The Hadronic Light-by-Light Scattering Contribution to the Muon $g-2$

Andreas Nyffeler

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Nyffeler, Phys. Rev. D 79, 073012 (2009), arXiv:0901.1172 [hep-ph]

Jegerlehner + Nyffeler, arXiv:0902.3360 [hep-ph] (to appear in Physics Reports)

Instituto de Física Corpuscular, Valencia
June 4, 2009

1. Introduction: Basics of the anomalous magnetic moment

For a spin 1/2 particle:

$$\vec{\mu} = g\frac{e}{2m}\vec{s}, \qquad \underbrace{g = 2}_{\text{Dirac}}(1 + a), \qquad a = \frac{1}{2}(g - 2)$$: anomalous magnetic moment

Long interplay between experiment and theory $\longrightarrow$ structure of fundamental forces

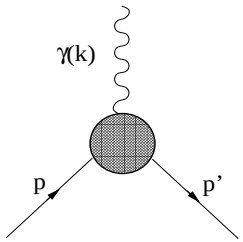
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In Quantum Field Theory (with C,P invariance):



$$= \quad (-ie)\bar{u}(p')\left[\gamma^{\mu}\underbrace{F_1(k^2)}_{\text{Dirac}} + \frac{i\sigma^{\mu\nu}k_{\nu}}{2m}\underbrace{F_2(k^2)}_{\text{Pauli}}\right] u(p)$$

$$F_1(0) = 1 \quad \text{and} \quad F_2(0) = a$$

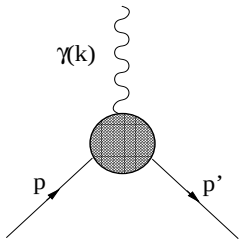
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a_e: $a_e^{\text{exp.}} = (11\,596\,521\,80.73 \pm 0.28) \times 10^{-12}$ [0.24ppb] [Hanneke et al. '08]

Most precise determination of $\alpha = e^2/4\pi$: $\alpha^{-1} = 137.035999084(51)$[0.37ppb]

a_μ: $a_\mu^{\text{exp}} = (11\,659\,2080 \pm 63) \times 10^{-11}$ [0.5ppm] [$g-2$ experiment Brookhaven, '06]

All sectors of Standard Model (SM) contribute: sensitive to electroweak and hadronic corrections and to possible contributions from New Physics:

$a_l \sim \left(\frac{m_l}{m_{\text{NP}}}\right)^2 \Rightarrow \left(\frac{m_\mu}{m_e}\right)^2 \sim 40000$ more sensitive than a_e [exp. precision $\longrightarrow$ "only" factor 19]

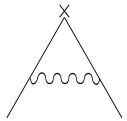
2. Muon $g-2$ in the Standard Model

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{had}}$$

QED contributions (from leptons) and electroweak contributions (EW) under control

QED (up to 5 loops !):

$$a_\mu^{\text{QED}} = (116\,584\,718.101 \pm 0.148) \times 10^{-11}$$



+ . . .

[Schwinger '48; . . . ; Kinoshita et al.; Laporta + Remiddi '96; Czarnecki + Marciano 2000; . . . ; Kinoshita et al. '07, '08 **!**]

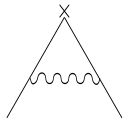
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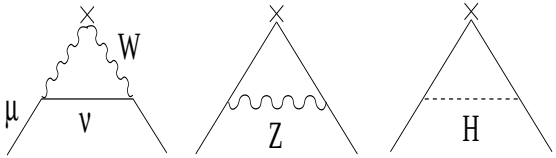


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Electroweak contributions (up to 2 loops):

$$a_\mu^{\text{EW},(1)} = (194.82 \pm 0.02) \times 10^{-11}$$



$$a_\mu^{\text{EW},(2)} = (-42.08 \pm 1.80) \times 10^{-11},$$

large since $\sim G_F m_\mu^2 \frac{\alpha}{\pi} \ln \frac{M_Z}{m_\mu}$

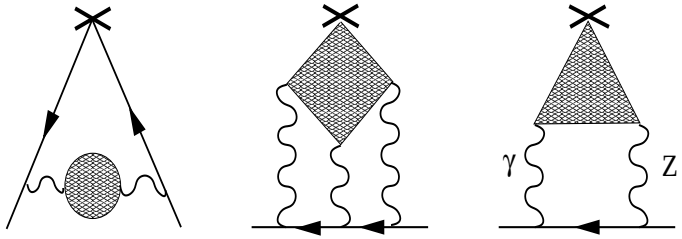
$$a_\mu^{\text{EW}} = (153.2 \pm 1.8) \times 10^{-11}$$

[Brodsky + Sullivan '67; . . . ; Knecht et al. '02; Czarnecki, Marciano, Vainshtein '02]

Hadronic contributions to a_μ

- QCD: quarks bound by strong gluonic interactions into hadronic states
- In particular for the light quarks $u, d, s \longrightarrow$ cannot use perturbation theory !
- Largest source of error in a_μ

Different types of contributions:



(a) (b) (c)

Light quark loop not well defined
$\longrightarrow$ Hadronic "blob"

(a) Hadronic vacuum polarization $\mathcal{O}(\alpha^2), \mathcal{O}(\alpha^3)$

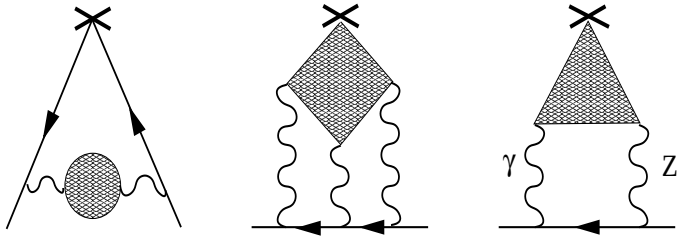
(b) Hadronic light-by-light scattering $\mathcal{O}(\alpha^3)$

(c) 2-loop electroweak contributions $\mathcal{O}(\alpha G_F m_\mu^2)$

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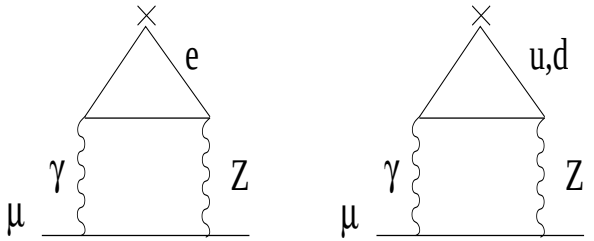
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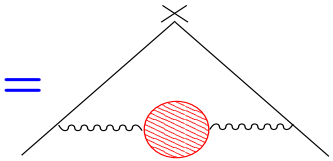
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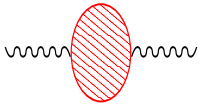
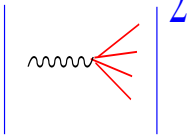
2-Loop EW
Small hadronic uncertainty $\sim 1 \times 10^{-11}$
from triangle diagrams
Anomaly cancellation within each generation !
Cannot separate leptons and quarks !



Hadronic vacuum polarization $\mathcal{O}(\alpha^2)$

At lowest order: can use optical theorem (unitarity)

$a_\mu^{\text{had. v.p.}} =$ 

Im  $\sim$  $\sim \sigma(e^+e^- \longrightarrow \gamma^* \longrightarrow \text{hadrons})$

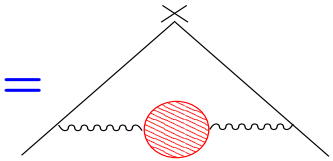
Starting from spectral representation for 2-point function:

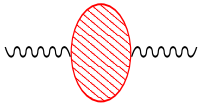
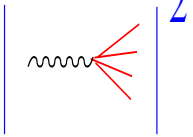
$$a_\mu^{\text{had. v.p.}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s}\, K(s)\, R(s), \qquad R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}$$

$K(s)$ slowly varying and positive; low-energy data very important due to factor $1/s$
[Bouchiat + Michel '61; Durand '62; Brodsky + de Rafael '68; Gourdin + de Rafael '69]

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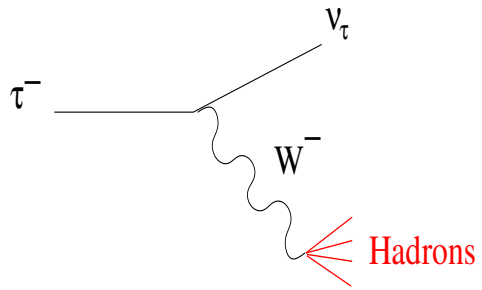
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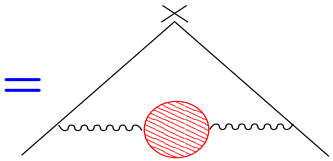
- Hadronic τ-decays e.g. $\tau^- \longrightarrow \nu_\tau \pi^- \pi^0$

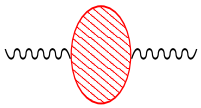
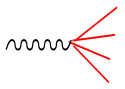
Problem: Corrections due to violation of isospin: $m_u \neq m_d$, electromagnetic radiative corrections
[Alemany et al. '98; Cirigliano et al. '01, '02; Davier et al. '03]



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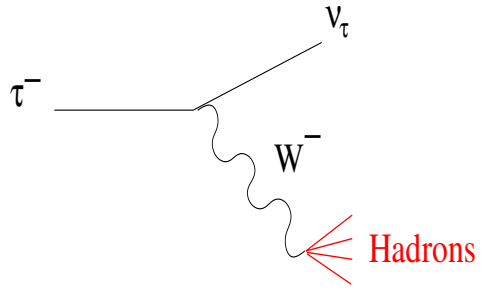
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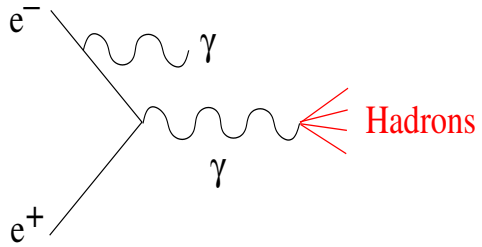
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- "Radiative return" at colliders with fixed center of mass energy (DAΦNE, B-Factories, LEP, ...)
[Binner, Kühn, Melnikov '99; Czyż et al. '00-'03]



Values for $a_\mu^{\text{had v.p.}}$

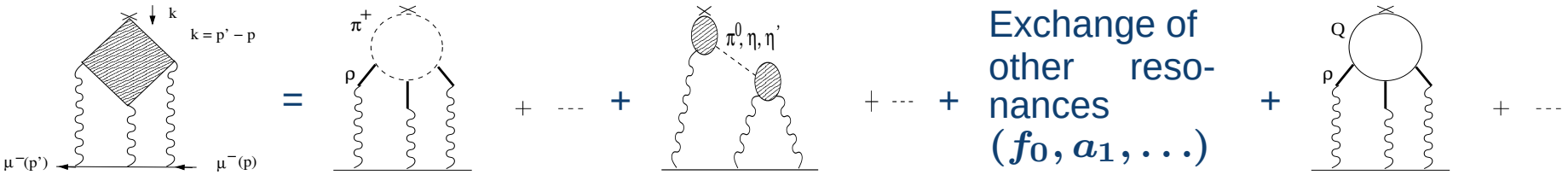
Selection of recent evaluations:

