$\eta - \eta'$ *Mixing from the Chiral Lagrangian*

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V. Mathieu, V. Vento
“ Pseudoscalar Glueball and $\eta - \eta'$ Mixing”
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arXiv:0910.0212 [hep-ph]

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“$\eta - \eta'$ Mixing in the Flavor Basis and Large N”
arXiv:1003.2119 [hep-ph]
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Motivation

Quark Model with 3 light quarks predicts $3^2 = 9$ $q\bar{q}$ mesons

$$0^{-+} \quad 3\pi,\ 4K,\ \eta \text{ and } \eta'$$

$$1^{--} \quad 3\rho,\ 4K^*,\ \omega \text{ and } \phi$$

$U(3)$ Decomposition leads to 2 isoscalars $[q\bar{q} = (u\bar{u} + d\bar{d})/\sqrt{2}]$

$$\eta_8 = \frac{1}{\sqrt{6}}(\sqrt{2}q\bar{q} - 2s\bar{s}) \qquad \eta_0 = \frac{1}{\sqrt{3}}(\sqrt{2}q\bar{q} + s\bar{s})$$

η and η' mixings of η_8 and η_0 (for vector $\omega - \phi$ mixing)
Decay Properties $\omega \to \pi^+\pi^-$, $\phi \to K^+K^-$

$$\omega(782) \sim q\bar{q} \qquad \phi(1020) \sim s\bar{s}$$

NOT the case for η and η' What is the quark content of η and η' ?
Possible small "glue" content in η'

$$\frac{\Gamma(J/\psi \to \eta'\gamma)}{\Gamma(J/\psi \to \eta\gamma)} = \left(\frac{\langle 0|G\tilde{G}|\eta'\rangle}{\langle 0|G\tilde{G}|\eta\rangle}\right)^2 \left(\frac{M^2_{J/\psi} - M^2_{\eta'}}{M^2_{J/\psi} - M^2_{\eta}}\right)^3 = 4.81 \pm 0.77$$

QCD = gauge theory with the color group $\mathcal{SU}(3)$

$$
\begin{aligned}
\mathcal{L}_{QCD} &= -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \sum \bar{q}(\gamma^\mu D_\mu - m)q \\
F_{\mu\nu} &= \partial_\mu A_\mu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]
\end{aligned}
$$

Degrees of freedom at High Energies: Quarks q and Gluons A_μ
Degrees of freedom at Low Energies: Pions, Kaons,...

Goldstone bosons of Chiral Symmetry Breaking
(Global) Chiral Symmetry: $U(3)_V \otimes U(3)_A$

$$
\begin{aligned}
U(3)_V : &\quad q \rightarrow \exp(i\theta_a \lambda^a)q \\
U(3)_A : &\quad q \rightarrow \exp(i\gamma_5 \theta_{5a} \lambda^a)q
\end{aligned}
$$

$U(3)_A$ broken spontaneously by quark condensate $\langle 0|\bar{q}q|0\rangle \neq 0$
$U(3)_A$ broken explicitly by quark masses m
$\rightarrow$ 9 Goldstone pseudoscalar bosons with a small mass $\propto m\langle 0|\bar{q}q|0\rangle$

3π, $4K$ and 2η

Observation of only 8 Goldstone bosons
Anomaly: Classical symmetry broken by quantum effects

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}A \exp(\mathcal{L}_{QCD} + J \cdot A + \bar{\eta}q + \eta\bar{q}) \\
\mathcal{D}q\mathcal{D}\bar{q} &\xrightarrow{U(1)_A} \mathcal{D}q\mathcal{D}\bar{q} \exp(\theta\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} F_{\alpha\beta})
\end{aligned}
$$

Variation is a total derivative

$$
\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} F_{\alpha\beta} = \partial_\mu K^\mu
$$

But non trivial gauge configuration (Instanton) with different winding number θ (topological charge)

$U(1)_A$ is not a symmetry of QCD
$\rightarrow \eta'$ is NOT a Goldstone boson $M_{\eta'} \sim 958$ MeV $> M_\eta \sim 548$ MeV

But Anomaly vanishes for large N, for a gauge group $SU(N \rightarrow \infty)$

Chiral Lagrangian

Lagrangian with 9 Golstone Boson π^a in the large N limit
Nonlinear parametrization

$$U = \exp(i\sqrt{2}\pi/f) \qquad \pi = \sqrt{2}\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_0}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_8}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} + \frac{\eta_0}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta_8 + \frac{\eta_0}{\sqrt{3}} \end{pmatrix}$$

$$U \in U(3) \qquad UU^\dagger = 1 \qquad U \to LUR^\dagger \qquad L \in U(3)_L, \quad R \in U(3)_R$$

Effective Lagrangian based on Symmetry with a Momentum Expansion p^2

$$\text{Kinetic term at } \mathcal{O}(p^2) \quad \frac{f^2}{8}\left\langle \partial_\mu U^\dagger \partial^\mu U \right\rangle \quad \text{but not} \quad \epsilon_1 \frac{f^2}{8}\left\langle U^\dagger \partial_\mu U \right\rangle \left\langle U \partial^\mu U^\dagger \right\rangle$$

