\dagger\phi'\right]$$

Assembling the pieces

$$\begin{aligned}\mathcal{L}_{\mathsf{EM}} =& \mathcal{L}_{\mathsf{kin}} + \mathcal{L} + \mathcal{L}_{\bar{\phi}\phi} + \mathcal{L}_{\bar{\phi}\phi\bar{\phi}\phi} + \mathcal{L}_{\mu} + \mathcal{L}_{\bar{\phi}_0\phi} + \mathcal{L}_{\bar{\phi}\phi_0} + \mathcal{L}_{\bar{\phi}_0\phi_0} \\ &+ \mathcal{L}_{\bar{\phi}_0\phi\bar{\phi}\phi} + \mathcal{L}_{\bar{\phi}\phi\bar{\phi}\phi_0} + \mathcal{L}_{\bar{\phi}_0\phi\bar{\phi}\phi_0} + \mathcal{L}_{\bar{\phi}_0\phi\bar{\phi}_0\phi} + \mathcal{L}_{\bar{\phi}\phi_0\bar{\phi}\phi_0} + \mathcal{L}_{\bar{\phi}\phi\bar{\phi}_0\phi_0}\end{aligned}$$

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$$\mathcal{L}_{\bar{\phi}\phi} = \sum_{n=3}^{\infty}\sum_{j=2}^{n-1}\int_{(1,\ldots,n)} dP(x)\beta_j(p_1,\ldots,p_n)\tilde{\phi}^{\dagger}(p_1)\tilde{V}(p_2)\ldots\tilde{V}(p_{j-1})\times$$
$$\times\tilde{\bar{V}}(p_j)\tilde{V}(p_{j+1})\ldots\tilde{V}(p_{n-1})\tilde{\phi}(p_n)$$

Assembling the pieces

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$$\times \tilde{\bar{V}}(p_j) \tilde{V}(p_{j+1}) \ldots \tilde{V}(p_{n-1}) \tilde{\phi}(p_n)$$

$$\beta_j(p_1, \ldots, p_n) = -\left(i\sqrt{2}\right)^{n-2} \frac{\langle p_1 p_j \rangle^2 \langle p_j p_n \rangle^2}{\langle p_1 p_2 \rangle \langle p_2 p_3 \rangle \ldots \langle p_{n-1} p_n \rangle \langle p_n p_1 \rangle}$$

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- We know a prescription to bring gauge theories into MHV form
- But:

  We have to adress the caracteristics of the desired theory

$$\int \mathcal{D}\phi\mathcal{D}\Psi \exp \int d^4x\, ik\{\mathrm{tr}[\tfrac{1}{2}\phi\Delta^{-1}\phi + K\phi]\}$$

$$\phi \rightarrow \phi' = \phi + \Delta K \Rightarrow \phi = \phi' - \Delta K$$

$$\int \mathcal{D}\phi\mathcal{D}\Psi \exp \int d^4x\, ik\{\mathrm{tr}[\frac{1}{2}(\phi' - \Delta K)\Delta^{-1}(\phi' - \Delta K) + K(\phi' - \Delta K)]$$

$$= \int \mathcal{D}\phi\mathcal{D}\Psi \exp \int d^4x\, ik\{\mathrm{tr}[\frac{1}{2}(\phi'\Delta^{-1}\phi' - \phi\Delta^{-1}\Delta K - \Delta K\Delta^{-1}\phi' +$$

$$+ \Delta K\Delta^{-1}\Delta K) + K\phi' - K\Delta K]\}$$

$$= \int \mathcal{D}\phi\mathcal{D}\Psi \exp \int d^4x\, ik\{\mathrm{tr}[\frac{1}{2}\phi'\Delta^{-1}\phi' - \frac{1}{2}\phi' K - \frac{1}{2}\Delta K\Delta^{-1}\phi' +$$

$$+ \frac{1}{2}\Delta K \cdot K + K\phi' - K\Delta K]\}$$

$$= \int \mathcal{D}\phi\mathcal{D}\Psi \exp \int d^4x\, ik\{\mathrm{tr}[\frac{1}{2}\phi'\Delta^{-1}\phi' + \frac{1}{2}\phi' K - \frac{1}{2}\Delta K\Delta^{-1}\phi' - \frac{1}{2}K\Delta K$$

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$$= \int \mathcal{D}\phi\mathcal{D}\Psi \exp \int d^4x\, ik\{\mathrm{tr}[\frac{1}{2}\phi'\Delta^{-1}\phi' - \frac{1}{2}K\Delta K]\}$$

Mapping $\mathbf{q} = (q_1 \dots q_n)$, $\mathbf{p} = (p_1 \dots p_n) \to \mathbf{Q} = (Q_1 \dots Q_n)$, $\mathbf{P} = (P_1 \dots P_n)$. $\mathbf{p}$, $\mathbf{P}$ are the canonical momenta.

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \mathbf{q}} \\ \frac{\partial H}{\partial \mathbf{p}} \end{pmatrix} \leftrightarrow \begin{pmatrix} \dot{\mathbf{Q}} \\ \dot{\mathbf{P}} \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial K}{\partial \mathbf{Q}} \\ \frac{\partial P}{\partial \mathbf{P}} \end{pmatrix}$$

Action has to be stationary:

$$\delta \int_{t_1}^{t_2} \left(\dot{q}^i p_i - H \right) dt = 0$$

$$\delta \int_{t_1}^{t_2} \left(\dot{Q}^i P_i - K \right) dt = 0$$

The above equations may only differ by a total derivative:

$$\dot{q}^i p_i - H = \dot{Q}^i P_i - K + \frac{dM}{dt}$$

M generating function.

$$M(\mathbf{q}, \mathbf{P}) = f^i(\mathbf{q}) P_i - Q^i P_i$$

Calculate derivative with respect to time:

$$\frac{dM}{dt} = \dot{q}^i \frac{\partial f^j}{\partial q^i} P_j + f^i(\mathbf{q}) \dot{P}_i - \dot{Q}^i P_i - Q^i \dot{P}_i$$

Plug in:

$$\dot{q}^i p_i - H = -K + \dot{q}^i \frac{\partial f^j}{\partial q^i} P_j + f^i(\mathbf{q}) \dot{P}_i - Q^i \dot{P}_i$$

Compare coefficients

$$\begin{aligned} H =& K \\ f^i(\mathbf{q}) =& Q^i \\ \Rightarrow p_i =& \frac{\partial f^j}{\partial q^i} P_j = \frac{\partial Q^j}{\partial q^i} P_j \end{aligned}$$