Authors	Contribution to $a_\mu^{\text{had v.p.}} \times 10^{11}$
Davier et al. '03 (e^+e^-)	$6963 \pm 62_{\text{exp}} \pm 36_{\text{rad}}$ $[\pm 72]$
Davier et al. '03 ($e^+e^- + \tau$)	$7110 \pm 50_{\text{exp}} \pm 8_{\text{rad}} \pm 28_{\text{SU(2)}}$ $[\pm 58]$
de Troconiz, Yndurain '05 (e^+e^-)	6935 ± 59
de Troconiz, Yndurain '05 ($e^+e^- + \tau$)	7018 ± 58
Davier et al. '06 (e^+e^-)	6909 ± 44
Hagiwara et al. '07 (e^+e^-, inclusive)	6894 ± 46
Jegerlehner '08 (e^+e^-)	6903.0 ± 52.6

- The last three entries include recent data from SND, CMD-2, BaBar
- Jegerlehner '08 also includes latest data from BaBar and KLOE
- Systematic discrepancy (2-3 σ) of spectral functions obtained from e^+e^- and from τ-data, in particular above the peak of the ρ-meson
- This could be due to an additional source of isospin violation
  Davier '03; Ghozzi + Jegerlehner '04; Benayoun '08
- Because of this uncertainty, τ-data have not been used in recent evaluations of $a_\mu^{\text{had v.p.}}$

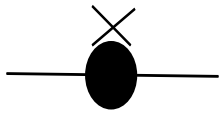
3. Hadronic light-by-light scattering in the muon g-2: anno 2008

Classification of contributions (de Rafael '94)



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

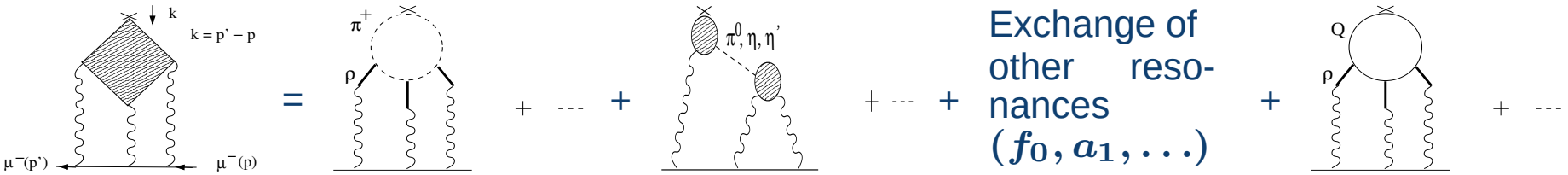
Last term corresponds to local "counterterm" $\sim \bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ in pure EFT (ChPT) approach to had. LbyL scattering.



Relevant scales ~ 500 MeV $- 2$ GeV. No direct relation to exp. data, in contrast to had. VP $\longrightarrow$ need hadronic (resonance) model

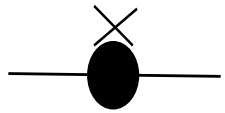
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Contribution to $a_\mu \times 10^{11}$

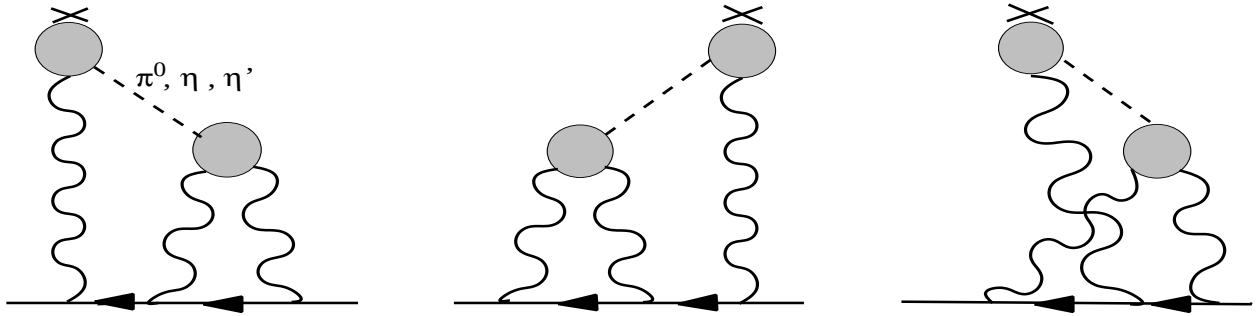
HKS:	+90 (15)	-5 (8)	+83 (6)	+1.7 (1.7) [a_1]	+10 (11)
BPP:	+83 (32)	-19 (13)	+85 (13)	-4 (3) [f_0, a_1]	+21 (3)
KN:	+80 (40)		+83 (12)		
MV:	+136 (25)	0 (10)	+114 (10)	+22 (5) [a_1]	0
2007:	+110 (40)				
		ud.: -45	ud.: $+\infty$		ud.: +60

HKS = Hayakawa, Kinoshita, Sanda; BPP = Bijnens, Pallante, Prades; KN = Knecht, Nyffeler
MV = Melnikov, Vainshtein; 2007 = Bijnens, Prades; Miller, de Rafael, Roberts

ud. = undressed, i.e. point vertices without form factors

4. Pion-exchange contribution: a new short-distance constraint

Pseudoscalar-exchange contribution to a_μ:



Shaded blobs represent off-shell form factor $\mathcal{F}_{\mathrm{PS}^*\gamma^*\gamma^*}$ where $\mathrm{PS} = \pi^0, \eta, \eta'$

Consider π^0. Definition of form factor:

$$\int d^4x\, d^4y\, e^{i(q_1\cdot x + q_2\cdot y)} \langle 0|T\{j_\mu(x) j_\nu(y) P^3(0)\}|0\rangle$$

$$= \varepsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\, \frac{i\langle\overline{\psi}\psi\rangle}{F_\pi}\, \frac{i}{(q_1+q_2)^2 - m_\pi^2}\, \mathcal{F}_{\pi^0*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$$

Up to small mixing effects with the states η and η'

j_μ = light quark part of the electromagnetic current:

$j_\mu(x) = (\overline{\psi}\hat{Q}\gamma_\mu\psi)(x)$, $\psi \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$, $\hat{Q} = \mathrm{diag}(2,-1,-1)/3$: charge matrix

$P^3 = \overline{\psi} i\gamma_5 \frac{\lambda^3}{2} \psi = \left(\overline{u} i\gamma_5 u - \overline{d} i\gamma_5 d\right)/2$, $\langle\overline{\psi}\psi\rangle$ = single flavor quark condensate

Bose symmetry: $\mathcal{F}_{\pi^0*\gamma^*\gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) = \mathcal{F}_{\pi^0*\gamma^*\gamma^*}((q_1+q_2)^2, q_2^2, q_1^2)$

Pion-exchange contribution

Projection onto the muon $g-2$ leads to (Knecht + Nyffeler '01):

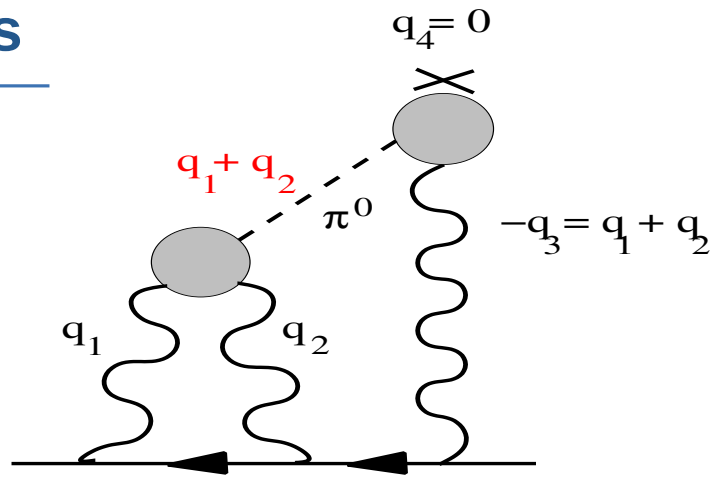
$$
\begin{aligned}
a_\mu^{\mathrm{LbyL};\pi^0} &= -e^6 \int \frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\,\frac{1}{q_1^2 q_2^2 (q_1+q_2)^2[(p+q_1)^2-m_\mu^2][(p-q_2)^2-m_\mu^2]} \\
&\quad\times\left[\frac{\mathcal{F}_{\pi^0*\gamma^**}(q_2^2,q_1^2,(q_1+q_2)^2)\;\mathcal{F}_{\pi^0*\gamma^*\gamma}(q_2^2,q_2^2,0)}{q_2^2-m_\pi^2}\,T_1(q_1,q_2;p)\right. \\
&\quad\left.+\,\frac{\mathcal{F}_{\pi^0*\gamma^**}((q_1+q_2)^2,q_1^2,q_2^2)\;\mathcal{F}_{\pi^0*\gamma^*\gamma}((q_1+q_2)^2,(q_1+q_2)^2,0)}{(q_1+q_2)^2-m_\pi^2}\,T_2(q_1,q_2;p)\right]
\end{aligned}
$$

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T_1(q_1,q_2;p) &= \frac{16}{3}(p\cdot q_1)(p\cdot q_2)(q_1\cdot q_2) - \frac{16}{3}(p\cdot q_2)^2 q_1^2 \\
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&\quad + \frac{16}{3}m_\mu^2 q_1^2 q_2^2 - \frac{16}{3}m_\mu^2(q_1\cdot q_2)^2 \\
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&\quad + \frac{8}{3}m_\mu^2 q_1^2 q_2^2 - \frac{8}{3}m_\mu^2(q_1\cdot q_2)^2
\end{aligned}
$$

where $p^2 = m_\mu^2$ and the external photon has now zero four-momentum (soft photon)

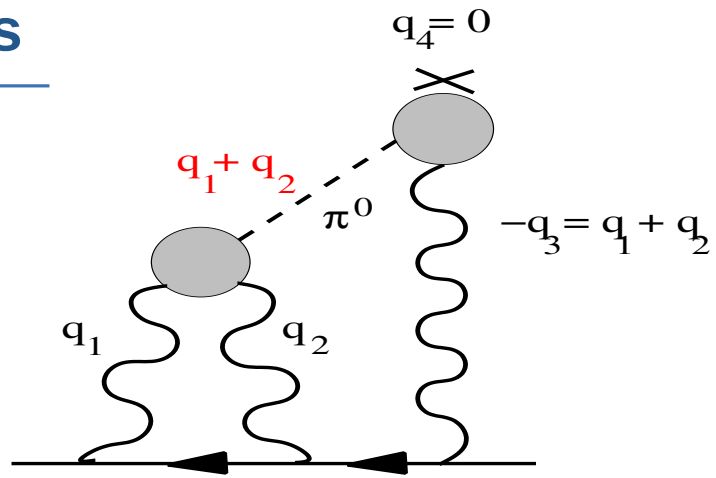
Off-shell versus on-shell form factors

- Jegerlehner '07, '08 pointed out that one should use off-shell form factors $\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ for the evaluation of the pion-exchange contribution. Seems to have been overlooked in the recent literature.
  From T_1:



$$\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0 * \gamma^* \gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

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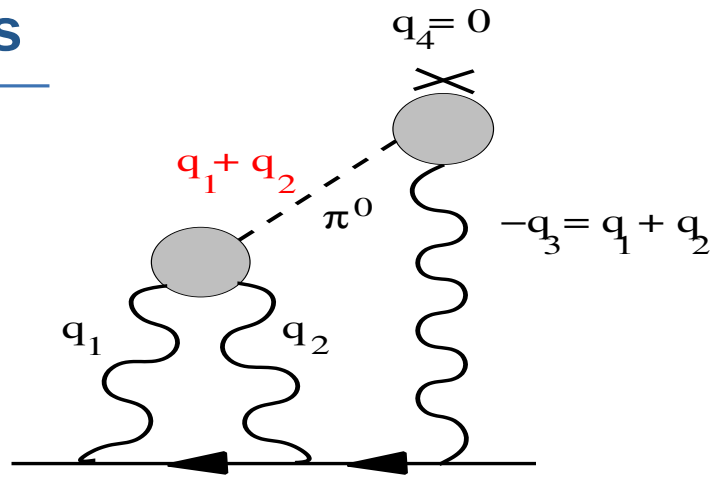
- On the other hand, Knecht + Nyffeler '01, Bijnens + Persson '01 and maybe other references used on-shell form factors:

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  Often the following notation is used: $\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0 \gamma^* \gamma^*}(m_\pi^2, q_1^2, q_2^2)$

- But form factor at external vertex $\mathcal{F}_{\pi^0 \gamma^* \gamma}(m_\pi^2, (q_1+q_2)^2, 0)$ for $(q_1+q_2)^2 \neq m_\pi^2$ violates momentum conservation, since momentum of external soft photon vanishes !