Symmetry breaking terms

Explicit breaking $\qquad B\frac{f^2}{8}\left\langle mU^\dagger + Um^\dagger \right\rangle$

$U(1)$ Anomaly $\qquad \frac{\alpha_0}{6}\left[\frac{f}{4}\left\langle \ln\left(\frac{\det U}{\det U^\dagger}\right)\right\rangle\right]^2 \sim -\frac{1}{2}\alpha_0\eta_0^2$

Chiral Lagrangian at leading order (in p^2 and $1/N$)

$$\mathcal{L}^{(p^2)} = \frac{f^2}{8}\left\langle \partial_\mu U^\dagger \partial^\mu U + B(mU^\dagger + Um^\dagger)\right\rangle - \frac{1}{2}\alpha_0 \eta_0^2$$

Expand $U = 1 + \sqrt{2}\pi_a\lambda^a/f - (\pi_a\lambda^a)^2/f^2 + \cdots$

$$\mathcal{L}^{(p^2)} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^b \delta_{ab} - \frac{1}{2}B\pi^a\pi^b \langle \lambda_a \lambda_b m\rangle - \frac{1}{2}\alpha_0\eta_0^2$$

Isospin Symmetry $m = \mathrm{diag}\ (\tilde{m}, \tilde{m}, \tilde{m}_s) \longrightarrow m_\pi^2 = B\tilde{m}$ and $m_K^2 = B(\tilde{m} + m_s)/2$

Mass matrix in the octet-singlet $(\eta_8 - \eta_0)$

$$\mathcal{M}_{80}^2 = \frac{1}{3}\begin{pmatrix} 4m_K^2 - m_\pi^2 & -2\sqrt{2}(m_K^2 - m_\pi^2) \\ -2\sqrt{2}(m_K^2 - m_\pi^2) & 2m_K^2 + m_\pi^2 + 3\alpha_0 \end{pmatrix}$$

Mass matrix in the flavor basis $(\eta_q - \eta_s)$

$$\mathcal{M}_{qs}^2 = \begin{pmatrix} m_\pi^2 & \sqrt{2}\alpha_0 \\ \sqrt{2}\alpha_0 & 2m_K^2 - m_\pi^2 \end{pmatrix}$$

Anomaly only source of mixing

No anomaly for vector $1^{--} \longrightarrow \omega$ and ϕ ideal mixed

Mass Matrix at Leading Order

Effective Lagrangian for vector (without anomaly)

$$\mathcal{L}^{(p^2)} = \frac{f^2}{8} \left\langle \partial_\mu V^\dagger \partial^\mu V + B(mV^\dagger + Vm^\dagger) \right\rangle$$

Mass matrix in the flavor basis ($\eta_q - \eta_s$)

$$\mathcal{M}^2_{qs} = \begin{pmatrix} m^2_\rho & 0 \\ 0 & 2m^2_{K^*} - m^2_\rho \end{pmatrix}$$

Physical states are $q\bar{q}$ and $s\bar{s}$

2 mass predictions at leading order:

$$m^2_\omega = m^2_\rho$$

Satisfied at 3%
Gell-Mann−Okubo mass formula

$$m^2_\omega + m^2_\phi = 2m^2_{K^*}$$

Satisfied at 8%

Mass Matrix at Leading Order

Mass matrix in the flavor basis $(\eta_q - \eta_s)$

$$\mathcal{M}^2_{qs} = \begin{pmatrix} m^2_\pi & \sqrt{2}\alpha_0 \\ \sqrt{2}\alpha_0 & 2m^2_K - m^2_\pi \end{pmatrix}$$

Rotation to Physical States

$$R^\dagger(\phi)\mathcal{M}^2_{qs}R(\phi) = \begin{pmatrix} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{pmatrix}$$

ϕ determines Decay Properties

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_s \end{pmatrix}.$$

Conservation of trace and determinant

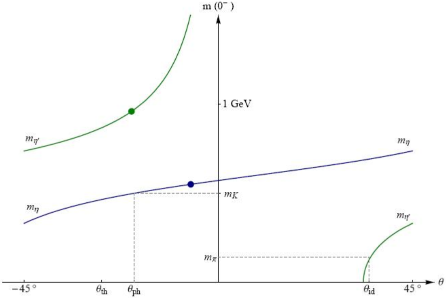
$$\begin{aligned} m^2_\eta + m^2_{\eta'} &= 2m^2_K + \alpha_0 \\ m^2_\eta m^2_{\eta'} &= (4m^2_K - m^2_\pi)\alpha_0/3 + 2m^2_K m^2_\pi \end{aligned}$$

Only 1 parameter α_0 but 2 states (or 2 invariants)

Trace $\alpha_0 = m_\eta^2 + m_{\eta'}^2 - 2m_K^2$

Determinant $\alpha_0 = 3\dfrac{m_\eta^2 m_{\eta'}^2 - 2m_K^2 m_\pi^2}{4m_K^2 - m_\pi^2}$

! Not Equal !