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- Melnikov + Vainshtein '03 had already observed this inconsistency and proposed to use

$$\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(m_\pi^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0 \gamma \gamma}(m_\pi^2, m_\pi^2, 0)$$

  i.e. a constant form factor at the external vertex given by the Wess-Zumino-Witten term
- However, this prescription will only yield the so-called pion-pole contribution and not the full pion-exchange contribution !

Off-shell versus on-shell form factors (cont.)

- In general, any evaluation e.g. using some resonance Lagrangian will lead to off-shell form factors at the vertices and will therefore not yield the pion-pole contribution only.

- Note: strictly speaking, the identification of the pion-exchange contribution is only possible, if the pion is on-shell (or nearly on-shell).

  If one is (far) off the mass shell of the exchanged particle, it is not possible to separate different contributions to the $g-2$, unless one uses some particular model where elementary pions can propagate.

- Following Jegerlehner, we will use off-shell form factors

$$\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((q_1+q_2)^2, q_1^2, q_2^2) \times \mathcal{F}_{\pi^0 * \gamma^* \gamma}((q_1+q_2)^2, (q_1+q_2)^2, 0)$$

  We view our evaluation as being a part of a full calculation of the hadronic light-by-light scattering contribution using a resonance Lagrangian along the lines of the Resonance Chiral Theory (Ecker et al. '89, . . .), which also fulfills all the relevant QCD short-distance constraints.

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$

- 2-loop integral $\longrightarrow$ 8-dim. integral. Integration over 3 angles can be done easily
- 5 non-trivial integrations: 2 moduli: $|q_1|$, $|q_2|$, 3 angles: $p\cdot q_1$, $p\cdot q_2$, $q_1\cdot q_2$ (recall $p^2 = m_\mu^2$).
- Observation: $p\cdot q_1$, $p\cdot q_2$ do not appear in the model-dependent form factors $\mathcal{F}_{\pi^0 *\gamma^* \gamma^*}$
- Can perform those two angular integrations by averaging expression for $a_\mu^{\mathrm{LbyL};\pi^0}$ over the direction of p (Jegerlehner + Nyffeler '09)

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Method of Gegenbauer polynomials (hyperspherical approach)

(Baker, Johnson, Willey '64, '67; Rosner '67; Levine, Roskies '74; Levine, Remiddi, Roskies '79)

Denote by $\hat{K}$ unit vector of four-momentum vector K in Euclidean space

Propagators in Euclidean space:

$$\frac{1}{(K-L)^2+M^2} = \frac{Z_{KL}^M}{|K||L|}\sum_{n=0}^{\infty}\left(Z_{KL}^M\right)^n C_n(\hat{K}\cdot\hat{L})$$

$$Z_{KL}^M = \frac{K^2+L^2+M^2-\sqrt{(K^2+L^2+M^2)^2-4K^2L^2}}{2|K||L|}$$

Use orthogonality conditions of Gegenbauer polynomials:

$$\int \mathrm{d}\Omega(\hat{K})\, C_n(\hat{Q}_1\cdot\hat{K})\, C_m(\hat{K}\cdot\hat{Q}_2) = 2\pi^2\,\frac{\delta_{nm}}{n+1}\,C_n(\hat{Q}_1\cdot\hat{Q}_2)$$

$$\int \mathrm{d}\Omega(\hat{K})\, C_n(\hat{Q}\cdot\hat{K})\, C_m(\hat{K}\cdot\hat{Q}) = 2\pi^2\,\delta_{nm}$$

$\hat{Q}_1\cdot\hat{K}$ = Cosine of angle between the four-dimensional vectors Q_1 and K

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

Average over direction $\hat{P}$ (note: $P^2 = -m_\mu^2$):

$$\langle \cdots \rangle = \frac{1}{2\pi^2} \int \mathrm{d}\Omega(\hat{P}) \, \cdots$$

After reducing numerators in the functions T_i in $a_\mu^{\mathrm{LbyL};\pi^0}$ against denominators of propagators, one is left with the following integrals, denoting propagators by $(4) \equiv (P+Q_1)^2 + m_\mu^2$, $(5) \equiv (P-Q_2)^2 + m_\mu^2$:

$$\begin{aligned}
\langle \frac{1}{(4)} \frac{1}{(5)} \rangle &= \frac{1}{m_\mu^2 R_{12}} \arctan\left(\frac{zx}{1-zt}\right) \\
\langle (P\cdot Q_1) \frac{1}{(5)} \rangle &= -(Q_1\cdot Q_2) \frac{(1-R_{m2})^2}{8m_\mu^2} \\
\langle (P\cdot Q_2) \frac{1}{(4)} \rangle &= (Q_1\cdot Q_2) \frac{(1-R_{m1})^2}{8m_\mu^2} \\
\langle \frac{1}{(4)} \rangle &= -\frac{1-R_{m1}}{2m_\mu^2} \\
\langle \frac{1}{(5)} \rangle &= -\frac{1-R_{m2}}{2m_\mu^2}
\end{aligned}$$

$Q_1 \cdot Q_2 = Q_1\, Q_2\, \cos\theta$, $t = \cos\theta$ (θ = angle between Q_1 and Q_2), $Q_i \equiv |Q_i|$

$R_{mi} = \sqrt{1+4m_\mu^2/Q_i^2}$, $x = \sqrt{1-t^2}$, $R_{12} = Q_1\, Q_2\, x$, $z = \frac{Q_1 Q_2}{4m_\mu^2}\,(1-R_{m1})\,(1-R_{m2})$

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

In this way one obtains (Jegerlehner + Nyffeler '09)

$$a_\mu^{\mathrm{LbL};\pi^0} = -\frac{2\alpha^3}{3\pi^2}\int_0^\infty \mathrm{d}Q_1\,\mathrm{d}Q_2 \int_{-1}^{+1} \mathrm{d}t\,\sqrt{1-t^2}\,Q_1^3\,Q_2^3$$
$$\times\left[\frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}(-Q_2^2,-Q_1^2,-Q_3^2)\,\mathcal{F}_{\pi^0*\gamma^*\gamma}(-Q_2^2,-Q_2^2,0)}{(Q_2^2+m_\pi^2)}\,I_1(Q_1,Q_2,t)\right.$$
$$\left.+\frac{\mathcal{F}_{\pi^0*\gamma^*\gamma^*}(-Q_3^2,-Q_1^2,-Q_2^2)\,\mathcal{F}_{\pi^0*\gamma^*\gamma}(-Q_3^2,-Q_3^2,0)}{(Q_3^2+m_\pi^2)}\,I_2(Q_1,Q_2,t)\right]$$

where $Q_3^2=(Q_1+Q_2)^2$, $\quad Q_1\cdot Q_2 = Q_1Q_2\cos\theta$, $\quad t=\cos\theta$

3-dimensional integral representation for $a_\mu^{\mathrm{LbyL};\pi^0}$ (cont.)

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where $Q_3^2=(Q_1+Q_2)^2$, $\quad Q_1\cdot Q_2 = Q_1Q_2\cos\theta$, $\quad t=\cos\theta$

$$I_1(Q_1,Q_2,t) = X(Q_1,Q_2,t)\Big(8\,P_1\,P_2\,(Q_1\cdot Q_2) - 2\,P_1\,P_3\,(Q_2^4/m_\mu^2 - 2\,Q_2^2) - 2\,P_1\,(2 - Q_2^2/m_\mu^2 + 2\,(Q_1\cdot Q_2)/m$$
$$+4\,P_2\,P_3\,Q_1^2 - 4\,P_2 - 2\,P_3\,(4 + Q_1^2/m_\mu^2 - 2\,Q_2^2/m_\mu^2) + 2/m_\mu^2\Big)$$
$$-2\,P_1\,P_2\,(1 + (1-R_{m1})\,(Q_1\cdot Q_2)/m_\mu^2) + P_1\,P_3\,(2 - (1-R_{m1})\,Q_2^2/m_\mu^2) + P_1\,(1-R_{m1})/m_\mu^2$$
$$+P_2\,P_3\,(2 + (1-R_{m1})^2\,(Q_1\cdot Q_2)/m_\mu^2) + 3\,P_3\,(1-R_{m1})/m_\mu^2$$

$$I_2(Q_1,Q_2,t) = X(Q_1,Q_2,t)\Big(4\,P_1\,P_2\,(Q_1\cdot Q_2) + 2\,P_1\,P_3\,Q_2^2 - 2\,P_1 + 2\,P_2\,P_3\,Q_1^2 - 2\,P_2 - 4P_3 - 4/m_\mu^2\Big)$$
$$-2\,P_1\,P_2 - 3\,P_1\,(1-R_{m2})/(2m_\mu^2) - 3\,P_2\,(1-R_{m1})/(2m_\mu^2) - P_3\,(2 - R_{m1} - R_{m2})/(2m_\mu^2)$$
$$+P_1\,P_3\,(2 + 3\,(1-R_{m2})\,Q_2^2/(2m_\mu^2) + (1-R_{m2})^2\,(Q_1\cdot Q_2)/(2m_\mu^2))$$
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where $P_1^2 = 1/Q_1^2$, $P_2^2 = 1/Q_2^2$, $P_3^2 = 1/Q_3^2$, $\quad X(Q_1,Q_2,t) = \frac{1}{Q_1Q_2\,x}\arctan\left(\frac{zx}{1-zt}\right)$

Experimental constraints on $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}$

1. Any hadronic model of the form factor has to reproduce the decay amplitude

$$\mathcal{A}(\pi^0 \to \gamma\gamma) = -\frac{e^2 N_C}{12\pi^2 F_\pi}[1 + \mathcal{O}(m_q)]$$

Fixed by the Wess-Zumino-Witten (WZW) term (chiral corrections small)

This leads to the constraint

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2. Information on the form factor with one on-shell and one off-shell photon from the process $e^+e^- \to e^+e^-\pi^0$

Experimental data (CELLO '90, CLEO '98) fairly well confirm the Brodsky-Lepage behavior:

$$\lim_{Q^2 \to \infty} \mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0) \sim -\frac{2F_\pi}{Q^2}$$

Maybe with slightly different prefactor

QCD short-distance constraints from OPE on $\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}$

Knecht + Nyffeler, EPJC '01 studied QCD Green's function $\langle VVP \rangle$ (order parameter of chiral symmetry breaking) in chiral limit and assuming octet symmetry (partly based on Moussallam '95; Knecht et al. '99)

1. If the space-time arguments of all three currents approach each other one obtains (up to corrections $\mathcal{O}(\alpha_s)$):

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((\lambda q_1 + \lambda q_2)^2, (\lambda q_1)^2, (\lambda q_2)^2) = \frac{F_0}{3} \frac{1}{\lambda^2} \frac{q_1^2 + q_2^2 + (q_1 + q_2)^2}{q_1^2 q_2^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

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2. When the space-time arguments of the two vector currents in $\langle VVP \rangle$ approach each other (OPE leads to Green's function $\langle AP \rangle$):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^0 * \gamma^* \gamma^*}(q_2^2,(\lambda q_1)^2,(q_2-\lambda q_1)^2) = \frac{2F_0}{3}\,\frac{1}{\lambda^2}\,\frac{1}{q_1^2}+\mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

Higher twist corrections have been worked out in Shuryak + Vainshtein '82, Novikov et al. '84 (in chiral limit):

$$\lim_{\lambda\to\infty} \frac{\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(0,(\lambda q_1)^2,(\lambda q_1)^2)}{\mathcal{F}_{\pi^0 \gamma\gamma}(0,0,0)} = -\frac{8}{3}\pi^2 F_0^2\left\{\frac{1}{\lambda^2 q_1^2}+\frac{8}{9}\,\frac{\delta^2}{\lambda^4 q_1^4}+\mathcal{O}\left(\frac{1}{\lambda^6}\right)\right\}$$

δ^2 parametrizes the relevant higher-twist matrix element

The sum-rule estimate in Novikov et al. '84 yielded $\delta^2 = (0.2 \pm 0.02)$ GeV^2

A new short-distance constraint at the external vertex in $a_\mu^{\mathrm{LbyL};\pi^0}$

3. When the space-time argument of one of the vector currents approaches the argument of the pseudoscalar density in $\langle VVP\rangle$ one obtains (Knecht + Nyffeler, EPJC '01):

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((\lambda q_1 + q_2)^2, (\lambda q_1)^2, q_2^2) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(q_2^2) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

where the vector-tensor two-point function Π_{VT} is defined by

$$\delta^{ab}(\Pi_{\mathrm{VT}})_{\mu\rho\sigma}(p) = \int d^4x e^{ip\cdot x}\langle 0|T\{V_\mu^a(x)(\overline{\psi}\,\sigma_{\rho\sigma}\frac{\lambda^b}{2}\,\psi)(0)\}|0\rangle$$

with $\sigma_{\rho\sigma} = \frac{i}{2}[\gamma_\rho, \gamma_\sigma]$. Conservation of the vector current and parity invariance then give

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At the external vertex in light-by-light scattering the following limit is relevant (soft photon $q_2 \to 0$)

$$\lim_{\lambda\to\infty} \mathcal{F}_{\pi^0 * \gamma^* \gamma}((\lambda q_1)^2, (\lambda q_1)^2, 0) = -\frac{2}{3}\frac{F_0}{\langle\overline{\psi}\psi\rangle_0}\Pi_{\mathrm{VT}}(0) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Note that there is no fall-off in this limit, unless $\Pi_{\mathrm{VT}}(0)$ vanishes !

A constituent quark model for the form factor would lead to a $1/q_1^2$ fall-off instead

New short-distance constraint at the external vertex (cont.)

Ioffe + Smilga ’84 defined quark condensate magnetic susceptibility χ of QCD in presence of constant external electromagnetic field

$$\langle 0|\bar{q}\sigma_{\mu\nu}q|0\rangle_F = e\,e_q\,\chi\,\langle\overline{\psi}\psi\rangle_0\,F_{\mu\nu}, \qquad e_u = 2/3,\ e_d = -1/3$$

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Belyaev + Kogan '84 then showed that

$$\Pi_{VT}(0) = -\frac{\langle\overline{\psi}\psi\rangle_0}{2}\,\chi$$

with our conventions for Π_{VT}, see also Mateu + Portoles '07

Therefore we get at the external vertex in $a_\mu^{\mathrm{LbyL};\pi^0}$

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^0*\gamma^*\gamma}((\lambda q_1)^2,(\lambda q_1)^2,0) = \frac{F_0}{3}\,\chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

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with our conventions for Π_{VT}, see also Mateu + Portoles '07

Therefore we get at the external vertex in $a_\mu^{\mathrm{LbyL};\pi^0}$

$$\lim_{\lambda\to\infty}\mathcal{F}_{\pi^0*\gamma^*\gamma}((\lambda q_1)^2,(\lambda q_1)^2,0) = \frac{F_0}{3}\,\chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

- Unfortunately there is no agreement in the literature what the actual value of χ should be.
- In comparing different results one has to keep in mind that χ actually depends on the renormalization scale μ.
- A quantitative comparison of various results for $\chi(\mu)$ is difficult, since one cannot use perturbative QCD down to scales of $\mu \sim 0.5$ GeV at which some estimates are given.
- Finally, it is not clear what would be the "correct" scale μ in the context of hadronic light-by-light scattering in the muon $g-2$!

Estimates for the quark condensate magnetic susceptibility χ

Authors	Method	$\chi(\mu)$ [GeV]$^{-2}$	Footnote
Ioffe + Smilga '84	QCD sum rules	$\chi(\mu = 0.5\ \text{GeV}) = -(8.16^{+2.95}_{-1.91})$	[1]
Narison '08	QCD sum rules	$\chi = -(8.5 \pm 1.0)$	[2]
Vainshtein '03	OPE for $\langle VVA \rangle$	$\chi = -N_C/(4\pi^2 F_\pi^2) = -8.9$	[3]
Balitsky + Yung '83	LMD	$\chi = -2/M_V^2 = -3.3$	[4]
Belyaev + Kogan '84	QCD sum rules	$\chi(0.5\ \text{GeV}) = -(5.7 \pm 0.6)$	[5]
Balitsky et al. '85	QCD sum rules	$\chi(1\ \text{GeV}) = -(4.4 \pm 0.4)$	[5]
Ball et al. '03	QCD sum rules	$\chi(1\ \text{GeV}) = -(3.15 \pm 0.30)$	[5]

[1]: QCD sum rule evalation of nucleon magnetic moments
[2]: Recent reanalysis of these sum rules for nucleon magnetic moments. At which scale μ ?
[3]: Probably at low scale $\mu \sim 0.5$ GeV, since pion dominance was assumed in derivation
[4]: The leading short-distance behavior of Π_{VT} is given by (Craigie + Stern '81)

$$\lim_{\lambda \to \infty} \Pi_{VT}((\lambda p)^2) = -\frac{1}{\lambda^2} \frac{\langle \bar{\psi}\psi \rangle_0}{p^2} + \mathcal{O}\left(\frac{1}{\lambda^4}\right)$$

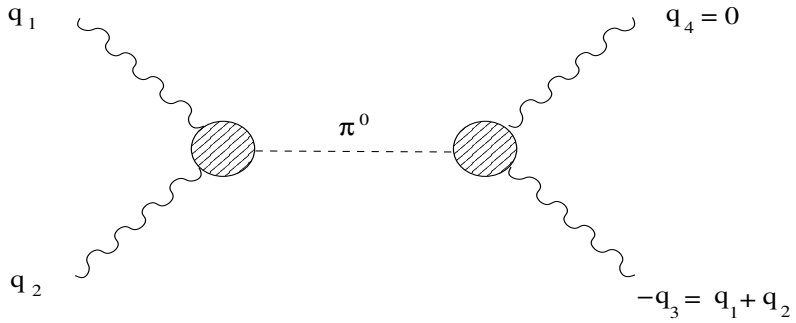
Assuming that the two-point function Π_{VT} is well described by the multiplet of the lowest-lying vector mesons (LMD) and satisfies this OPE constraint leads to the ansatz (Balitsky + Yung '83, Belyaev + Kogan '84, Knecht + Nyffeler, EPJC '01)

$$\Pi_{VT}^{\text{LMD}}(p^2) = -\langle \bar{\psi}\psi \rangle_0 \frac{1}{p^2 - M_V^2} \quad \Rightarrow \quad \chi^{\text{LMD}} = -\frac{2}{M_V^2} = -3.3\ \text{GeV}^{-2}$$

Not obvious at which scale. Maybe $\mu = M_V$ as for low-energy constants in ChPT.
[5]: LMD estimate later improved by taking more resonance states $\rho', \rho'', \ldots$ in QCD sum rule analysis of Π_{VT}. Note that the last value is very close to original LMD estimate !

The short-distance constraint by Melnikov + Vainshtein

Melnikov + Vainshtein ’03 found QCD short-distance constraint on whole 4-point function:

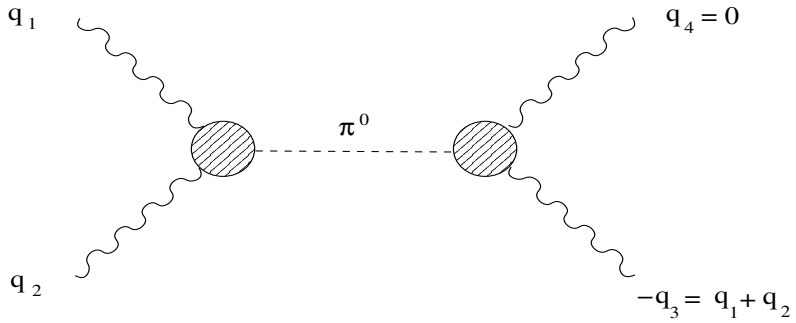


$$\langle \underbrace{VV}\, V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

OPE

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$$\langle \underbrace{VV}_{\text{OPE}} V|\gamma\rangle \overset{q_1^2 \sim q_2^2 \gg (q_1+q_2)^2}{\Rightarrow} \langle AV|\gamma\rangle$$

In this way they obtain as intermediate result for the LbyL scattering amplitude:

$$\mathcal{A}_{\pi^0} = \frac{3}{2F_\pi} \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)}{q_3^2 - m_\pi^2} (f_{2;\mu\nu} \tilde{f}_1^{\nu\mu})(\tilde{f}_{\rho\sigma} f_3^{\sigma\rho}) + \text{permutations}$$

$f_i^{\mu\nu} = q_i^\mu \epsilon_i^\nu - q_i^\nu \epsilon_i^\mu$ and $\tilde{f}_{i;\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_i^{\rho\sigma}$ for $i = 1, 2, 3$ = fi eld strength tensors of internal photons with polarization vectors ϵ_i, for external soft photon $f^{\mu\nu} = q_4^\mu \epsilon_4^\nu - q_4^\nu \epsilon_4^\mu$.

Except in $\tilde{f}_{\rho\sigma}$ the limit $q_4 \to 0$ is understood in $f_3^{\sigma\rho}$ and in the pion propagator.

- From the expression with on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) \equiv \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2, q_1^2, q_2^2)$ it is obvious that Melnikov + Vainshtein only consider the pion-pole contribution !
- Note the absence of a second form factor at the external vertex $\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)$. Replaced by a constant (WZW) form factor $\mathcal{F}_{\pi^0\gamma\gamma}(m_\pi^2, 0)$!
- Note the overall $1/q_3^2$ behavior for large q_3 (apart from $f_3^{\sigma\rho}$)

5. New evaluation of pion-exchange contribution in large-N_C QCD

Framework: Minimal hadronic approximation for Green's function in large-N_C QCD
(Peris et al. '98, . . .)