Degrande and Gérard, JHEP **0905** (2009) 043
$\phi \sim (40-45)^\circ$ angle in the flavor basis $(\eta_q - \eta_s)$
$\theta \sim -(15-10)^\circ$ angle in the $U(3)$ basis $(\eta_8 - \eta_0)$

$$\theta = \phi - \theta_i$$

with the ideal mixing angle $\theta_i = \arccos(1/\sqrt{3}) \sim 54.7^\circ$

Georgi Bound

$$\mathcal{L}^{(p^2)} = \frac{f^2}{8}\left\langle \partial_\mu U^\dagger \partial^\mu U + B(mU^\dagger + Um^\dagger)\right\rangle - \frac{1}{2}\alpha_0\eta_0^2$$

Only 1 parameter α_0 but 2 states

$$\mathcal{M}^2_{qs} = \begin{pmatrix} m_\pi^2 & \sqrt{2}\alpha_0 \\ \sqrt{2}\alpha_0 & 2m_K^2 - m_\pi^2 \end{pmatrix}$$

Bound by Georgi, Phys. Rev. D **49** (1994) 1666

$$\frac{M_1^2 - m_\pi^2}{M_2^2 - m_\pi^2} \leq \frac{3-\sqrt{3}}{3+\sqrt{3}} = 0.268 < \left(\frac{m_{\eta'}^2 - m_\pi^2}{m_\eta^2 - m_\pi^2}\right)_{\text{PHYS.}} = 0.313$$

f related to decay constants ; current $A^a_\mu = -f\partial_\mu \pi^a$

$$\left\langle 0|A^a_\mu(x)|\pi^b\right\rangle = -if_\pi p_\mu \delta^{ab} e^{-ipx}.$$

Same Decay Constant $f_\pi = f_K = f$ for All Goldstone Bosons

$$\frac{f_K}{f_\pi} \sim 1.2 \neq 1$$

Solutions ?

1. INCLUDING A THIRD GLUONIC STATE

$$\mathcal{M}^2_{qs} = \begin{pmatrix} m_\pi^2 & \sqrt{2}\alpha_0 \\ \sqrt{2}\alpha_0 & 2m_K^2 - m_\pi^2 \end{pmatrix} \Rightarrow 3 \times 3 \text{ Matrix } \mathcal{M}^2_{qsg}$$

Description of the "glue" content in η'

$$\frac{\Gamma(J/\psi \to \eta'\gamma)}{\Gamma(J/\psi \to \eta\gamma)} = \left(\frac{\langle 0|G\tilde{G}|\eta'\rangle}{\langle 0|G\tilde{G}|\eta\rangle}\right)^2 \left(\frac{M^2_{J/\psi} - M^2_{\eta'}}{M^2_{J/\psi} - M^2_{\eta}}\right)^3 = 4.81 \pm 0.77$$

V. Mathieu, V. Vento, Phys. Rev. D**81**, 034004 (2010)

2. NEXT TO LEADING ORDER $\mathcal{O}(p^4)$
   Different decay constants $f_\pi \neq f_K$

   J. Schechter, A. Subbaraman and H. Weigel, Phys. Rev. D **48**, 339 (1993)

   J. M. Gerard and E. Kou, Phys. Lett. B **616**, 85 (2005)

   V. Mathieu, V. Vento, arXiv:1003.2119 [hep-ph]

3. NEXT TO LEADING ORDER $\mathcal{O}(p^4)$ AND A THIRD STATE
   ...in preparation...

Axial Anomaly $\rightarrow$ mixing with η_0

$$\partial^\mu\left(\bar{q}\gamma_\mu\gamma_5 q\right) = 2im\bar{q}\gamma_5 q + \frac{\alpha_S}{4\pi} G_{\mu\nu}\widetilde{G}^{\mu\nu} \rightarrow \langle gg|\partial^\mu J^0_{\mu 5}|0\rangle \neq 0$$

$$\Rightarrow |\eta^{(\prime)}\rangle = X_{\eta^{(\prime)}}|\eta_q\rangle + Y_{\eta^{(\prime)}}|\eta_s\rangle + Z_{\eta^{(\prime)}}|gg\rangle$$

Mixing Scheme with pseudoscalar glueball

$$\begin{pmatrix}|\eta\rangle \\ |\eta'\rangle \\ |G\rangle\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 \\ 0 & \cos\phi_G & -\sin\phi_G \\ 0 & \sin\phi_G & \cos\phi_G\end{pmatrix}\begin{pmatrix}|\eta_q\rangle \\ |\eta_s\rangle \\ |gg\rangle\end{pmatrix}$$

Measurement of $Z_G^2 = \langle\eta'|gg\rangle^2$

$V \rightarrow P\gamma$ and $P \rightarrow V\gamma \Rightarrow$	0.04 ± 0.09	Escribano and Nadal (2007)
$\dfrac{Br(\phi(1020) \rightarrow \eta'\gamma)}{Br(\phi(1020) \rightarrow \eta\gamma)} \Rightarrow$	0.12 ± 0.04	KLOE collaboration (2008)
$J/\Psi \rightarrow VP \Rightarrow$	0.28 ± 0.21	Escribano (2008)

Where is the physical pseudoscalar glueball $|G\rangle$?

$$\eta(1295), \quad \eta(1405), \quad \eta(1475), \quad \ldots$$

Cheng *et al.*, Phys. Rev D **79** (2008) 014024
He *et al.*, arXiv:0903.5032v1 [hep-ph]

Third Gluonic State

What are the theoretical constraints on R ?