- In leading order in N_C, an infinite tower of narrow resonances contributes in each channel of a particular Green's function.
- The low-energy and short-distance behavior of these Green's functions is then matched with results from QCD, using ChPT and the OPE, respectively.
- It is assumed that taking the lowest few resonances in each channel gives a good description of the Green's function in the real world (generalization of VMD)

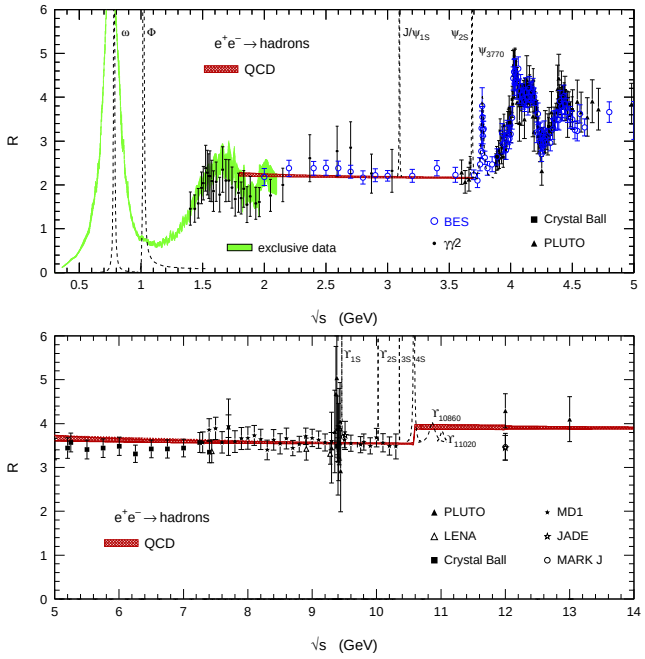
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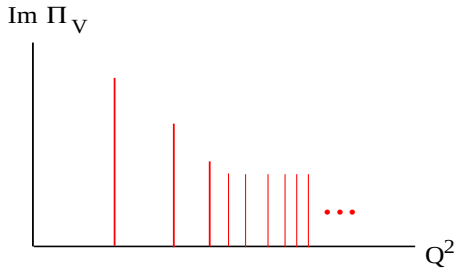
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Example: 2-point function $\langle VV \rangle \longrightarrow$ spectral function $\mathrm{Im}\Pi_V \sim \sigma(e^+e^- \longrightarrow \text{hadrons})$

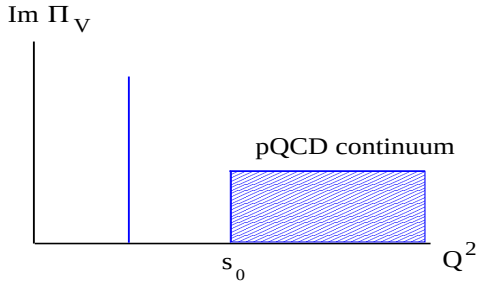
Real world (Davier et al., '03)



Large-N_C QCD ('t Hooft '74)



Minimal Hadronic Approximation (MHA)



Scale s_0 fixed by the OPE

Form factor $\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}$ in large-N_C QCD

Knecht + Nyffeler, EPJC '01: wrote down such minimal hadronic ansätze for on-shell $\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(m_\pi^2, q_1^2, q_2^2)$ and off-shell form factors $\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$

- With one multiplet of vector resonances (LMD) or two multiplets ρ and ρ' (LMD+V).
- Both fulfill all OPE short-distance constraints, but LMD ansatz does not reproduce Brodsky-Lepage fall-off.

Off-shell LMD+V form factor:

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0 * \gamma^* \gamma^*}(q_3^2, q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2\, q_2^2\, (q_1^2 + q_2^2 + q_3^2) + P_H^V(q_1^2, q_2^2, q_3^2)}{(q_1^2 - M_{V_1}^2)\,(q_1^2 - M_{V_2}^2)\,(q_2^2 - M_{V_1}^2)\,(q_2^2 - M_{V_2}^2)}$$

$$P_H^V(q_1^2, q_2^2, q_3^2) = h_1\,(q_1^2+q_2^2)^2 + h_2\, q_1^2\, q_2^2 + h_3\,(q_1^2+q_2^2)\, q_3^2 + h_4\, q_3^4 + h_5\,(q_1^2+q_2^2) + h_6\, q_3^2 + h_7$$

where $q_3^2 = (q_1+q_2)^2$

Fixing the LMD+V model parameters

- h_7: Determined through the normalization with decay width $\pi^0 \to \gamma\gamma$

  $\Rightarrow h_7 = -N_C\, M_{V_1}^4\, M_{V_2}^4 / (4\pi^2\, F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$

- h_1: In order to reproduce the Brodsky-Lepage behavior

  $\Rightarrow h_1 = 0$ GeV^2

- h_5: Knecht + Nyffeler, EPJC '01 have fitted the on-shell form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$ to the CLEO data

  $\Rightarrow h_5 = 6.93 \pm 0.26$ GeV$^4 - h_3 m_\pi^2$

- h_2: Melnikov + Vainshtein '03 pointed out that h_2 can be obtained from higher-twist corrections in OPE

  $\Rightarrow h_2 = -4\,(M_{V_1}^2 + M_{V_2}^2) + (16/9)\,\delta^2 \simeq -10.63$ GeV^2

  where we used $M_{V_1} = M_\rho = 775.49$ MeV, $M_{V_2} = M_{\rho'} = 1.465$ GeV and $\delta^2 = 0.2$ GeV^2

Fixing the LMD+V model parameters (cont.)

- h_3, h_4:

$$\Pi_{VT}^{LMD+V}(p^2) = -\langle\overline{\psi}\psi\rangle_0 \frac{p^2 + c_{VT}}{(p^2 - M_{V_1}^2)(p^2 - M_{V_2}^2)}, \qquad c_{VT} = \frac{M_{V_1}^2 M_{V_2}^2 \chi}{2}$$

(Knecht + Nyffeler, EPJC '01)

OPE constraint for form factor leads to relation:

$$h_1 + h_3 + h_4 = 2c_{VT} = M_{V_1}^2 M_{V_2}^2 \chi$$

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Assuming that LMD/LMD+V framework is self-consistent, we will take

$$\chi = (-3.3 \pm 1.1) \text{ GeV}^{-2} \;\Rightarrow\; h_3 + h_4 = (-4.3 \pm 1.4) \text{ GeV}^2 \qquad (*)$$

with a typical large-N_C uncertainty of 30% for χ
Vary h_3 in range ± 10 GeV^2 and determine h_4 from relation $(*)$ and vice versa

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- Short-distance constraint by Melnikov + Vainshtein '03

First $q_1^2 \sim q_2^2 \gg q_3^2$ and then q_3^2 large, one obtains at external vertex:

$$\frac{3}{F_\pi} \mathcal{F}_{\pi^0 * \gamma^* \gamma}^{LMD+V}(q_3^2, q_3^2, 0) \overset{q_3^2 \to \infty}{\to} \frac{h_1 + h_3 + h_4}{M_{V_1}^2 M_{V_2}^2} = \frac{2c_{VT}}{M_{V_1}^2 M_{V_2}^2} = \chi$$

With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with M+V '03 !

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With pion propagator this leads to overall $1/q_3^2$ behavior. Agrees qualitatively with M+V '03 !
With constant (WZW) form factor at external vertex we would get (in chiral limit):

$$\frac{3}{F_\pi} \mathcal{F}_{\pi^0 \gamma\gamma}^{LMD+V}(0,0,0) = \frac{h_7}{M_{V_1}^4 M_{V_2}^4} = -\frac{N_C}{4\pi^2 F_\pi^2} \simeq -8.9\text{ GeV}^{-2}$$

With $\chi = -N_C/(4\pi^2 F_\pi^2)$ from Vainshtein '03, we would precisely satisfy the short-distance constraint from M+V '03. Problem: M+V '03 only consider pion-pole contribution !

Fixing the LMD+V model parameters (cont.)

- h_6:
  - Final result for $a_\mu^{\mathrm{LbyL};\pi^0}$ is very sensitive to value of h_6. We can get some indirect information on size and sign of h_6 as follows.
  - Estimates of low-energy constants in chiral Lagrangians via exchange of resonances work quite well. However, we may get some corrections, if we consider the exchange of heavier resonances as well. Typically, a large-N_C error of 30% can be expected.

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  - In $\langle VVP \rangle$ appear 2 combinations of low-energy constants from the chiral Lagrangian of odd intrinsic parity at $\mathcal{O}(p^6)$, denoted by A_{V,p^2} and $A_{V,(p+q)^2}$ in Knecht + Nyffeler, EPJC '01.

$$A_{V,p^2}^{\mathrm{LMD}} = \frac{F_\pi^2}{8M_V^4} - \frac{N_C}{32\pi^2 M_V^2} = -1.11\,\frac{10^{-4}}{F_\pi^2}$$

$$A_{V,p^2}^{\mathrm{LMD+V}} = \frac{F_\pi^2}{8M_{V_1}^4}\frac{h_5}{M_{V_2}^4} - \frac{N_C}{32\pi^2 M_{V_1}^2}\left(1 + \frac{M_{V_1}^2}{M_{V_2}^2}\right) = -1.36\,\frac{10^{-4}}{F_\pi^2}$$

The relative change is only about 20%, well within expected large-N_C uncertainty !

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$$A_{V,(p+q)^2}^{\mathrm{LMD}} = -\frac{F_\pi^2}{8M_V^4} = -0.26\,\frac{10^{-4}}{F_\pi^2}, \qquad A_{V,(p+q)^2}^{\mathrm{LMD+V}} = -\frac{F_\pi^2}{8M_{V_1}^4 M_{V_2}^4}h_6$$

Note that $A_{V,(p+q)^2}^{\mathrm{LMD}}$ is "small" compared to A_{V,p^2}^{LMD}. About same size as absolute value of the shift in A_{V,p^2} when going from LMD to LMD+V !

  - Assuming again that LMD/LMD+V framework is self-consistent, but allowing for a 100% uncertainty of $A_{V,(p+q)^2}^{\mathrm{LMD}}$, we get the range $h_6 = (5 \pm 5)$ GeV^4

New estimate for the pion-exchange contribution

$a_\mu^{\mathrm{LbyL};\pi^0} \times 10^{11}$ with the off-shell LMD+V form factor

	$h_6 = 0$ GeV^4	$h_6 = 5$ GeV^4	$h_6 = 10$ GeV^4
$h_3 = -10$ GeV^2	68.4	74.1	80.2
$h_3 = 0$ GeV^2	66.4	71.9	77.8
$h_3 = 10$ GeV^2	64.4	69.7	75.4
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$h_4 = 0$ GeV^2	67.3	72.8	78.8
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$\chi = -3.3$ GeV^{-2}, $h_1 = 0$ GeV^2, $h_2 = -10.63$ GeV^2 and $h_5 = 6.93$ GeV^4 $- h_3 m_\pi^2$

When varying h_3 (upper half of the table), the parameter h_4 is fixed by the constraint $h_3 + h_4 = M_{V_1}^2 M_{V_2}^2 \chi$. In the lower half the procedure is reversed.