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \\ |G\rangle \end{pmatrix} = \mathcal{R} \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \\ |gg\rangle \end{pmatrix}$$

Inclusion of a third gluonic term in the chiral lagrangian *via* the θ−term and the anomaly : $-\frac{1}{2}\beta\eta_g(\sqrt{2}\eta_q + \eta_s)$

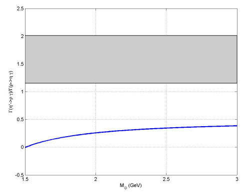
$$\mathcal{M}^2_{qsg} = \begin{pmatrix} m_\pi^2 + 2\alpha & \sqrt{2}\alpha & \sqrt{2}\beta \\ \sqrt{2}\alpha & 2m_K^2 - m_\pi^2 + \alpha & \beta \\ \sqrt{2}\beta & \beta & \gamma \end{pmatrix}$$

Diagonalization $\quad \mathcal{R}^\dagger \mathcal{M}^2_{qsg} \mathcal{R} = \mathcal{D}$
$\mathcal{D} = \mathrm{diag}\ (M_\eta^2, M_{\eta'}^2, M_G^2)$ with M_G^2 unknown
Three unknowns α, β, γ but three rotation invariants

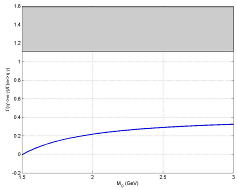
$$\alpha, \beta, \gamma \equiv F(M_\eta^2, M_{\eta'}^2, M_{\eta_\theta}^2) \qquad \Rightarrow \mathcal{R}(M_\eta^2, M_{\eta'}^2, M_G^2) \Rightarrow \ \mathrm{Decays}(M_G^2)$$

$$\beta^2 \geq 0 \longrightarrow M_G^2 \geq \frac{1}{3}(4M_K^2 - M_\pi^2) + \frac{8}{3}\frac{(M_K^2 - M_\pi^2)^2}{4M_K^2 - M_\pi^2 - 3M_\eta^2} \sim (1.5\ \mathrm{GeV})^2$$

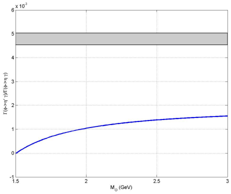
$$\frac{\Gamma(\eta' \to \rho\gamma)}{\Gamma(\rho \to \eta\gamma)}$$

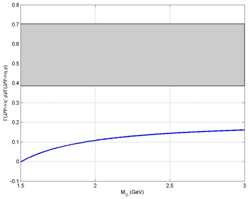


$$\frac{\Gamma(\eta' \to \omega\gamma)}{\Gamma(\omega \to \eta\gamma)}$$

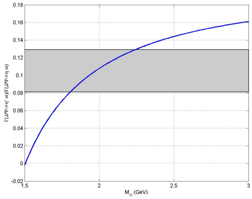


$$\frac{\Gamma(\phi \to \eta'\gamma)}{\Gamma(\phi \to \eta\gamma)}$$

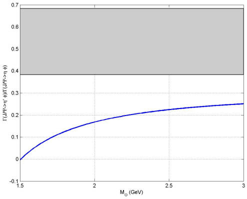




$$\frac{\Gamma(J/\Psi \to \rho\eta')}{\Gamma(J/\Psi \to \rho\eta)}$$



$$\frac{\Gamma(J/\Psi \to \omega\eta')}{\Gamma(J/\Psi \to \omega\eta)}$$



$$\frac{\Gamma(J/\Psi \to \phi\eta')}{\Gamma(J/\Psi \to \phi\eta)}$$

Decays with Octet-Glueball Coupling

Addition of a octet-glueball coupling $-\frac{1}{2}\delta\eta_g(\eta_q - \sqrt{2}\eta_s)$
Mass matrix becomes

$$\mathcal{M}^2_{qsg} = \begin{pmatrix} m_\pi^2 + 2\alpha & \sqrt{2}\alpha & \sqrt{2}\beta + \delta \\ \sqrt{2}\alpha & 2m_K^2 - m_\pi^2 + \alpha & \beta - \sqrt{2}\delta \\ \sqrt{2}\beta + \delta & \beta - \sqrt{2}\delta & \gamma \end{pmatrix}$$

Four unknowns $\alpha, \beta, \gamma, \delta$ but three rotation invariants
Rotation with only two angles

$$\mathcal{R} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_G & -\sin\phi_G \\ 0 & \sin\phi_G & \cos\phi_G \end{pmatrix}$$