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- Uncertainty in h_6 affects result by $\pm 6.4 \times 10^{-11}$. If we use $h_6 = (0 \pm 10)$ GeV^4, the result would vary by $\pm 12 \times 10^{-11}$!
- The variation with h_3 (with h_4 determined from the constraint or vice versa) is much smaller, at most $\pm 2.5 \times 10^{-11}$.
- The variation of h_5 by ± 0.26 GeV^4 only leads to changes of $\pm 0.6 \times 10^{-11}$.
- Within scanned region:
  - Minimal value: 63.2×10^{-11} [$\chi = -2.2$ GeV^{-2}, $h_3 = 10$ GeV^2, $h_6 = 0$ GeV^4]
  - Maximum value: 83.3×10^{-11} [$\chi = -4.4$ GeV^{-2}, $h_4 = 10$ GeV^2, $h_6 = 10$ GeV^4]

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- The variation of h_5 by ± 0.26 GeV^4 only leads to changes of $\pm 0.6 \times 10^{-11}$.
- Within scanned region:
  - Minimal value: 63.2×10^{-11} [$\chi = -2.2$ GeV^{-2}, $h_3 = 10$ GeV^2, $h_6 = 0$ GeV^4]
  - Maximum value: 83.3×10^{-11} [$\chi = -4.4$ GeV^{-2}, $h_4 = 10$ GeV^2, $h_6 = 10$ GeV^4]

Take average of results for $h_6 = 5$ GeV^4 for $h_3 = 0$ GeV^2 and $h_4 = 0$ GeV^2 as estimate:

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

Added errors from χ, h_3 (or h_4) and h_6 linearly (don't follow Gaussian distribution !)

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i

The h_i enter the form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma^*}^{\mathrm{LMD+V}}$ linearly in the numerator, therefore (Nyffeler '09):

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} = \left(\frac{\alpha}{\pi}\right)^3 \left[\sum_{i=1}^{7} c_i\, \tilde{h}_i + \sum_{i=1}^{7}\sum_{j=i}^{7} c_{ij}\, \tilde{h}_i\, \tilde{h}_j\right]$$

with dimensionless coefficients c_i, c_{ij}, if we measure the h_i in appropriate units of GeV: $\tilde{h}_i = h_i/\mathrm{GeV}^2$ for $i = 1,2,3,4$, $\tilde{h}_i = h_i/\mathrm{GeV}^4$ for $i = 5,6$ and $\tilde{h}_7 = h_7/\mathrm{GeV}^6$.

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$c_i \times 10^4$	−1.4530	0	−1.4530	−1.4530	0.4547	0.4547	−1.2048

$c_{ij} \times 10^4$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
$i=1$	0.4447	0.0729	0.7428	0.7120	−0.3620	−0.3332	0.9916
$i=2$	···	0	0.0730	0.0729	−0.0557	−0.0557	0.2221
$i=3$	···	···	0.2980	0.5653	−0.1967	−0.1679	0.1796
$i=4$	···	···	···	0.2672	−0.1679	−0.1391	0.1162
$i=5$	···	···	···	···	0.1215	0.1796	−0.8072
$i=6$	···	···	···	···	···	0.0581	−0.3052
$i=7$	···	···	···	···	···	···	1.6122

There is no term without the constants h_i, because of the form factor $\mathcal{F}_{\pi^{0*}\gamma^*\gamma}(q_3^2, q_3^2, 0)$ with a soft photon at the external vertex. This also leads to $c_2 = 0$ and $c_{22} = 0$.

h_1, h_3, h_4 are not independent, but must obey the relation $h_1 + h_3 + h_4 = M_{V_1}^2\, M_{V_2}^2\, \chi$, because of the new short-distance constraint on the form factor at the external vertex.

Parametrization of $a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0}$ for arbitrary model parameters h_i (cont.)

Since the parameters h_1, h_2, h_5, h_7 are quite well known:

- $h_1 = 0$ GeV^2 (Brodsky-Lepage behavior of $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$)
- $h_2 = -4\,(M_{V_1}^2 + M_{V_2}^2) + (16/9)\,\delta^2 \simeq -10.63$ GeV^2 (higher-twist corr. in OPE)
- $h_5 = (6.93 \pm 0.26)$ GeV$^4 - h_3 m_\pi^2$ (fit to CLEO data for $\mathcal{F}_{\pi^0\gamma^*\gamma}^{\mathrm{LMD+V}}(m_\pi^2, -Q^2, 0)$
- $h_7 = -N_C\, M_{V_1}^4\, M_{V_2}^4/(4\pi^2 F_\pi^2) - h_6 m_\pi^2 - h_4 m_\pi^4$ (normalization to $\Gamma(\pi^0 \to \gamma\gamma)$)

one can write down the simplified expression:

$$
\begin{aligned}
a_{\mu;\mathrm{LMD+V}}^{\mathrm{LbyL};\pi^0} \;=\; \left(\frac{\alpha}{\pi}\right)^3 \Big[& 503.3764 - 6.5223\,\tilde{h}_3 - 5.0962\,\tilde{h}_4 + 7.8557\,\tilde{h}_6 \\
& + 0.3017\,\tilde{h}_3^2 + 0.5683\,\tilde{h}_3\,\tilde{h}_4 - 0.1747\,\tilde{h}_3\,\tilde{h}_6 \\
& + 0.2672\,\tilde{h}_4^2 - 0.1411\,\tilde{h}_4\,\tilde{h}_6 + 0.0642\,\tilde{h}_6^2 \Big] \times 10^{-4}
\end{aligned}
$$

where only h_3, h_4, h_6 enter as free parameters, with the constraint $h_3 + h_4 = M_{V_1}^2\, M_{V_2}^2\, \chi$.

In this way, one can easily update our result, if the not so precisely known model parameters h_3, h_4, h_6 and the exact value (and relevant scale) of the magnetic susceptibility $\chi(\mu)$ can be determined more precisely in the future.

Update for η and η' using VMD form factors

- Short-distance analysis of LMD+V form factor in Knecht + Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
- Simplified approach using VMD form factors (as done in many other papers)

$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2, q_1^2, q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}, \quad \mathrm{PS} = \eta, \eta'$$

normalized to experimental decay width $\Gamma(\mathrm{PS} \to \gamma\gamma)$
$\Rightarrow F_{\eta,\mathrm{eff}} = 93.0$ MeV, $\quad F_{\eta',\mathrm{eff}} = 74.0$ MeV

Update for η and η' using VMD form factors

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  normalized to experimental decay width $\Gamma(\mathrm{PS}\to\gamma\gamma)$
  $\Rightarrow F_{\eta,\mathrm{eff}} = 93.0$ MeV, $\quad F_{\eta',\mathrm{eff}} = 74.0$ MeV

- Problem with VMD form factor: damping is too strong, behaves like $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2,-Q^2,-Q^2)\sim 1/Q^4$, instead of $\sim 1/Q^2$ deduced from OPE.
  However, final result is not too sensitive to high-energy behavior (see analysis in Knecht + Nyffeler '01). It seems more important to have good description at small and intermediate energies below 1 GeV, e.g. by reproducing slope of form factor $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2,0)$ at origin.
  CLEO fi tted VMD ansatz for $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2,0)$ with adjustable parameter Λ_{PS} instead of M_V.
  $\Rightarrow \Lambda_\eta = 774 \pm 29$ MeV, $\quad \Lambda_{\eta'} = 859 \pm 28$ MeV. We take these masses for our evaluation.

Update for η and η' using VMD form factors

- Short-distance analysis of LMD+V form factor in Knecht + Nyffeler, EPJC '01, performed in chiral limit and assuming octet symmetry $\Rightarrow$ not valid anymore for η and η' !
- Simplified approach using VMD form factors (as done in many other papers)

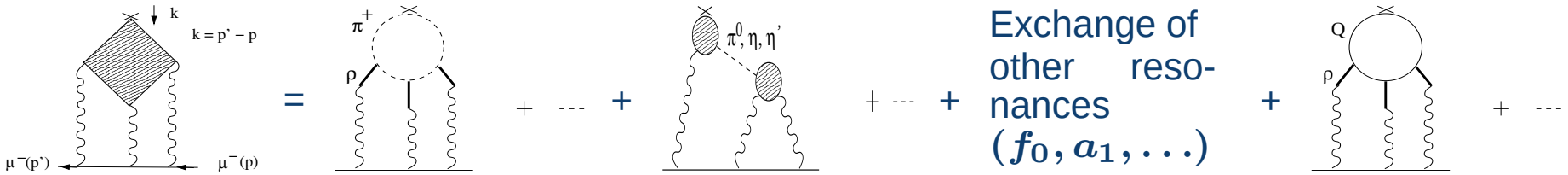
$$\mathcal{F}^{\mathrm{VMD}}_{\mathrm{PS}^*\gamma^*\gamma^*}(q_3^2,q_1^2,q_2^2) = -\frac{N_c}{12\pi^2 F_{\mathrm{PS}}}\frac{M_V^2}{(q_1^2-M_V^2)}\frac{M_V^2}{(q_2^2-M_V^2)}, \quad \mathrm{PS}=\eta,\eta'$$

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  CLEO fi tted VMD ansatz for $\mathcal{F}_{\mathrm{PS}\gamma^*\gamma}(-Q^2,0)$ with adjustable parameter Λ_{PS} instead of M_V.
  $\Rightarrow \Lambda_\eta = 774 \pm 29$ MeV, $\quad \Lambda_{\eta'} = 859 \pm 28$ MeV. We take these masses for our evaluation.
- With VMD form factors at both vertices (i.e. not taking pole-approximation as done in Melnikov+Vainshtein '03) we get $a_\mu^{\mathrm{LbyL};\eta} = 14.5\times10^{-11}$ and $a_\mu^{\mathrm{LbyL};\eta'} = 12.5\times10^{-11}$
- Our final estimate for the sum of all pseudoscalars:

$$a_\mu^{\mathrm{LbyL;PS}} = (99\pm16)\times10^{-11}$$

  where we have assumed a 16% error, as inferred for pion-exchange contribution.
  A detailed analysis for η, η' is needed along the lines of LMD+V.

6. Summary of hadronic light-by-light scattering results



Chiral counting:	p^4	p^6	p^8	p^8
N_C-counting:	1	N_C	N_C	N_C

- Evaluations of full had. LbyL scattering contribution:
  - BPP = Bijnens, Pallante, Prades '95, '96, '02
    Use mainly ENJL (Extended Nambu-Jona-Lasinio) model; but for some contributions also other models
  - HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02
    Use mainly HLS (Hidden Local Symmetry) model; often HLS = VMD
- Partial evaluation:
  - MV = Melnikov, Vainshtein '03 (large-N_C QCD: LMD, LMD+V)
- Recent summaries on had. LbyL scattering:
  - BP = Bijnens, Prades '07
  - PdRV = Prades, de Rafael '09 (arXiv:0901.0306)
    Analyzed results obtained by different groups with various models and suggested new estimates for some individual contributions (shifted central values, enlarged errors) to cover range of results.
  - JN = Jegerlehner, Nyffeler '09 (arXiv:0902.3360)
    Includes new evaluation of pseudoscalar-exchange contribution.