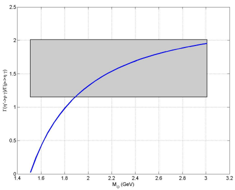
Same constraint on the mass

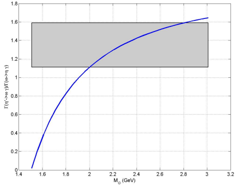
$$\beta^2 \geq 0 \longrightarrow M_G^2 \geq \frac{1}{3}(4M_K^2 - M_\pi^2) + \frac{8}{3}\frac{(M_K^2 - M_\pi^2)^2}{4M_K^2 - M_\pi^2 - 3M_\eta^2} \sim (1.5\text{ GeV})^2$$

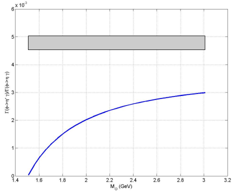
Decays ?

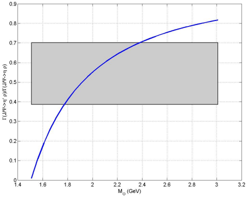
Decay with Octet-Glueball Coupling

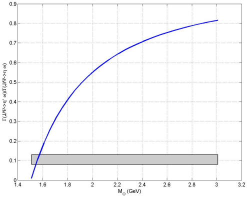
$$\frac{\Gamma(\eta' \to \rho\gamma)}{\Gamma(\rho \to \eta\gamma)} \qquad \frac{\Gamma(\eta' \to \omega\gamma)}{\Gamma(\omega \to \eta\gamma)} \qquad \frac{\Gamma(\phi \to \eta'\gamma)}{\Gamma(\phi \to \eta\gamma)}$$

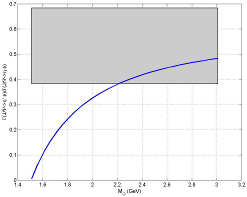












$$\frac{\Gamma(J/\Psi \to \rho\eta')}{\Gamma(J/\Psi \to \rho\eta)} \qquad \frac{\Gamma(J/\Psi \to \omega\eta')}{\Gamma(J/\Psi \to \omega\eta)} \qquad \frac{\Gamma(J/\Psi \to \phi\eta')}{\Gamma(J/\Psi \to \phi\eta)}$$

Octet-Glueball Coupling

Decays more in agreement with data $\Rightarrow$ octet-glueball coupling needed

$$M_G \sim 2.2 \text{ GeV}$$

Mixing scheme for $M_G = 2.2$ GeV

$$\begin{pmatrix} |\eta\rangle \\ |\eta'\rangle \\ |G\rangle \end{pmatrix} = \begin{pmatrix} 0.66 & -0.75 & 0 \\ 0.61 & 0.54 & 0.59 \\ 0.44 & 0.39 & -0.81 \end{pmatrix} \begin{pmatrix} |\eta_q\rangle \\ |\eta_s\rangle \\ |gg\rangle \end{pmatrix}$$

BUT only leading order
At next to leading order, glueball is not needed

$\Rightarrow$ Results might change drastically at NLO

What can we learn from chiral Lagrangian at NLO without glueball ?

3 $\mathcal{O}(p^4)$ Corrections at Large N [Gerard and E. Kou, Phys. Lett. B **616** (2005) 85]

$$\mathcal{L}^{(p^4)} = \mathcal{L}^{(p^2)} + \frac{f^2}{8}\left[-\frac{B}{\Lambda^2}\left\langle m\partial^2 U^\dagger \right\rangle + \frac{B^2}{2\Lambda_1^2}\left\langle m^\dagger U m^\dagger U \right\rangle + \frac{B}{2\Lambda_2^2}\left\langle m^\dagger U \partial_\mu U \partial^\mu U^\dagger \right\rangle \right] + \text{h.c}$$

3 more Low-Energy Constants (LEC)

$$\begin{aligned}
\frac{f_K}{f_\pi} &= 1 + (m_K^2 - m_\pi^2)\left(\frac{1}{\Lambda^2} + \frac{1}{\Lambda_1^2}\right) \\
M_\pi^2 &= m_\pi^2\left[1 + m_\pi^2\left(\frac{2}{\Lambda_1^2} - \frac{1}{\Lambda_2^2}\right)\right] \\
M_K^2 &= m_K^2\left[1 + m_K^2\left(\frac{2}{\Lambda_1^2} - \frac{1}{\Lambda_2^2}\right)\right] \\
M_\eta^2 &= f_\eta(M_\pi^2, M_K^2, \alpha_0, \Lambda_1, \Lambda_2) \\
M_{\eta'}^2 &= f_{\eta'}(M_\pi^2, M_K^2, \alpha_0, \Lambda_1, \Lambda_2)
\end{aligned}$$

Enough parameters to reproduce $M_\pi, M_K, M_\eta, M_{\eta'}, f_\pi, f_K$ and ϕ

But Numerical Procedure

Low Energy Constant do not have a clear physical meaning
Express the mass matrix in term of $y = f_q/f_s$ (related to f_K/f_π)
Difficult from

$$\mathcal{L}^{(p^4)} = \mathcal{L}^{(p^2)} + \frac{f^2}{8}\Big[-\frac{B}{\Lambda^2}\left\langle m\partial^2 U^\dagger \right\rangle + \frac{B^2}{2\Lambda_1^2}\left\langle m^\dagger U m^\dagger U \right\rangle + \frac{B}{2\Lambda_2^2}\left\langle m^\dagger U \partial_\mu U \partial^\mu U^\dagger \right\rangle \Big] + \text{h.c}$$

Rotation preserving unitarity $U^\dagger U = 1$ at $\mathcal{O}(p^4)$

$$U \longrightarrow U - \frac{B}{2\Lambda^2}(m - U m^\dagger U)$$

to kill $\left\langle m\partial^2 U^\dagger \right\rangle$ + h.c.

Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{(p^4)} =& \frac{f^2}{8}\left[\left(\frac{B^2}{2\Lambda_1^2} + \frac{B^2}{2\Lambda^2}\right)\left\langle m^\dagger U m^\dagger U \right\rangle + \left(\frac{B}{2\Lambda_2^2} + \frac{B}{\Lambda^2}\right)\left\langle m^\dagger U \partial_\mu U \partial^\mu U^\dagger \right\rangle + \text{h.c.}\right] \\ &+ \mathcal{L}^{(p^2)} + \alpha_0 \frac{B}{12\Lambda^2}(\sqrt{2}\eta_q + \eta_s)(\sqrt{2}\tilde{m}\eta_q + m_s\eta_s)\end{aligned}$$

Modified Mass Matrix

New Lagrangian after rotation

$$\mathcal{L}^{(p^4)} = \frac{f^2}{8}\left[\left(\frac{B^2}{2\Lambda_1^2} + \frac{B^2}{2\Lambda^2}\right)\left\langle m^\dagger U m^\dagger U\right\rangle + \left(\frac{B}{2\Lambda_2^2} + \frac{B}{\Lambda^2}\right)\left\langle m^\dagger U \partial_\mu U \partial^\mu U^\dagger\right\rangle + \text{h.c.}\right]$$
$$+ \mathcal{L}^{(p^2)} + \alpha_0 \frac{B}{12\Lambda^2}(\sqrt{2}\eta_q + \eta_s)(\sqrt{2}\tilde{m}\eta_q + m_s\eta_s)$$

Rewriting Λ, Λ_1 and Λ_2 in terms of f_q and f_s
Expanding in the fields the kinetic term reads only in the flavor basis

$$\frac{1}{2}\left(\frac{f_q}{f}\right)^2 \partial_\mu\eta_q\partial^\mu\eta_q + \frac{1}{2}\left(\frac{f_s}{f}\right)^2 \partial_\mu\eta_s\partial^\mu\eta_s$$

$\langle\lambda_a\lambda_b m\rangle\,\partial_\mu\pi^a\partial^\mu\pi^b \to \partial_\mu\eta_8\partial^\mu\eta_0$ in the $U(3)$ basis
[Herrera-Siklody, Latorre, Pascual and Taron, Nucl. Phys. B **497** (1997) 345]

Mass matrix with only 2 parameters α and $y = f_q/f_s$

$$\mathcal{M}^2_{qs} = \begin{pmatrix} M_\pi^2 + 2\alpha & \alpha y\sqrt{2} \\ \alpha y\sqrt{2} & M_{ss}^2 + \alpha y^2 \end{pmatrix} + \mathcal{O}\left(\frac{\alpha}{\Lambda^2}\right)$$

Neglecting Flavor mixing term $(\sqrt{2}\eta_q + \eta_s)(\sqrt{2}\tilde{m}\eta_q + m_s\eta_s)$
We can solve analytically with $M_{ss}^2 = 2M_K^2 - M_\pi^2$

Mass Matrix at Leading Order

Chiral Lagrangian at leading order (in p^2 and $1/N$)

$$\mathcal{L}^{(p^2)} = \frac{f^2}{8}\left\langle \partial_\mu U^\dagger \partial^\mu U + B(mU^\dagger + Um^\dagger)\right\rangle - \frac{1}{2}\alpha_0\eta_0^2$$

Expand $U = 1 + \sqrt{2}\pi_a\lambda^a/f - (\pi_a\lambda^a)^2/f^2 + \cdots$

$$\mathcal{L}^{(p^2)} = \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^b\delta_{ab} - \frac{1}{2}B\pi^a\pi^b\langle\lambda_a\lambda_b m\rangle - \frac{1}{2}\alpha_0\eta_0^2$$

Isospin Symmetry $m = \text{diag}\ (\tilde{m}, \tilde{m}, \tilde{m}_s) \longrightarrow m_\pi^2 = B\tilde{m}$ and $m_K^2 = B(\tilde{m} + m_s)/2$