Pseudoscalar exchanges

Model for $\mathcal{F}_{P^*\gamma^*\gamma^*}$	$a_\mu(\pi^0) \times 10^{11}$	$a_\mu(\pi^0, \eta, \eta') \times 10^{11}$
Point coupling	$+\infty$	$+\infty$
ENJL (modified) [BPP]	59(9)	85(13)
VMD / HLS [HKS,HK]	57(4)	83(6)
LMD+V (on-shell, $h_2 = 0$) [KN]	58(10)	83(12)
LMD+V (on-shell, $h_2 = -10$ GeV^2) [KN]	63(10)	88(12)
LMD+V (on-shell, constant FF at ext. vertex) [MV]	77(7)	114(10)
nonlocal χQM (off-shell) [DB]	65(2)	–
LMD+V (off-shell) [N]	72(12)	99(16)
AdS/QCD (on-shell) [HoK]	68	102
[PdRV]	–	114(13)
[JN]	72(12)	99(16)

KN = Knecht, Nyffeler '01; DB = Dorokhov, Broniowski '08 (χQM = Chiral Quark Model); N = Nyffeler '09; HoK = Hong, Kim '09

- BPP use rescaled VMD result for η, η'. Also all LMD+V evaluations use VMD for η, η'.
- Result for pion with off-shell LMD+V form factors at both verties is not too far from value given by M+V '03, but this is pure coincidence ! M+V '03 use on-shell form factors, i.e. a constant factor at the external vertex !

  Note: Following M+V '03 and using $h_2 = -10$ GeV^2 we actually obtain 79.8×10^{-11} for the pion-pole contribution, close to the value 79.6×10^{-11} given in B+P '07 and 79.7×10^{-11} in D+B '08
- In nonlocal χQM there is a strong damping for off-shell pions $\longrightarrow$ result always smaller than pion-pole contribution. Error probably underestimates intrinsic hadronic model uncertainty !

Axial-vector exchanges

Model for $\mathcal{F}_{A^*\gamma^*\gamma^*}$	$a_\mu(a_1) \times 10^{11}$	$a_\mu(a_1, f_1, f_1') \times 10^{11}$
ENJL-VMD [BPP] (nonet symmetry)	2.5(1.0)	—
ENJL-like [HKS,HK] (nonet symmetry)	1.7(1.7)	—
LMD [MV] (f_1 pure octet, f_1' pure singlet)	5.7	17
LMD [MV] (ideal mixing)	5.7	22(5)
[PdRV]	—	15(10)
[JN]	—	22(5)

- M+V '03: derived QCD short-distance constraint for axial-vector pole contribution (with on-shell form factor $\mathcal{F}_{A\gamma^*\gamma^*}$ at both vertices)
- Simple VMD ansatz: short-distance constraints forbids form factor at external vertex. Assuming all axial-vectors in the nonet have same mass M leads to

$$a_\mu^{\mathrm{AV}} = \left(\frac{\alpha}{\pi}\right)^3 \frac{m_\mu^2}{M^2} N_C \,\mathrm{Tr}\left[\hat{Q}^4\right] \left(\frac{71}{192} + \frac{81}{16} S_2 - \frac{7\pi^2}{144}\right) + \ldots \approx 1010 \frac{m_\mu^2}{M^2} \times 10^{-11}$$

($\hat{Q} = \mathrm{diag}(2/3, -1/3, -1/3)$, $S_2 = 0.26043$)

Strong dependence on mass M:
$M = 1300$ MeV: $a_\mu^{\mathrm{AV}} = 7 \times 10^{-11}$, $M = M_\rho$: $a_\mu^{\mathrm{AV}} = 28 \times 10^{-11}$ (with $+\ldots$)

- More sophisticated LMD ansatz (Czarnecki, Marciano, Vainshtein '03), see Table. Now there is form factor at external vertex. Dressing leads to lower effective mass M. Furthermore f_1, f_1' have large coupling to photons $\longrightarrow$ huge enhancement compared to BPP, HKS.

Scalar exchanges

Model for $\mathcal{F}_{S^*\gamma^*\gamma^*}$	a_μ(scalars) $\times 10^{11}$
Point coupling	$-\infty$
ENJL [BPP]	$-7(2)$
[PdRV]	$-7(7)$
[JN]	$-7(2)$

- Within ENJL model: scalar exchange contribution related by Ward identities to (constituent) quark loop $\longrightarrow$ HK argued that effect of (broad) scalar resonances below several hundred MeV might already be included in sum of (dressed) quark loops and (dressed) $\pi + K$ loops !
- Potential double-counting is definitely an issue for the broad sigma meson $f_0(600)$ ($\leftrightarrow \pi^+\pi^-; \pi^0\pi^0$). Ongoing debate whether the scalar resonances $f_0(980), a_0(980)$ are two-quark or four-quark states.
- It is not clear which scalar resonances are described by ENJL model. Model parameters fixed by fitting various low-energy observables and resonance parameters, among them $M_S = 980$ MeV. However, model then yields $M_S^{\rm ENJL} = 620$ MeV.
- Can the usually broad scalar resonances be described by a simple resonance Lagrangian which works best in large-N_C limit, i.e. for very narrow states ?

Charged pion and kaon loops

Model $\pi^+\pi^-\gamma^*(\gamma^*)$	$a_\mu(\pi^\pm)\times 10^{11}$	$a_\mu(\pi^\pm,K^\pm)\times 10^{11}$
Point coupling (scalar QED)	-45.3	-49.8
VMD [KNO, HKS]	-16	$-$
full VMD [BPP]	$-18(13)$	$-19(13)$
HLS [HKS,HK]	-4.45	$-4.5(8.1)$
[MV] (all N_C^0 terms !)	$-$	$0(10)$
[PdRV]	$-$	$-19(19)$
[JN]	$-$	$-19(13)$

- Dressing leads to a rather huge suppression compared to scalar QED !
- Debate between BPP and HKS, how to maintain chiral and gauge invariance when dressing the point coupling with form factors $\longrightarrow$ potentially large model dependence !
- M+V '03 studied HLS model via expansion in $(m_\pi/M_\rho)^2$ and $(m_\mu - m_\pi)/m_\pi$:

$$a_{\mu;\mathrm{HLS}}^{\mathrm{LbL};\pi^\pm} = \left(\frac{\alpha}{\pi}\right)^3 \sum_{i=0}^{\infty} f_i\left[\frac{m_\mu - m_\pi}{m_\pi}, \ln\left(\frac{M_\rho}{m_\pi}\right)\right]\left(\frac{m_\pi^2}{M_\rho^2}\right)^i = \left(\frac{\alpha}{\pi}\right)^3 (-0.0058)$$
$$= (-46.37 + 35.46 + 10.98 - 4.70 - 0.3 + \ldots)\times 10^{-11} = -4.9(3)\times 10^{-11}$$

- Large cancellation between first three terms in series. Expansion converges only very slowly. Main reason: typical momenta in the loop integral are of order $\mu = 4m_\pi \approx 550$ MeV and the effective expansion parameter is μ/M_ρ, not m_π/M_ρ.
- M+V '03: Final result is very likely suppressed, but also very model dependent $\longrightarrow$ chiral expansion looses predictive power $\longrightarrow$ lumped together all terms subleading in N_C.

Dressed quark loops

Model	$a_\mu(\mathbf{quarks}) \times 10^{11}$
Point coupling	$\mathbf{62(3)}$
VMD [HKS, HK]	$\mathbf{9.7(11.1)}$
ENJL + bare heavy quark [BPP]	$\mathbf{21(3)}$
[PdRV] (Bare c-quark only !)	$\mathbf{2.3}$
[JN]	$\mathbf{21(3)}$

- de Rafael '94 argued that the dressed quark loops can be interpreted as irreducible contribution to the 4-point function $\langle VVVV \rangle$. They also appear as short-distance complement of the ENJL and HLS low-energy models. Absorb as counterterms the remaining cutoff dependences of low-energy effective models.
- Quark-hadron duality: the quark loops also model contributions from exchanges and loops of heavier hadronic states, like $\pi', a_0', f_0', p, n, \ldots$.
- Again very large model-dependent effect of the dressing to the photons.
- BPP employ the ENJL model up to some cutoff μ and then add the bare quark loop with constituent quark mass $M_Q = \mu$. The latter contribution simulates the high-momentum component of the quark loop which is non-negligible.
- Recently, PdRV '09 argued that the (dressed) light-quark loops should not be included as separate contribution. They assume them to be already covered by using the short-distance constraint from M+V '03 for the pseudoscalar-pole contribution.

Hadronic light-by-light scattering in the muon $g-2$: anno 2009

Some results for the various contributions to $a_\mu^{\mathrm{LbyL;had}} \times 10^{11}$:

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops +subl. N_C	–	–	–	0 ± 10	–	–	–
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '01; MV = Melnikov, Vainshtein '03; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09

- Pseudoscalar-exchange contribution dominates numerically. But other contributions are not negligible. Note cancellation between π, K-loops and quark loops !
- $(80 \pm 40) \times 10^{-11}$ not in KN '01; estimate used by Marseille group before MV '03.
- PdRV: Do not consider dressed light quark loops as separate contribution ! Assume it is already taken into account by using short-distance constraint of MV '03 on pseudoscalar-pole contribution. Why should this be the case ?
  Added all errors in quadrature ! Like HK(S). Too optimistic ?
- N, JN: Evaluation of the axial vectors by MV '03 is definitely some improvement over earlier calculations. It seems, however, again to be only the axial-vector pole contribution.
  Added all errors linearly. Like BPP, MV, BP, MdRR. Too pessimistic ?

Summary of contributions to a_μ (Jegerlehner + Nyffeler '09)

- Leptonic QED contributions: $a_\mu^{\rm QED} = (116\,584\,718.10 \pm 0.15) \times 10^{-11}$
- Electroweak contributions: $a_\mu^{\rm EW} = (\underbrace{153.2}_{?} \pm \underbrace{1.8}_{?}) \times 10^{-11}$
- Hadronic contributions:
  - Vacuum Polarization: $a_\mu^{\text{had. v.p.}}(e^+e^-) = (\underbrace{6903.0}_{??} \pm \underbrace{52.6}_{??} - (100.3 \pm 2.2)) \times 10^{-11}$

    $a_\mu^{\text{had. v.p.}}(\tau) = (\underbrace{7110}_{??} \pm \underbrace{58}_{??} - (100.3 \pm 2.2)) \times 10^{-11}$
  - Light-by-Light scattering: $a_\mu^{\rm LbyL} = (\underbrace{116 \pm 39}_{??}) \times 10^{-11}$
- Total SM contribution: $a_\mu^{\rm SM}(e^+e^-) = (116\,591\,790.0 \pm 52.6 \pm 40 \pm 1.8\ [\pm 66.2]) \times 10^{-11}$

  $a_\mu^{\rm SM}(\tau) = (116\,591\,997.0 \pm \underbrace{58}_{\text{v.p.}} \pm \underbrace{40}_{\text{LbyL}} \pm \underbrace{1.8}_{\text{QED + EW}}\ [\pm 70.5]) \times 10^{-11}$
- Experimental value: $a_\mu^{\rm exp} = (116\,592\,080.0 \pm 63.0) \times 10^{-11}$

$$a_\mu^{\rm exp} - a_\mu^{\rm SM}(e^+e^-) = (290 \pm 92) \times 10^{-11} \qquad [3.2\,\sigma]$$

$$a_\mu^{\rm exp} - a_\mu^{\rm SM}(\tau) = (83 \pm 95) \times 10^{-11} \qquad [0.9\,\sigma]$$

τ-data: Evaluation problematic because of isospin violations

e^+e^--data: Discrepancy real ? Sign for New Physics ?

Probably more work is needed in Theory and Experiment !

7. Conclusions and outlook

- Jegerlehner '07, '08: one should use off-shell form factors $\mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((q_1+q_2)^2, q_1^2, q_2^2)$ to evaluate pion-exchange contribution.
  Prescription by Melnikov + Vainshtein '03 to use a constant (WZW) form factor at the external vertex only yields pion-pole contribution with on-shell form factors $\mathcal{F}_{\pi^0 \gamma^* \gamma^*}(m_\pi^2, q_1^2, q_2^2)$.
- We derived a new short-distance constraint at external vertex, relating the off-shell form factor to quark condensate magnetic susceptibility χ:

$$\lim_{\lambda \to \infty} \mathcal{F}_{\pi^0 * \gamma^* \gamma^*}((\lambda q_1)^2, (\lambda q_1)^2, 0) = \frac{F_0}{3} \chi + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

  Problem: value of $\chi(\mu)$ and relevant scale μ not precisely known.
- We performed new evaluation of pion-exchange contribution within large-N_C approximation using form factors that fulfill all QCD short-distance constraints.
  Framework with two multiplets of vector resonances (LMD+V) for form factor and two-point function Π_{VT}:

$$a_\mu^{\mathrm{LbyL};\pi^0} = (72 \pm 12) \times 10^{-11}$$

  Important imputs: $\chi = -(3.3 \pm 1.1)$ GeV^{-2} (via $\chi^{\mathrm{LMD}} = -2/M_V^2$), $h_6 = (5 \pm 5)$ GeV^4 (via LEC $A^{\mathrm{LMD}}_{V,(p+q)^2}$).
- Updated values for η and η' (using simple VMD form factors):

$$a_\mu^{\mathrm{LbyL;PS}} = (99 \pm 16) \times 10^{-11}$$

- Combined with evaluations of the other contributions we get:

$$a_\mu^{\mathrm{LbyL;had}} = (116 \pm 40) \times 10^{-11}$$

Outlook on hadronic light-by-light scattering

Can we match precision of potential future $g-2$ experiment with error $\sim 15 \times 10^{-11}$?

- Some progress made in recent years for pseudoscalars and axial-vector contributions, implementing many experimental and theoretical constraints. More uncertainty for exchanges of scalars (and heavier resonances) and for (dressed) pion + kaon and (dressed) quark loops !
- Soon results from Lattice QCD ? $\langle VVVV \rangle$ needs to be integrated over phase space of 3 off-shell photons $\longrightarrow$ much more complicated than hadronic vacuum polarization. But even result with 50% error (but reliable !) would be very helpful !

Suggested way forward in the meantime:

- Important to have unified consistent framework (model) which deals with all contributions. Partly achieved by BPP (ENJL model) and HKS (HLS model), but not fully.
- Purely phenomenological approach: resonance Lagrangian where all couplings are fixed from experiment. Non-renormalizable Lagrangian: how to achieve matching with pQCD ?
- Large N_C framework: matching Green's functions with QCD short-distance constraints (e.g. using Resonance Chiral Theory $\longrightarrow$ many unknown couplings enter).
- In both approaches: experimental information on various on-shell and off-shell hadronic form factors would be very helpful.
- Problems: how to go beyond leading order in N_C with such resonance Lagrangians, e.g. loops with resonances, finite width effects ? Some progress made in recent years. What about broad states, like the scalars ? Issue of double-counting.
- Supplement these approaches with model-independent low-energy theorems in some particular limits (e.g. $m_\mu \longrightarrow 0, m_\pi \longrightarrow 0$ with m_μ/m_π fixed).

Backup slides

Hadronic light-by-light scattering: ChPT approach

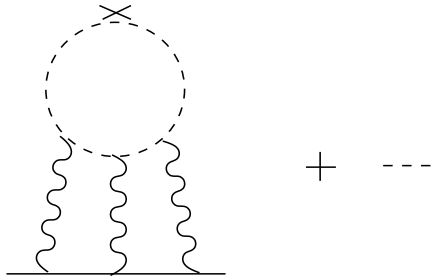
EFT for $E \ll 1$ GeV with pions, photons and muons

[de Rafael '94; M. Knecht, A.N., M. Perrottet, E. de Rafael, '02; Ramsey-Musolf + Wise '02]

Note: chiral counting here refers to contribution to a_μ. Differs from counting in de Rafael '94 !

Contributions to $a_\mu^{\text{LbyL;had}}$

$\mathcal{O}(p^6)$: charged pion loop
(finite, subleading in $1/N_C$)



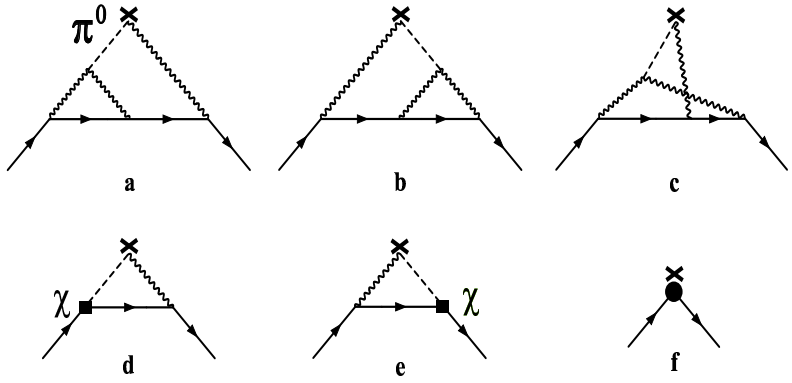
$\mathcal{O}(p^8)$: pion-pole (leading in $1/N_C$)

Divergent 2-loop contribution

$\longrightarrow$ need counterterms

1. One-loop graphs with insertion of χ
   (■) = coupling $\overline{\psi}\,\gamma_\mu\gamma_5\,\psi\,\partial^\mu\pi^0$

2. Local counterterm (●)



$\Rightarrow a_\mu^{\text{LbyL;had}}$ cannot be obtained in (pure) EFT framework
$\longrightarrow$ resonance models for form factors

Hadronic light-by-light scattering: Large log's

Renormalization group in EFT $\Rightarrow$ leading "large" logarithm $\ln^2(\mu_0/m_\mu)$

$$a_\mu^{\text{LbyL;had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f\left(\frac{m_{\pi^\pm}}{m_\mu}, \frac{m_{K^\pm}}{m_\mu}\right) \right. \quad \text{(loops with pions and kaons)}$$

$$\left. + \overbrace{c \approx 0.025 \text{ (universal)}}^{} \; N_C \left(\frac{m_\mu^2}{16\pi^2 F_\pi^2} \frac{N_C}{3}\right) \left[\ln^2 \frac{\mu_0}{m_\mu} + \overbrace{\chi(\mu_0)}^{(\pi^0 \to e^+e^-)} \ln \frac{\mu_0}{m_\mu} + c_0 \right] + \ldots \right.$$

$f = -0.038$; $\mu_0 \sim M_\rho$: hadronic scale, $\ln \frac{M_\rho}{m_\mu} \sim 2$

Problem: π^0-exchange $\longrightarrow$ cancellation between $\ln^2$ and $\ln$:

$$\begin{aligned} a_\mu^{\text{LbyL};\pi^0}\Big|_{\text{VMD}} &= \left(\frac{\alpha}{\pi}\right)^3 c \left[\ln^2 \frac{M_\rho}{m_\mu} + c_1 \ln \frac{M_\rho}{m_\mu} + c_0 \right] \\ &\overset{\text{Fit}}{=} \left(\frac{\alpha}{\pi}\right)^3 c \left[3.94 - 3.30 + 1.08 \right] \\ &= \left[12.3 - 10.3 + 3.4 \right] \times 10^{-10} \\ &= 5.4 \times 10^{-10} \end{aligned}$$

Hadronic vacuum polarization in large-N_C QCD (de Rafael)

Adler function (Minimal Hadronic Approximation)

$$\mathcal{A}(Q^2) \equiv -Q^2 \frac{\partial \Pi_V(Q^2)}{\partial Q^2}$$

$$\mathcal{A}(Q^2)\big|_{\text{MHA}} = \left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right) e^2 \left\{ 2 f_V^2 M_V^2 \frac{Q^2}{(Q^2+M_V^2)^2} + \frac{N_C}{16\pi^2}\frac{4}{3}\frac{Q^2}{Q^2+s_0}(1+\ldots)\right\}$$

Chiral loops (two-pion states) subleading in $1/N_C$

No $1/Q^2$ term in the OPE $\Rightarrow$ fixes s_0: $\quad 2 f_V^2 M_V^2 = \frac{N_C}{16\pi^2}\frac{4}{3} s_0 \left(1+\frac{3}{8}\frac{\alpha_s(s_0)}{\pi}+\ldots\right)$

General relation:

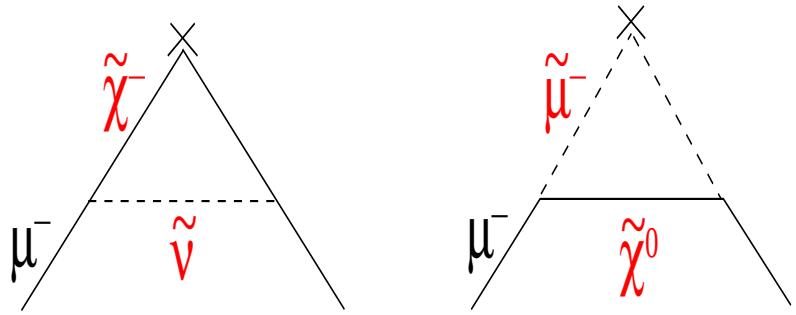
$$a_\mu^{\text{had. v.p.}} = \frac{\alpha}{\pi}\int_0^1 \frac{dx}{x}(1-x)\left(1-\frac{x}{2}\right)\mathcal{A}\left(\frac{x^2}{1-x}m_\mu^2\right)$$

$$a_\mu^{\text{had. v.p.}}\big|_{\text{MHA}} = (5700 \pm 1700)\times 10^{-10} \qquad \text{(30\% systematic error)}$$

Compare with evaluation using experimental data (Davier et al. '03)

$$a_\mu^{\text{had. v.p.}} = (6963 \pm 72)\times 10^{-10}$$

SUSY contributions to a_μ



Chargino $\tilde{\chi}^-$ contribution dominates over neutralino $\tilde{\chi}^0$

Large $\tan\beta$ limit (Czarnecki + Marciano '01):

$$|a_\mu^{\text{SUSY}}| \approx \frac{\alpha(M_Z)}{8\pi \sin^2\theta_W} \frac{m_\mu^2}{M_{\text{SUSY}}^2} \tan\beta \left(1 - \frac{4\alpha}{\pi} \ln \frac{M_{\text{SUSY}}}{m_\mu}\right)$$

$$\approx 130 \times 10^{-11} \left(\frac{100\ \text{GeV}}{M_{\text{SUSY}}}\right)^2 \tan\beta$$

Compare: $a_\mu^{\text{EW}} \approx 150 \times 10^{-11}$

To explain $\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}}(e^+e^-) \approx 290 \times 10^{-11}$

$$\Rightarrow M_{\text{SUSY}} \approx 135 - 425\ \text{GeV} \qquad (4 < \tan\beta < 40)$$