Mass matrix in the octet-singlet $(\eta_8 - \eta_0)$

$$\mathcal{M}_{80}^2 = \frac{1}{3}\begin{pmatrix} 4m_K^2 - m_\pi^2 & -2\sqrt{2}(m_K^2 - m_\pi^2) \\ -2\sqrt{2}(m_K^2 - m_\pi^2) & 2m_K^2 + m_\pi^2 + 3\alpha_0 \end{pmatrix}$$

Mass matrix in the flavor basis $(\eta_q - \eta_s)$

$$\mathcal{M}_{qs}^2 = \begin{pmatrix} m_\pi^2 & \sqrt{2}\alpha_0 \\ \sqrt{2}\alpha_0 & 2m_K^2 - m_\pi^2 \end{pmatrix}$$

Anomaly only source of mixing

No anomaly for vector $1^{--} \longrightarrow \omega$ and ϕ ideal mixed

Solving Analytically $\eta - \eta'$ Mass Matrix

Mass matrix with only 2 parameters α and $y = f_q/f_s$

$$\mathcal{M}_{qs}^2 = \begin{pmatrix} M_\pi^2 + 2\alpha & \alpha y\sqrt{2} \\ \alpha y\sqrt{2} & M_{ss}^2 + \alpha y^2 \end{pmatrix}$$

2 equations (trace and determinant) and 2 parameters

$$\begin{aligned} y^2 &= 2\frac{M_\eta^2 M_{\eta'}^2 - M_{ss}^2(M_\eta^2 + M_{\eta'}^2 - M_{ss}^2)}{M_\pi^2(M_\eta^2 + M_{\eta'}^2 - M_\pi^2) - M_\eta^2 M_{\eta'}^2} \\ \alpha &= \frac{M_\eta^2 + M_{\eta'}^2 - M_\pi^2 - M_{ss}^2}{2 + y^2} \\ \sin 2\phi &= \frac{2\sqrt{2}\alpha y}{M_{\eta'}^2 - M_\eta^2} \end{aligned}$$

Predicted value for the decay constant ratio and mixing angle

$$\frac{f_K}{f_\pi} = \sqrt{\frac{1+y^2}{2y^2}} = 1.15 \qquad \phi = 41.4^\circ$$

Physical value $f_K/f_\pi \sim 1.19$ or $y^2 \sim 0.7$ and $\phi \sim (35-45)^\circ$

J/ψ decays:

$$\frac{\Gamma(J/\psi \to \eta' \rho)}{\Gamma(J/\psi \to \eta \rho)} = (\tan\phi)^2 \left(\frac{k^\rho_{\eta'}}{k^\rho_\eta}\right)^3 = 0.54 \pm 0.16 \to \phi = (39 \pm 2.9)^\circ$$

With k_P^V the meson momenta

Radiative meson decays:

$$\begin{aligned}\frac{\Gamma(\eta' \to \rho\gamma)}{\Gamma(\rho \to \eta\gamma)} &= 3(\tan\phi)^2 \left(\frac{M^2_{\eta'} - M^2_\rho}{M^2_\rho - M^2_\eta}\right)^3 \left(\frac{M_{\eta'}}{M_\rho}\right)^3 \\ &= 1.35 \pm 0.24 \to \phi = (35.5 \pm 5.5)^\circ\end{aligned}$$

Good agreement with the predicted theoretical value $\phi = 41.4^\circ$

$$\begin{aligned}\frac{\Gamma(\eta \to \gamma\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} &= \frac{1}{3}\left(\frac{M_\eta}{M_{\pi^0}}\right)^3 [\cos\phi - y\sin\phi]^2 \,, \\ \frac{\Gamma(\eta' \to \gamma\gamma)}{\Gamma(\pi^0 \to \gamma\gamma)} &= \frac{1}{3}\left(\frac{M_{\eta'}}{M_{\pi^0}}\right)^3 [\sin\phi + y\cos\phi]^2\end{aligned}$$

leads to $y^2 = 0.65$ not so far from the predicted value $y^2 = 0.71$

FKS Hypothesis

Feldmann, Kroll and Stech, Phys. Rev. D **58** (1998) 114006
Prediction of $\phi = 42.4^\circ$ based current algebra

$$\langle 0 | J^{q,s}_{5\mu}(x) | \eta^{(\prime)} \rangle = -i f^{q,s}_{\eta^{(\prime)}} p_\mu e^{-ipx}$$

$$\langle 0 | J^{q(s)}_{5\mu}(x) | \eta_{q(s)} \rangle = -i f_{q(s)} p_\mu e^{-ipx}$$

Decay constants in the flavor basis follow the particle state mixing

$$\begin{pmatrix} f^q_\eta & f^s_\eta \\ f^q_{\eta'} & f^s_{\eta'} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} f_q & 0 \\ 0 & f_s \end{pmatrix}$$

Non flavor transition

$$J^q_{5\mu} | \eta_s \rangle = 0 \qquad J^s_{5\mu} | \eta_q \rangle = 0$$

Mass matrix in the flavor Basis

$$\mathcal{M}^2_{\mathrm{FKS}} = \begin{pmatrix} M^2_{qq} + \frac{\sqrt{2}}{f_q} \langle 0 | G\tilde{G} | \eta_q \rangle & \frac{1}{f_s} \langle 0 | G\tilde{G} | \eta_q \rangle \\ \frac{\sqrt{2}}{f_q} \langle 0 | G\tilde{G} | \eta_s \rangle & M^2_{ss} + \frac{1}{f_s} \langle 0 | G\tilde{G} | \eta_s \rangle \end{pmatrix}$$

Symmetric matrix

$$y = \sqrt{2} \frac{\langle 0 | \frac{\alpha_s}{4\pi} G\tilde{G} | \eta_s \rangle}{\langle 0 | \frac{\alpha_s}{4\pi} G\tilde{G} | \eta_q \rangle} = \frac{f_q}{f_s}$$

FKS Hypothesis

Feldmann, Kroll and Stech, Phys. Rev. D **58** (1998) 114006

$$\begin{pmatrix} f_\eta^q & f_\eta^s \\ f_{\eta'}^q & f_{\eta'}^s \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} f_q & 0 \\ 0 & f_s \end{pmatrix}$$

Non flavor transition

$$J_{5\mu}^q|\eta_s\rangle = 0 \qquad J_{5\mu}^s|\eta_q\rangle = 0$$

Mass matrix in the flavor Basis

$$\mathcal{M}^2_{\text{FKS}} = \begin{pmatrix} M_{qq}^2 + \frac{\sqrt{2}}{f_q}\langle 0|G\tilde{G}|\eta_q\rangle & \frac{1}{f_s}\langle 0|G\tilde{G}|\eta_q\rangle \\ \frac{\sqrt{2}}{f_q}\langle 0|G\tilde{G}|\eta_s\rangle & M_{ss}^2 + \frac{1}{f_s}\langle 0|G\tilde{G}|\eta_s\rangle \end{pmatrix}$$

Symmetric matrix

$$y = \sqrt{2}\frac{\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta_s\rangle}{\langle 0|\frac{\alpha_s}{4\pi}G\tilde{G}|\eta_q\rangle} = \frac{f_q}{f_s}$$

Inputs from Chiral Lagrangian:

$$M_{ss}^2 = 2M_K^2 - M_\pi^2$$

Prediction for ϕ and y with inputs M_K, M_π, M_η, $M_{\eta'}$

FKS from Chiral Lagrangian

FKS mass matrix in the flavor Basis

$$\mathcal{M}^2_{\rm FKS} = \begin{pmatrix} M^2_{qq} + \frac{\sqrt{2}}{f_q}\langle 0|G\tilde{G}|\eta_q\rangle & \frac{1}{f_s}\langle 0|G\tilde{G}|\eta_q\rangle \\ \frac{\sqrt{2}}{f_q}\langle 0|G\tilde{G}|\eta_s\rangle & M^2_{ss} + \frac{1}{f_s}\langle 0|G\tilde{G}|\eta_s\rangle \end{pmatrix}$$

Chiral Lagrangian:

$$\left(\partial^\mu A^0_\mu\right)_{\rm anomaly} = f\alpha_0(\sqrt{2}\eta_q + \eta_s) \equiv 3G\tilde{G}$$

Normalization of the fields $\langle 0|\eta_q|\eta_q\rangle = f/f_q$ and $\langle 0|\eta_s|\eta_s\rangle = f/f_s$

$$\langle 0|G\tilde{G}|\eta_q\rangle = \alpha\sqrt{2}f_q \qquad \langle 0|G\tilde{G}|\eta_s\rangle = \alpha y^2 f_s$$

Translation with previous notation

$$\mathcal{M}^2_{\rm FKS} = \begin{pmatrix} M^2_\pi + 2\alpha & \alpha y^2 \\ \alpha y\sqrt{2} & M^2_{ss} + \alpha y^2 \end{pmatrix}$$

Chiral Lagrangian mass matrix without the approximation concerning $f^{q(s)}_{\eta^{(\prime)}}$

Chiral Lagrangian explains FKS

Summary

Symmetry $\rightarrow \omega(782) \sim q\bar{q} \qquad \phi(1020) \sim s\bar{s}$

Anomaly in pseudoscalar $\rightarrow$ no ideal mixing for η and η'

Chiral Lagrangian at LO not enough to describe η and η'

Inclusion of glueball in chiral Lagrangian

Decays ok but octet-glueball coupling and $M_G \sim 2.2$ GeV

Good description at NLO without glueball but numerical procedure

Small mixing $\langle 0|J^q_{5\mu}|\eta_s\rangle \ll 1$ and $\langle 0|J^s_{5\mu}|\eta_q\rangle \ll 1 \rightarrow$ analytic results

Prediction $\phi = 41.4^\circ$ and $f_K/f_\pi = 1.15$ OK with data

Explanation of FKS formalism

Future developments:

Inclusion of pseudoscalar glueball in the chiral Lagrangian at NLO
to explain glue content of η'